Classical Mechanics 2, Spring 2014 CMI
Problem set 14
Due by 10am on Tuesday Apr 15, 2014
Rotation matrices and axis of rotation

1. $\langle \mathbf{8} \rangle$ Consider the 2D rotation matrix $R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ where $s = \sin\theta$ and $c = \cos\theta$.

   (a) $\langle \mathbf{2} \rangle$ Find the eigenvalues $\lambda_\pm$ of R.

   (b) $\langle \mathbf{4} \rangle$ Find the corresponding unit norm eigenvectors $v_\pm$. In what sense are they orthogonal?

   (c) $\langle \mathbf{2} \rangle$ All points on an axis of rotation must be left unchanged by the rotation. Assuming R is a non-trivial rotation $\theta \neq 0$, try to find a vector in $\mathbb{R}^2$ that qualifies as an axis of rotation[1].

2. $\langle \mathbf{4} \rangle$ Now consider a rotation of 3D space of the form $R = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where s, c are as above.

   Find the eigenvalues and corresponding eigenvectors. Give an axis of rotation and specify the angle and sense of rotation.

3. $\langle \mathbf{12} \rangle$ More generally, suppose $R \in SO(3)$ is a rotation matrix, i.e., $R^t R = I$ and $\det R = 1$.

   (a) $\langle \mathbf{4} \rangle$ Show that every such rotation matrix R must have at least one one real eigenvalue and that non-real eigenvalues must come in complex-conjugate pairs.

   (b) $\langle \mathbf{2} \rangle$ Show that if a rotation has 3 real eigenvalues $\{\lambda, \mu, \nu\}$, then the set of eigenvalues must be either $\{1, 1, 1\}$ or $\{1, -1, -1\}$. Hint: In its eigenbasis, R is diagonal with entries $\{\lambda, \mu, \nu\}$.

   (c) $\langle \mathbf{3} \rangle$ What sort of rotations do the two cases in the previous question correspond to? Mention in words an axis and angle of rotation in each case.

   (d) $\langle \mathbf{3} \rangle$ Assuming that R does not have 3 real eigenvalues, show that it has precisely one real eigenvalue and find it. Hint: The condition $R^t R = I$ will in general *not* hold in its eigenbasis since it involves a *complex* change of basis. However, the condition $R^\dagger R = I$ where $R^\dagger = (R^t)^*$ is the complex conjugate transpose holds in any basis as does $\det R = 1$. The eigenvector corresponding to the real eigenvalue may be taken as the axis of rotation.

[1] If $R = I$ is the identity, then any non-zero vector in $\mathbb{R}^2$ qualifies as an axis of rotation with zero angle of rotation.