Classical Mechanics (PG), Autumn 2013 CMI
Problem set 3
Due at the beginning of lecture on Monday August 19, 2013
Collision, oscillations

1. $\langle \mathbf{5} \rangle$ Consider a particle of mass m in a $V(r) = -\frac{\alpha}{r^n}$ potential for $n > 0$. It starts at $t = 0$ at a radial distance r_0 with a radially inward directed velocity (≥ 0). Will a collision occur in finite time t_c? Why? If so, find the asymptotic behavior of the radial distance $r(t)$ near the collision time $t \approx t_c$.

2. $\langle \mathbf{25} \rangle$ Consider the matrix $M = \begin{pmatrix} -5 & 4 \\ -9 & 7 \end{pmatrix}$ representing a linear transformation acting on $\mathbb{R}^2$ in the standard basis.

   (a) $\langle \mathbf{1} \rangle$ Find $\det M$, $\operatorname{tr} M$.

   (b) $\langle \mathbf{1} \rangle$ Deduce its eigenvalues and also find the characteristic polynomial of M.

   (c) $\langle \mathbf{2} \rangle$ Find linearly independent eigenvectors u corresponding to each eigenvalue. To avoid fractions, normalize all eigenvectors so that their upper component is 2.

   (d) $\langle \mathbf{2} \rangle$ Can M be diagonalized by a change of basis? Why?

   (e) $\langle \mathbf{2} \rangle$ There is a basis for $\mathbb{R}^2$ consisting of eigenvectors and generalized eigenvectors of M in which M takes a simple form. A generalized eigenvector $v \neq 0$ is one which is annihilated by $(M - \lambda I)^2$ in this case. Find all such v and pick one really simple $\tilde{v}$ that is linearly independent of the genuine eigenvectors u.

   (f) $\langle \mathbf{2} \rangle$ Check that $(M - \lambda I)\tilde{v} = cu$ is proportional to an eigenvector u with eigenvalue λ. Why did this have to be the case?

   (g) $\langle \mathbf{5} \rangle$ Now find the matrix $\tilde{J}$ representing M in the new basis consisting of $u, \tilde{v}$. Mention any plausible features of $\tilde{J}$ that are manifest. Hint: The matrix $\tilde{J} = S^{-1}MS$ where the columns of S are $u, \tilde{v}$ *in that order*. S is the similarity transformation from the old to the new basis.

   (h) $\langle \mathbf{4} \rangle$ To get the Jordan normal form J of M we want the north-east corner entry in J to be equal to 1 (first super diagonal). This is achieved by picking a generalized eigenvector v so that the above proportionality constant is $c = 1$. Can you see why this is the case? Find a generalized eigenvector v so that $(M - \lambda I)v = u$. Show that such a v is not unique. So for definiteness, choose the upper component of v to be 1.

   (i) $\langle \mathbf{3} \rangle$ Find the Jordan normal form of M, $J = S^{-1}MS$ using the above choice of v.

   (j) $\langle \mathbf{3} \rangle$ Among other things, the Jordan normal form helps us understand what M does when it acts repeatedly, which will be useful in analyzing systems subject to forces that oscillate periodically in time. Find M^{100} using the similarity transformation to Jordan normal form. Notice that this is substantially easier than direct calculation of M^{100}.