Classical Mechanics 2, Spring 2016 CMI

Problem set 7

Due by the beginning of lecture on Monday Mar 7, 2016

Canonical Transformations, Liouville's Theorem, Phase portraits

1. $\langle \mathbf{9} \rangle$ Consider a free particle of mass m moving on a line with coordinate q, canonically conjugate momentum p and hamiltonian $H = \frac{p^2}{2m}$. Suppose we make the transformation to new variables $Q = q + kp, P = p$.

   (a) $\langle \mathbf{3} \rangle$ What is the physical dimension of constant k? Argue why the transformation is canonical. Under what circumstances is it an infinitesimal canonical transformation?

   (b) $\langle \mathbf{2} \rangle$ Find a generating function $f(q,p)$ for the corresponding infinitesimal canonical transformation (up to an additive constant).

   (c) $\langle \mathbf{4} \rangle$ Physically interpret the above infinitesimal CT and its generator by a suitable choice of k in terms of familiar physical quantities.

2. $\langle \mathbf{9} \rangle$ Consider a particle of mass m moving in one dimension subject to the double-well potential $V(x) = g(x^2 - a^2)^2$ with $g, a > 0$.

   (a) $\langle \mathbf{3} \rangle$ Give the algebraic equation (in terms of p and x) for the separatrices that divide the phase space between trajectories that are restricted to a single well from those that explore both wells. Specify the energy on the separatrices and draw a figure of the separatrices on the phase plane.

   (b) $\langle \mathbf{3} \rangle$ Find an expression (in terms of x) for the possible slopes of the separatrices dp/dx.

   (c) $\langle \mathbf{2} \rangle$ Find the slopes dp/dx of the separatrices at the origin of phase space ($x = p = 0$) in terms of the parameters a, m, g. Check that the slope has the correct dimension.

   (d) $\langle \mathbf{1} \rangle$ Does the separatrix have a discontinuous derivative at the origin or not?

3. $\langle \mathbf{7} \rangle$ Liouville theorem for infinitesimal CTs for one degree of freedom. Consider the infinitesimal CT $Q = q + \delta q, P = p + \delta p$ generated by $\epsilon f(q,p)$.

   (a) $\langle \mathbf{4} \rangle$ Express $\delta q, \delta p$ as derivatives of f. Write the 2×2 Jacobian matrix $J = \frac{\partial(Q,P)}{\partial(q,p)}$ explicitly with entries expressed as partial derivatives of f. Write $J = I + \epsilon F$ for an appropriate matrix F.

   (b) $\langle \mathbf{3} \rangle$ Express $\det J$ as a quadratic polynomial in ϵ and show that the linear term is equal to the trace of F. Thus find $\det J$ ignoring quadratic terms in ϵ.