Classical Mechanics (PG), Autumn 2013 CMI
Problem set 9
Due at the beginning of lecture on Monday Sept 23, 2013
Variational principle for shortest time

1. $\langle \mathbf{25} \rangle$ Consider a particle of mass m sliding down a frictionless wire (curve) in a vertical plane subject to Earth's vertically downward gravitational acceleration g. The x axis is chosen horizontal and rightwards and y axis vertically **downwards** and it starts **from rest** at point $A : (x, y) = (0, 0)$. The potential energy is defined to be zero at the height $y = 0$. The wire ends at a fixed point B below (starting from rest, the particle cannot climb higher than its initial height, so $y \geq 0$) and to the right of A. We wish to find the shape $y(x)$ of the wire which minimizes the time taken to reach $B : (x_f, y_f)$.

   (a) $\langle \mathbf{2} \rangle$ Draw a rough diagram of the above situation indicating the axes and the points mentioned and a proposed curve of shortest time (we will see that it is not a straight line).

   (b) $\langle \mathbf{1} \rangle$ What is the numerical value of the conserved energy of the particle?

   (c) $\langle \mathbf{5} \rangle$ Show that the time taken to go from A to B is

$$T_{AB} = \int_0^{x_f} \sqrt{\frac{1 + y'(x)^2}{2gy}} dx. \tag{1}$$

   (d) $\langle \mathbf{5} \rangle$ Define the 'Lagrange' function $L(y, y') = \sqrt{\frac{1+y'(x)^2}{2gy}}$ and obtain the Euler-Lagrange equation for extremization of T_{AB}. Show that you get

$$2yy'' + y'^2 + 1 = 0. \tag{2}$$

   (e) $\langle \mathbf{3} \rangle$ We will treat this as a sort of Newton equation and try to integrate it to find $y(x)$. Find an integrating factor (like for Newton's equation) and integrate this equation once, show that you get (here R is a constant length of integration)

$$y(1 + y'^2) = R \tag{3}$$

   (f) $\langle \mathbf{3} \rangle$ Reduce this 1st order ODE to quadrature and integrate it by the obvious trigonometric substitution $y = R \sin^2 \theta$ from $(x, y) = (0, 0)$ to (x, y). Show that you get

$$\frac{x}{R} = \arcsin \sqrt{\frac{y}{R}} - \sqrt{\frac{y}{R}\left(1 - \frac{y}{R}\right)} \tag{4}$$

   (g) $\langle \mathbf{3} \rangle$ Show that the equation for the curve of shortest time is not a straight line, but in parametric form is the equation for a cycloid (given below). How is ϕ related to θ? Plot the cycloid on a graph, showing R, bearing in mind that y increases downwards.

$$x = \frac{R}{2}(\phi - \sin \phi) \quad \text{and} \quad y = \frac{R}{2}(1 - \cos \phi). \tag{5}$$

   (h) $\langle \mathbf{3} \rangle$ What are the coordinates of the bottom of the cycloid, and what is the time taken to go from $(0, 0)$ to the bottom of the cycloid?