**Classical Mechanics (PG), Autumn 2013 CMI**
Problem set 13
Due at the beginning of lecture on Tuesday Oct 15, 2013
Canonical transformations, Liouville's theorem

1. ⟨**4**⟩ Suppose $J = I + \epsilon F$ where $J$ is a $2n \times 2n$ matrix as in the discussion of Liouville's theorem and $\epsilon$ is a small parameter. Recall the identity $e^{\,\text{tr} \, \log J} = \det J$ where $\log[I + \epsilon F]$ is defined by the logarithmic series. Find the *quadratic* Taylor polynomial for $\det J$ regarded as a series in $\epsilon$.

2. ⟨**6**⟩ For one degree of freedom, find all infinitesimal *linear* canonical transformations $(q,p) \mapsto (Q,P)$ that fix the origin $(0,0) \mapsto (0,0)$. What is the dimension of the space of such infinitesimal linear CTs? Find the generating function $f(q,p)$ for the most general such infinitesimal linear canonical transformation.

3. ⟨**5**⟩ For one degree of freedom, find all (finite) *linear* canonical transformations $(q,p) \mapsto (Q,P)$ that fix the origin. Identify the matrix group of such CTs. What is the dimension of this group as a manifold?

4. ⟨**10**⟩ Consider the finite canonical transformation, corresponding to a rotation of the phase plane
$$Q = cq - sp \quad \text{and} \quad P = sq + cp \quad \text{where} \quad s = \sin\theta \quad \text{and} \quad c = \cos\theta. \tag{1}$$
   (a) ⟨**2**⟩ We seek a generating function of type-II $W(q,P)$ for the above finite CT. Find the PDEs that $W(q,P)$ must satisfy to ensure it generates the above CT. What sort of PDEs are they?
   (b) ⟨**5**⟩ Integrate the PDEs and give a simple formula for the generating function $W(q,P)$.
   (c) ⟨**1**⟩ Verify that your proposed function $W(q,P)$ indeed generates the above finite rotation.
   (d) ⟨**2**⟩ Find a generating function of type $F_1(q,Q)$ that generates the same finite rotation via an appropriate Legendre transform from $W(q,P)$.