Classical Mechanics 2, Spring 2014 CMI

Problem set 4

Due by the beginning of lecture on Monday Jan 27, 2014

Harmonic oscillator and action principle

1. $\langle \mathbf{5} \rangle$ Recall that the general solution of $\ddot{x} = -\omega^2 x$ is $x(t) = a\cos\omega t + b\sin\omega t$ where a, b are constants of integration. Find the unique classical trajectory connecting $x(t_i) = x_i$ and $x(t_f) = x_f$ assuming $\omega\,\Delta t \neq n\pi$ for any integer n. Here $\Delta t = t_f - t_i$. You may use the abbreviations $c_i = \cos\omega t_i$, $s_f = \sin\omega t_f$ etc.

2. $\langle \mathbf{17} \rangle$ Consider a particle of mass m in the potential $V(x) = \frac{1}{2} m\omega^2 x^2$. Suppose $x(t)$ is a trajectory between $x_i(t_i)$ and $x_f(t_f)$ and let $x(t) + \delta x(t)$ be a neighboring path with $\delta x(t_i) = \delta x(t_f) = 0$.

   (a) $\langle \mathbf{5} \rangle$ Write the classical action of the path $x + \delta x$ as a quadratic Taylor polynomial in δx. Show that you get the following expression. What can you say about S_1?

   $$S[x+\delta x] = S_0 + S_1 + S_2 = S[x] - \int_{t_i}^{t_f} (m\ddot{x} + m\omega^2 x)\,\delta x\,dt + \int_{t_i}^{t_f} \left[\frac{1}{2} m(\delta\dot{x})^2 - \frac{1}{2} m\omega^2 (\delta x)^2\right] dt$$

   (b) $\langle \mathbf{2} \rangle$ For what values of κ is $x(t) + \delta x(t)$ a legitimate neighboring path for the variation

   $$\delta x(t) = \epsilon\ \sin\kappa(t - t_i)\ ? \qquad (1)$$

   (c) $\langle \mathbf{3} \rangle$ Evaluate $S_2[\delta x]$ for all the allowed values of κ.

   (d) $\langle \mathbf{3} \rangle$ Take $\Delta t = t_f - t_i = 10\,\text{s}$ and $\omega = 1$ Hz. Find a path that can be made arbitrarily close to the trajectory $x(t)$, whose action is *less* than that of $x(t)$.

   (e) $\langle \mathbf{3} \rangle$ Take $\Delta t = t_f - t_i = 10\,\text{s}$ and $\omega = 1$ Hz. Find a path that can be made arbitrarily close to the trajectory $x(t)$, whose action is *more* than that of $x(t)$.

   (f) $\langle \mathbf{1} \rangle$ What sort of an extremum of action is the classical trajectory?