Classical Mechanics (PG), Autumn 2013 CMI

Problem set 5

Due at the beginning of lecture on Monday August 26, 2013

Time period of oscillations as a function of energy

1. $\langle \mathbf{3} \rangle$ For the Kepler problem, find the time period $T(E)$ as a function of energy $E < 0$ for the bound (necessarily elliptical) orbits. Show that $T(E)$ diverges as $E \to 0^-$ and find the manner in which it diverges. Hint: You do not need to evaluate any integrals, use the laws and formulae we have derived.

2. $\langle \mathbf{14} \rangle$ Consider a particle in the 1D potential $V(x) = -V_o \,\mathrm{sech}^2(x/l)$ where $V_o > 0$ and $l > 0$ is a length scale.

   (a) $\langle \mathbf{3} \rangle$ Roughly plot the potential, label the axes, mark l and V_0. Indicate an energy for which there is oscillatory motion.

   (b) $\langle \mathbf{1} \rangle$ For $0 > E \geq -V_0$, find the turning points $\pm x_0$ ($x_0 \geq 0$) of the oscillatory motion.

   (c) $\langle \mathbf{2} \rangle$ Show that the period of oscillations in this potential is given by

   $$T(E) = 2\sqrt{\frac{2m}{V_o}} \int_0^{x_0} \frac{\cosh(x/l)\; dx}{\sqrt{1 - f\cosh^2(x/l)}} \tag{1}$$

   where f is the energy fraction $f = |E|/V_o$, satisfying $0 < f \leq 1$.

   (d) $\langle \mathbf{5} \rangle$ Without evaluating any integrals explicitly, obtain a very simple formula for $T(E)$. Hint: The substitution $y = f\cosh^2(x/l)$ suggests itself. It also to helps to use results of problem set 2 to show that

   $$\pi = \int_V^W \frac{dE}{\sqrt{(E-V)(W-E)}} = \int_f^1 \frac{dy}{\sqrt{(1-y)(y-f)}}. \tag{2}$$

   Note: If you did not answer this question in problem set 3 correctly, then you must attempt it again.

   (e) $\langle \mathbf{3} \rangle$ How does the time period $T(E)$ depend on the depth of the potential V_o? Qualitatively indicate why the behaviour of $T(E)$ as $E \to 0^-$ is physically expected.