Classical Mechanics 2, Spring 2014 CMI
Problem set 2
Due by the beginning of lecture on Monday Jan 13, 2014
Conserved energy, simple harmonic oscillator

1. $\langle \mathbf{4} \rangle$ Derive a conserved energy for Newton's equation for three degrees of freedom $m\ddot{x}_i = f_i$ where $i = 1, 2, 3$ or $m\ddot{\mathbf{r}} = \mathbf{f}$ where the cartesian components of the force are $f_i = -\frac{\partial V}{\partial x_i}$. Proceed by finding a suitable integrating factor.

2. $\langle \mathbf{12} \rangle$ Recall that for motion of a particle of mass m on a line, the solutions $x(t)$ of Newton's equation with energy E and initial position x_0 was reduced to the integral

$$t - t_0 = \pm \int_{x_0}^{x} \frac{dy}{\sqrt{\frac{2}{m}(E - V(y))}} \tag{1}$$

Consider a simple harmonic oscillator potential $V(x) = \frac{1}{2}kx^2$ for which $E \geq 0$ and let $\omega = \sqrt{\frac{k}{m}}$.

(a) $\langle \mathbf{4} \rangle$ Evaluate the integral (use $\int \frac{du}{\sqrt{1-u^2}} = \arcsin u$) and solve for the trajectories with given E, x_0. Show that you get

$$x(t) = \pm \sqrt{\frac{2E}{k}} \sin\left(\omega(t - t_0) \pm \arcsin\left(\sqrt{\frac{k}{2E}}\, x_0\right)\right). \tag{2}$$

The upper signs correspond to one solution and the lower signs to another solution.

(b) $\langle \mathbf{2} \rangle$ Specialize to the case where the particle starts from the equilibrium position at $t_0 = 0$ and simplify the formula for $x(t)$. Also find the momentum $p(t)$.

(c) $\langle \mathbf{2} \rangle$ Show that $x(t)$ satisfies the 'initial conditions' $x(0) = 0$ and $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E$.

(d) $\langle \mathbf{1} \rangle$ For $E > 0$, indicate pictorially (in configuration space) how the two solutions obtained in the previous question differ.

(e) $\langle \mathbf{3} \rangle$ On the x-p phase plane, draw a phase portrait for the simple harmonic oscillator indicating at least two qualitatively different trajectories and the arrow of time. Indicate the initial portions of the trajectories corresponding to the two solutions obtained above.