Classical Mechanics 2, Spring 2014 CMI
Problem set 3
Due by the beginning of lecture on Monday Jan 20, 2014
Lagrangian

1. ⟨**15**⟩ Practice with polar coordinates. Consider a particle moving on the x, y plane $z = 0$ in a central potential $V(r)$. The Lagrangian is $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(r)$. Define plane polar coordinates for the particle's location via $x = r\cos\phi, y = r\sin\phi$. Abbreviate $\sin\phi = s, \cos\phi = c$. Recall that the unit vector in the radial direction is $\hat{r} = c\hat{x} + s\hat{y}$ and that linear momentum is $\mathbf{p} = m\dot{x}\hat{x} + m\dot{y}\hat{y}$. The Euler-Lagrange equations in polar coordinates were found to be $m\ddot{r} = mr\dot{\phi}^2 - V'(r)$ and $mr\ddot{\phi} = -2m\dot{r}\dot{\phi}$.

   (a) ⟨**2**⟩ Show that $\dot{r} = c\dot{x} + s\dot{y}$

   (b) ⟨**2**⟩ Show that $\dot{\phi} = \frac{1}{r^2}(x\dot{y} - y\dot{x})$.

   (c) ⟨**2**⟩ Show that the momentum $p_r = m\dot{r}$ conjugate to r, is just the radial component of linear momentum $\mathbf{p}\cdot\hat{r}$.

   (d) ⟨**2**⟩ Show that the momentum $p_\phi = mr^2\dot{\phi}$ conjugate to ϕ is the z-component of angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ by explicitly calculating the cross product.

   (e) ⟨**3**⟩ Draw the unit vector in the direction of increasing ϕ, called $\hat{\phi}$, in a diagram. Express $\hat{\phi}$ as a linear combination of $\hat{x}, \hat{y}$, using the diagram and an appropriate triangle. Choose $0 < \phi < \pi/2$. Check that $\hat{r}\cdot\hat{\phi} = 0$.

   (f) ⟨**2**⟩ Suppose we define the angular velocity as $\vec{\omega} = \dot{\phi}\hat{z}$ and a fictitious force $\vec{F}_c = -2\vec{\omega}\times\vec{p}$. Show that the $\hat{\phi}$ component of $\vec{F}_c$ is what appears on the rhs of the Euler-Lagrange equation $mr\ddot{\phi} = -2m\dot{r}\dot{\phi}$.

   (g) ⟨**2**⟩ $\hat{x}, \hat{y}$ are constant unit vectors, in the sense that they point in the same direction everywhere on configuration space and also at every point along a trajectory: $\frac{d\hat{x}}{dt} = \frac{d\hat{y}}{dt} = 0$. But $\hat{r}, \hat{\phi}$ change direction from place to place. Show that

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{r} \quad \text{and} \quad \frac{d\hat{r}}{dt} = \dot{\phi}\hat{\phi} \tag{1}$$

2. ⟨**7**⟩ Consider a particle whose dynamics is specified by the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 + b(q)\dot{q}. \tag{2}$$

   Here $b(q)$ is some differentiable function of q.

   (a) ⟨**3**⟩ Find the momentum conjugate to q and the equation of motion. What sort of motion does the Lagrangian describe?

   (b) ⟨**4**⟩ Explain the nature of this particle by examining the principle of extremal action $S = \int_{t_0}^{t_1} L\, dt$ for this Lagrangian. Can you relate this action to a more familiar one, how do they differ?