Classical Mechanics (PG), Autumn 2013 CMI
Problem set 16
Due at the beginning of lecture on Monday Nov 11, 2013
Double pendulum

1. $\langle \mathbf{26} \rangle$ consider a double pendulum with 'lower' bob of mass m suspended by a massless rod of length l from an 'upper' bob of mass m which is in turn suspended from a fixed pivot by a massless rod of length l (see figure in lecture notes). The system is free to move in a vertical plane subject to gravity. The rods make angles θ_1, θ_2 counterclockwise relative to the downward vertical. The cartesian coordinates of the two bobs are

$$\begin{aligned} \mathbf{r}_1 &= (x_1, y_1) \quad \text{where} \quad x_1 = l \sin\theta_1 \quad \text{and} \quad y_1 = -l\cos\theta_1, \quad \text{and} \\ \mathbf{r}_2 &= (x_2, y_2) \quad \text{where} \quad x_2 = l\sin\theta_1 + l\sin\theta_2 \quad \text{and} \quad y_2 = -l\cos\theta_1 - l\cos\theta_2. \end{aligned} \tag{1}$$

Choose the potential energy to vanish at the height of the pivot.

(a) $\langle \mathbf{4} \rangle$ Find simple expressions for the potential and kinetic energies V, T and the Lagrangian. Use the abbreviations $c_1 = \cos\theta_1, s_2 = \sin\theta_2,\ c_{12} = \cos(\theta_1 - \theta_2), s_{12} = \sin(\theta_1 - \theta_2)$ etc.

(b) $\langle \mathbf{3} \rangle$ Identify the configuration space of the double pendulum. Which manifold is it and what are a convenient set of coordinates on it?

(c) $\langle \mathbf{2} \rangle$ Identify an interesting continuous symmetry $\theta_i \to \theta_i + \delta\theta_i$ of the Lagrangian in the absence of the gravitational force.

(d) $\langle \mathbf{2} \rangle$ Find the momenta p_1, p_2 conjugate to θ_1, θ_2.

(e) $\langle \mathbf{3} \rangle$ Find the angular momenta $\mathbf{L}_1, \mathbf{L}_2$ of the two bobs.

(f) $\langle \mathbf{5} \rangle$ Try to find a relation among $p_1, p_2, \mathbf{L}_1$ and $\mathbf{L}_2$ and identify its physical meaning and significance in the context of the above symmetry.

(g) $\langle \mathbf{3} \rangle$ Express the generalized velocities $\dot\theta_i$ in terms of the generalized coordinates and momenta. Show that you get

$$\dot\theta_1 = \frac{p_1 - c_{12}p_2}{ml^2(1+s_{12}^2)} \quad \text{and} \quad \dot\theta_2 = \frac{2p_2 - c_{12}p_1}{ml^2(1+s_{12}^2)} \tag{2}$$

(h) $\langle \mathbf{4} \rangle$ Find the conserved total energy and hamiltonian, show that you get

$$\begin{aligned} E &= T + V = \frac{1}{2} ml^2 \left[2\dot\theta_1^2 + \dot\theta_2^2 + 2c_{12}\dot\theta_1\dot\theta_2\right] - mgl[2\cos\theta_1 + \cos\theta_2] \quad \text{and} \\ H &= \frac{1}{2ml^2(1+s_{12}^2)}\left[p_1^2 + 2p_2^2 - 2c_{12}p_1p_2\right] - mgl[2\cos\theta_1 + \cos\theta_2]. \end{aligned} \tag{3}$$