Classical Mechanics 1, Autumn 2022 CMI
Problem set 3
Due by 6pm, Friday Aug 26, 2022
Polar coordinates

1. $\langle \mathbf{3+6} \rangle$ Recall Cartesian (x, y) and plane polar coordinates (r, θ) discussed in the lecture. A particle moving on the plane has position vector $\boldsymbol{r}(t)$ relative to the fixed origin. (a) Find an expression for its velocity $\boldsymbol{v}(t)$ in plane polar coordinates (as a linear combination of $\hat{r}, \hat{\theta}$ with coefficients that depend on the polar coordinates and their time derivatives). Interpret the terms you get. (b) Similarly, find an expression for the acceleration $\boldsymbol{a} = \ddot{\boldsymbol{r}}$, again as a linear combination of $\hat{r}, \hat{\theta}$. Interpret the 4 terms that arise in $\ddot{\boldsymbol{r}}$ by giving them suitable names or indicating in words the nature of the terms.

2. $\langle \mathbf{4} \rangle$ Suppose we make a linear change from Cartesian coordinates (x, y) to new coordinates (u, v) on the plane, given by $u = ax + by$ and $v = cx + dy$ for some real constants a, b, c, d. It is possible to show that vectors pointing in the direction of increasing u and v holding the other fixed are given by $\boldsymbol{u} = a\hat{x} + b\hat{y}$ and $\boldsymbol{v} = c\hat{x} + d\hat{y}$ where $\hat{x}$ and $\hat{y}$ are the usual unit vectors in the directions of increasing x and y respectively. Find conditions on (a, b, c, d) to guarantee that $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthonormal. Interpret the answer in terms of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

3. $\langle \mathbf{5} \rangle$ As we will discuss in the lectures, spherical polar coordinates in 3d are defined via $z = r\cos\theta, x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$. In spherical polar coordinates (r, θ, ϕ), we define three vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ via linear combinations of the Cartesian unit vectors:

$$\begin{aligned} \hat{r} &= \cos\theta\hat{z} + \sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y}), \\ \hat{\theta} &= -\sin\theta\hat{z} + \cos\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) \quad \text{and} \\ \hat{\phi} &= -\sin\phi\hat{x} + \cos\phi\hat{y}. \end{aligned} \tag{1}$$

Verify that $(\hat{r}, \hat{\theta}, \hat{\phi})$ is a right-handed orthonormal system.

4. $\langle \mathbf{4} \rangle$ Spherical polar coordinates. Bearing in mind Equation (1), express $\dot{\hat{r}}$ along a particle's trajectory $(r(t), \theta(t), \phi(t))$ as a linear combination of $\hat{r}, \hat{\theta}$ and $\hat{\phi}$. Qualitatively explain the coefficient of $\hat{r}$.