Continuum Mechanics, Spring 2018 CMI
Problem set 3
Due at the beginning of lecture on Monday Feb 12, 2018
Wave equation and Heat equation

1. $\langle \mathbf{8} \rangle$ Suppose $u(x,t)$ and $\tilde{u}(x,t)$ are two solutions of the wave equation subject to the same initial conditions $u(x,0) = \tilde{u}(x,0) = h(x), \dot{u}(x,0) = \dot{\tilde{u}}(x,0) = v(x)$ and the same boundary conditions, say clamped at $x = 0, l$. Use the conservation of energy to show that $u(x,t) = \tilde{u}(x,t)$, i.e., that the solution of the initial-boundary value problem for the wave equation is unique.

2. $\langle \mathbf{10} \rangle$ We expressed the Lagrangian for the wave equation on an interval of length l with periodic boundary conditions in terms of Fourier coefficients of the wave height u and their velocities:

$$L = \int \left[\frac{1}{2}\rho u_t^2 - \frac{1}{2}\tau u_x^2\right] dx = \frac{\rho l}{2}\left[\dot{a}_0^2 + \frac{1}{2}\sum_{n\geq 1}\left(\dot{a}_n^2 + \dot{b}_n^2\right)\right] - \frac{\tau\pi^2}{l}\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2). \tag{1}$$

a_0 is the only cyclic coordinate and its conjugate momentum $\pi_0 = \rho l \dot{a}_0$ is conserved. We can find additional cyclic coordinates by going to 'polar-coordinates' in the a_n, b_n planes. Let $a_n = r_n \cos\theta_n$ and $b_n = r_n \sin\theta_n$.

   (a) $\langle \mathbf{4} \rangle$ Write the Lagrangian in terms of a_0, r_n, θ_n and their velocities.

   (b) $\langle \mathbf{6} \rangle$ Identify infinitely many cyclic coordinates. What are the corresponding conserved conjugate momenta? Express them in terms r_n and θ_n as well as a_n and b_n (and their velocities). Give a suitable descriptive name for these conserved quantities.

3. $\langle \mathbf{5} \rangle$ What are the dimensions of thermal conductivity k that appears in Fourier's law of heat conduction? What are the dimensions of specific heat c_v? As a consequence, check that diffusivity $\alpha = k/\rho c_v$ has dimensions of area per unit time.