Classical Mechanics (PG), Autumn 2013 CMI

Problem set 7

Due at the beginning of lecture on Wednesday Sept 4, 2013

Action principle, Legendre transform

1. $\langle \mathbf{6} \rangle$ Give an example from the Kepler problem to illustrate the fact that there may be more than one trajectory joining given initial and final locations at specified times. Illustrate with a figure showing $\mathbf{r}(t_i), \mathbf{r}(t_f)$ and more than one trajectory connecting them. You must give formulae for t_i, t_f, $\mathbf{r}(t_i)$ and $\mathbf{r}(t_f)$.

2. $\langle \mathbf{4} \rangle$ The first law of thermodynamics says that the change in internal energy of a gas is equal to the heat supplied to the gas minus the work done by the gas. For infinitesimal reversible changes, $dU = TdS - PdV$. Here dU is the increase in internal energy, P the pressure, dV the increase in volume dS the increase in entropy and T the absolute temperature.

   (a) $\langle \mathbf{1} \rangle$ What are the independent variables that U depends on?

   ANS: $U(S, V)$. You need two independent variables on the thermodynamic state space, say S, V. The equation of state relates P, V, T so you need only two of them to specify the thermodynamic state, say P, T. Instead of P we can use the conjugate variable V. Instead of T we may use the conjugate variable S.

   (b) $\langle \mathbf{1} \rangle$ Helmholtz free energy may be introduced via the formula $F = U - TS$. Find the independent variables that F depends on.

   (c) $\langle \mathbf{1} \rangle$ Express the pressure and entropy in terms of the Helmholtz free energy.

   (d) $\langle \mathbf{1} \rangle$ Write a formula for Helmholtz free energy as a Legendre transform of the internal energy. Indicate which variable to extremize in and give the condition for an extremum.

3. $\langle \mathbf{7} \rangle$ Find a smooth convex function $L(v)$ of a real variable v whose Legendre transform $H(p) = \mathrm{ext}_v(pv - L(v))$ is the same function as L. In other words, $H(x) = L(x)$ for any $x \in \mathbb{R}$. You may first try to guess such a function using physical knowledge. But you must also formulate the above condition as an equation and try to solve it to determine such a function.