Classical Mechanics (PG), Autumn 2013 CMI

Problem set 14

Due at the beginning of lecture on Monday Oct 28, 2013

Generating function for canonical transformations, action-angle variables for SHO

1. $\langle \mathbf{12} \rangle$ Consider the simple harmonic oscillator $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$ with spring constant $k = m\omega^2$. We follow the notation and conventions adopted in the lecture.

   (a) $\langle \mathbf{4} \rangle$ Use an abbreviated action integral to solve the time-independent HJ equation and find a generating function $W(q, I)$ for the canonical transformation to action-angle variables $I = E/\omega$ and $\theta = \arctan(m\omega q/p)$.

   (b) $\langle \mathbf{4} \rangle$ Verify that this function W generates a canonical transformation to the above action-angle variables. In other words, show explicitly that

$$\frac{\partial W}{\partial q} \stackrel{?}{=} p = \sqrt{2m(E-V)} \quad \text{and} \quad \frac{\partial W}{\partial I} \stackrel{?}{=} \theta = \arcsin\left(q\sqrt{\frac{m\omega}{2I}} \right). \qquad (1)$$

   (Why is this different-looking formula for θ, also valid?)

   (c) $\langle \mathbf{4} \rangle$ Use the above $W(q, I)$ to find a generating function of the first kind $F_1(q, \theta)$ for the same canonical transformation. Give the explicit formula for I as a function of θ, q as well as a simple formula for $F_1(q, \theta)$.