Classical Mechanics 2, Spring 2014 CMI

Problem set 10

Due by the beginning of lecture on Monday Mar 17, 2014

Generating functions for finite canonical transformations.

1. $\langle \mathbf{12} \rangle$ We seek a generator of type $F_3(p, Q, t)$ for a finite canonical transformation from old to new canonical variables and hamiltonian $(q, p; H) \to (Q, P; K)$.

   (a) $\langle \mathbf{5} \rangle$ Staring from appropriate action principles for Hamilton's equations in the old and new variables, express the equations of transformation in terms of F_3, i.e., find q, P, K in terms of F_3

   (b) $\langle \mathbf{4} \rangle$ By comparing the relations among differentials for F_1 and F_3, express F_3 as a Legendre transform of $F_1(q, Q)$

   (c) $\langle \mathbf{3} \rangle$ Find a generating function of type $F_3(p, Q)$ that generates the scaling CT $Q = \lambda q . P = p/\lambda$.

2. $\langle \mathbf{12} \rangle$ Consider the finite canonical transformation, corresponding to a rotation of the phase plane

$$Q = cq - sp \quad \text{and} \quad P = sq + cp \quad \text{where} \quad s = \sin\theta \quad \text{and} \quad c = \cos\theta. \tag{1}$$

   (a) $\langle \mathbf{2} \rangle$ We seek a generating function of type-II $W(q, P)$ for the above finite CT. Find the differential equations that $W(q, P)$ must satisfy to ensure it generates the above CT.

   (b) $\langle \mathbf{5} \rangle$ Integrate the differential equations and give a simple formula for the generating function $W(q, P)$.

   (c) $\langle \mathbf{1} \rangle$ Verify that your proposed function $W(q, P)$ indeed generates the above finite rotation.

   (d) $\langle \mathbf{2} \rangle$ Find a generating function of type $F_1(q, Q)$ that generates the same finite rotation via an appropriate Legendre transform from $W(q, P)$. This provides an example of a CT that admits a generator of both type I and II.

   (e) $\langle \mathbf{2} \rangle$ Try to find a generator of type I for the identity CT, by letting the angle of rotation go to zero. What do you find?