Classical Mechanics 2, Spring 2016 CMI

Problem set 3

Due by the beginning of lecture on Monday Jan 25, 2016

Coordinate invariance of EL equations and action principle

1. $\langle \mathbf{8} \rangle$ Consider a free particle on the positive half line $q > 0$ with Lagrangian $L(q, \dot{q}) = \frac{1}{2} m \dot{q}^2$ and equation of motion $\ddot{q} = 0$. Let us choose a new coordinate on configuration space $Q = q^2$.

   (a) $\langle \mathbf{3} \rangle$ Though there is no force, the equation of motion is not $\ddot{Q} = 0$. Find the correct equation of motion in terms of Q by transforming $\ddot{q} = 0$.

   (b) $\langle \mathbf{2} \rangle$ Transform the Lagrangian and express it as a function $\tilde{L}(Q, \dot{Q})$.

   (c) $\langle \mathbf{2} \rangle$ Find the EL equation in the new variable and express it as a 2nd order ODE for Q.

   (d) $\langle \mathbf{1} \rangle$ Verify that the EL equation for Q agrees with that obtained by transforming $\ddot{q} = 0$ to the new coordinate.

2. $\langle \mathbf{16} \rangle$ Consider a particle of mass m in the potential $V(x) = \frac{1}{2} m \omega^2 x^2$. Suppose $x(t)$ is a trajectory between $x_i(t_i)$ and $x_f(t_f)$ and let $x(t) + \delta x(t)$ be a neighboring path with $\delta x(t_i) = \delta x(t_f) = 0$.

   (a) $\langle \mathbf{4} \rangle$ Write the classical action of the path $x + \delta x$ as a quadratic Taylor polynomial in δx. Show that you get the following expression. What can you say about S_1?

   $$S[x+\delta x] = S_0 + S_1 + S_2 = S[x] - \int_{t_i}^{t_f} (m\ddot{x} + m\omega^2 x) \delta x dt + \int_{t_i}^{t_f} \left[\frac{1}{2} m (\delta \dot{x})^2 - \frac{1}{2} m \omega^2 (\delta x)^2 \right] dt$$

   (b) $\langle \mathbf{2} \rangle$ For what values of κ is $x(t) + \delta x(t)$ a legitimate neighboring path for the variation

   $$\delta x(t) = \epsilon \, \sin \kappa (t - t_i) \; ? \qquad (1)$$

   (c) $\langle \mathbf{3} \rangle$ Evaluate $S_2[\delta x]$ for all the allowed values of κ.

   (d) $\langle \mathbf{3} \rangle$ Take $\Delta t = t_f - t_i = 10$s and $\omega = 1$ Hz. Find a path that can be made arbitrarily close to the trajectory $x(t)$, whose action is *less* than that of $x(t)$.

   (e) $\langle \mathbf{3} \rangle$ Take $\Delta t = t_f - t_i = 10$s and $\omega = 1$ Hz. Find a path that can be made arbitrarily close to the trajectory $x(t)$, whose action is *more* than that of $x(t)$.

   (f) $\langle \mathbf{1} \rangle$ What sort of an extremum of action is the classical trajectory?