Classical Mechanics 2, Spring 2014 CMI

Problem set 8

Due by the beginning of lecture on Monday Feb 17, 2014

Poisson brackets, angular momentum Poisson brackets

1. $\langle \mathbf{6} \rangle$ Consider a free particle moving on the half line $q > 0$ with Lagrangian $L(q, \dot{q}) = \frac{1}{2} m \dot{q}^2$. Suppose we make the change of coordinate to $Q = q^2$.

   (a) $\langle \mathbf{2} \rangle$ Find the new Lagrangian $\tilde{L}(Q, \dot{Q})$.

   (b) $\langle \mathbf{2} \rangle$ Find the momentum P conjugate to Q. Express P as a function of Q and $\dot{Q}$ and as a function of q and p.

   (c) $\langle \mathbf{2} \rangle$ Find the Poisson bracket $\{Q, P\}$ and compare with $\{q, p\}$.

2. $\langle \mathbf{22} \rangle$ Angular momentum Poisson brackets from $\{r_i, p_j\} = \delta_{ij}$. We place all indices down-stairs in this problem, and sum repeated indices.

   (a) $\langle \mathbf{1} \rangle$ Define the Levi-Civita symbol ϵ_{ijk} for $1 \leq i, j, k \leq 3$ by the condition that it is anti-symmetric under interchange of any pair of *neighbouring* indices along with the 'initial' condition $\epsilon_{123} = 1$. Show that it is anti-symmetric under interchange of any pair of indices, (not necessarily neighbors).

   (b) $\langle \mathbf{3} \rangle$ Give the values of all the components of the ϵ symbol. How many components are there in all?

   (c) $\langle \mathbf{3} \rangle$ From $\vec{L} = \vec{r} \times \vec{p}$ write the three components of angular momentum L_x, L_y, L_z in terms of x, y, z, p_x, p_y, p_z and show that they may be summarized by the formula $L_i = \epsilon_{ijk} r_j p_k$. Repeated indices are summed.

   (d) $\langle \mathbf{7} \rangle$ Use the properties of the Poisson bracket and the identity

   $$\sum_{i=1}^{3} \epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}. \tag{1}$$

   to show that

   $$\{L_i, L_j\} = r_i p_j - r_j p_i. \tag{2}$$

   For uniformity of notation, begin by taking $L_i = \epsilon_{ikl} r_k p_l$ and $L_j = \epsilon_{jmn} r_m p_n$

   (e) $\langle \mathbf{2} \rangle$ Use the above formula for L_i to show that $\epsilon_{ijk} L_k = r_i p_j - r_j p_i$.

   (f) $\langle \mathbf{2} \rangle$ Conclude from the last two questions that

   $$\{L_i, L_j\} = \epsilon_{ijk} L_k \tag{3}$$

   Compare this with the 3 formulae derived in lecture: $\{L_x, L_y\} = L_z$ and cyclic permutations thereof. Do they agree?

   (g) $\langle \mathbf{2} \rangle$ Use the above results to show that

   $$\{\{L_i, L_j\}, L_k\} = \delta_{ik} L_j - \delta_{jk} L_i. \tag{4}$$

   (h) $\langle \mathbf{2} \rangle$ Show that the components of angular momentum satisfy the Jacobi identity

   $$\{\{L_i, L_j\}, L_k\} + \{\{L_j, L_k\}, L_i\} + \{\{L_k, L_i\}, L_j\} = 0. \tag{5}$$