Classical Mechanics (PG), Autumn 2013 CMI
Problem set 8
Due at the beginning of lecture on Wednesday Sept 18, 2013
Period of simple pendulum

1. $\langle \mathbf{16} \rangle$ To illustrate the use of series expansions for elliptic integrals, let us find a 'low energy' expansion for the period of a simple pendulum with a bob of mass m suspended from a rigid rod of length l from a fixed pivot and free to oscillate in a plane under Earth's gravitational acceleration g. Suppose the zero of potential energy is chosen at the level of the pivot. For low energies $E \gtrsim -mgl$, $\epsilon = E/mgl \gtrsim -1$ and $k = \sqrt{\frac{1}{2}(1+\epsilon)} \gtrsim 0$. We wish to find a power series for the period of oscillation $T(k)$. Define the complete elliptic integral of the first kind by

$$K(k) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \quad \text{for} \quad 0 \le k < 1. \tag{1}$$

(a) $\langle \mathbf{4} \rangle$ Show that the period of librational motion for $E < mgl$ is

$$T(k) = \frac{4}{\omega} K(k) \tag{2}$$

(b) $\langle \mathbf{5} \rangle$ Show that

$$K(k) = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\, k^{2n}}{2^n\, n!} I_n \quad \text{where} \quad I_n = \int_0^1 \frac{s^{2n}\, ds}{\sqrt{1-s^2}}. \tag{3}$$

(c) $\langle \mathbf{4} \rangle$ Use the 'standard' definite integral

$$\int_0^{\pi/2} \sin^{2n}\theta \, d\theta = \frac{\sqrt{\pi}\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)} \tag{4}$$

and properties of the Gamma function ($\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(n+1) = n\Gamma(n)$) to show that

$$I_n = \frac{\pi}{2} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n\, n!} \tag{5}$$

(d) $\langle \mathbf{3} \rangle$ Use the above results to obtain the following series expansion for the period of a pendulum

$$T(k) = 2\pi\sqrt{\frac{l}{g}} \left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + \cdots \right] \tag{6}$$

For small oscillations $k \to 0$ it reduces to the well known formula $T = 2\pi\sqrt{l/g}$.