Classical Mechanics (PG), Autumn 2013 CMI
Problem set 12
Due at the beginning of lecture on Monday Oct 7, 2013
Canonical transformations

1. $\langle \mathbf{5} \rangle$ Recall the scaling transformation for the Kepler problem $t \to \lambda t, \mathbf{r} \to \lambda^\gamma \mathbf{r}$ for $\gamma = -2/3$.

   (a) $\langle \mathbf{1} \rangle$ Is this transformation a symmetry of Newton's equation of motion?

   (b) $\langle \mathbf{2} \rangle$ How does momentum $\mathbf{p}$ change under this transformation? Is the transformation canonical?

   (c) $\langle \mathbf{2} \rangle$ How does the hamiltonian transform? Is it a symmetry of the hamiltonian? Does one expect a Noether conserved quantity?

2. $\langle \mathbf{9} \rangle$ We illustrate how to get a finite canonical transformation by composing infinitesimal ones. Consider the infinitesimal generator $f(q,p) = -\frac{1}{2}\delta\theta(q^2 + p^2)$ for a system with one degree of freedom and canonically conjugate phase space variables q, p.

   (a) $\langle \mathbf{2} \rangle$ Find the infinitesimal canonical transformation $q \to Q = q + \delta q,\ p \to P = p + \delta p$ generated by $f(q,p)$. Express the answer in matrix form and identify the matrix T:

$$\begin{pmatrix} Q \\ P \end{pmatrix} = [I + \delta\theta\ T] \begin{pmatrix} q \\ p \end{pmatrix}. \tag{1}$$

   (b) $\langle \mathbf{6} \rangle$ The effect of composing this infinitesimal CT twice is given by

$$\begin{pmatrix} Q \\ P \end{pmatrix} = [I + \delta\theta\ T][I + \delta\theta\ T] \begin{pmatrix} q \\ p \end{pmatrix}. \tag{2}$$

   To get a finite CT, we compose n infinitesimal CTs generated by f, and let $n \to \infty$. Find how $\delta\theta$ must behave as $n \to \infty$ to ensure a finite limiting CT. Find the limiting CT and express it in matrix form

$$\begin{pmatrix} Q \\ P \end{pmatrix} = A \begin{pmatrix} q \\ p \end{pmatrix}. \tag{3}$$

   Find a simple formula for the matrix A and show that the resulting finite CT is a rotation

$$Q = cq - sp \quad \text{and} \quad P = sq + cp \quad \text{where} \quad s = \sin\theta \quad \text{and} \quad c = \cos\theta. \tag{4}$$

   Relate the angle θ to $\delta\theta$ and n.

   (c) $\langle \mathbf{1} \rangle$ Verify that this CT preserves the fundamental Poisson bracket relations.