Continuum Mechanics, Spring 2018 CMI
Problem set 1
Due at the beginning of lecture on Monday Jan 22, 2018
Wave equation

1. ⟨**4**⟩ Express the potential energy of a stretched string $V = \int_0^L \frac{1}{2}\tau u_x^2 dx$ in terms of the second spatial derivative of the height u, assuming the tension τ is independent of location. Does the expression you obtain hold for clamped and/or open boundaries?

2. ⟨**4**⟩ Use the wave equation to directly verify that the energy $E = \int (\frac{1}{2}\rho u_t^2 + \frac{1}{2}\tau u_x^2)\, dx$ of a string is conserved subject to both clamped and open boundary conditions. You may assume that u is such that time derivatives can be taken inside the integral and that ρ and τ are both constants.

3. ⟨**15**⟩ For linear PDEs with constant coefficients such as the wave equation, we may associate a dispersion relation between angular frequency ω and wave number k (more generally wave vector $\mathbf{k}$). By this we mean the relation between ω and k that must hold for the traveling wave $e^{i(kx-\omega t)}$ to be a solution. Find the dispersion relation $\omega(k)$ for the following equations. In each case compute the phase speed $c_p = \omega/k$ and the group speed $c_g = \frac{d\omega}{dk}$. If the phase speed is independent of wave number, we say the equation is non-dispersive. Classify the equations as dispersive/non-dispersive.

   (a) ⟨**3**⟩ The d'Alembert wave equation $u_{tt} = c^2 u_{xx}$.

   (b) ⟨**3**⟩ The first order wave equation $u_t = c u_x$.

   (c) ⟨**3**⟩ The Fourier heat conduction equation $u_t = \alpha u_{xx}$.

   (d) ⟨**3**⟩ The linearized KdV equation $u_t = \beta u_{xxx}$.

   (e) ⟨**3**⟩ The Schrödinger wave equation $i\hbar\psi_t = -(\hbar^2/2m)\psi_{xx}$.

4. ⟨**15**⟩ Consider the wave equation on the interval $[0, L]$, $u_{tt} = c^2 u_{xx}$ with Dirichlet BCs $u(0,t) = u(L,t) = 0$. The string is released from rest after being plucked upwards to form an isosceles triangle of height s.

   (a) ⟨**6**⟩ What is $u(x, 0)$? Find the Fourier series representation of the initial height profile $u(x, 0)$. How fast do the Fourier coefficients decay?

   (b) ⟨**3**⟩ Write down the Fourier series for $u(x,t)$ obeying the above initial and boundary conditions.

   (c) ⟨**6**⟩ Suppose $c = a = L = 1$. Give the formula for the Fourier coefficients and for $u(x,t)$. Plot the approximate wave height $u(x, 0)$ at $t = 0$ by retaining 1,2,3 and 4 terms in the Fourier series. You may use a plotting software (online or offline) to produce the graphs and roughly reproduce them in hand-drawn sketches.