Classical Mechanics 2, Spring 2014 CMI

Problem set 12

Due by the beginning of lecture on Monday Mar 31, 2014

Inertia tensor of a rigid body with planes/axis of symmetry.

1. $\langle \mathbf{20} \rangle$ Consider a rigid body with mass density $\rho(\mathbf{r})$ and total mass M. Suppose it has two mutually orthogonal planes of reflection symmetry, i.e., the mass distribution is unchanged upon reflection in either plane. Examples are a conical top, a uniform right circular cylinder and a uniform right elliptical cylinder.

   (a) $\langle \mathbf{4} \rangle$ Set up (in a figure) a Cartesian coordinate system that is adapted to the above two planes of symmetry, which may be taken as the xz and yz planes. Write two formulas for the reflection invariance of $\rho(x, y, z)$.

   (b) $\langle \mathbf{4} \rangle$ Show that the center of mass must lie on the intersection of the two planes of symmetry. On which axis does the CM $(\bar{X}, \bar{Y}, \bar{Z})$ lie?

   (c) $\langle \mathbf{4} \rangle$ Show that in the above Cartesian basis, the inertia tensor $I_{ij} = \int d^3\mathbf{r}\, \rho\, (r^2 \delta_{ij} - r_i r_j)$ is diagonal, and write integral expressions for the three principal moments of inertia and specify the corresponding principal axes. Begin by writing out the inertia matrix of 9 components and consider the individual integrals.

   (d) $\langle \mathbf{4} \rangle$ Now suppose we restrict to a rigid body that has an axis of rotational symmetry, e.g. a right circular cylinder or conical top. The mass distribution is symmetric under rotation by any angle about the axis. How many pairs of orthogonal planes of reflection symmetry does such a rigid body have? Draw a picture.

   (e) $\langle \mathbf{4} \rangle$ Use rotation-invariance by $\pi/2$ $(x' = -y, y' = x)$ to show that the integrals for two of the principal moments of inertia of a rigid body with an axis of rotational symmetry are equal.