Classical Mechanics (PG), Autumn 2013 CMI
Problem set 11
Due at the beginning of lecture on Friday Oct 4, 2013
Poisson brackets, infinitesimal canonical transformations

1. $\langle \mathbf{3} \rangle$ Find the unequal time p.b. $\{q(0), q(t)\}$ for a free particle of mass m moving on a line.

2. $\langle \mathbf{10} \rangle$ Consider two infinitesimal CTs generated by functions f and g on phase space $\mathbb{R}^{2n}$. In general, the composition of a pair of CTs depends on the order of application: the group of CTs is non-abelian. Show that this non-abelian nature is captured to first approximation by the Poisson bracket of generators $\{f, g\}$. More precisely, suppose $(q,p) \to (Q_f, P_f)$ via the generator f and $(Q_f, P_f) \to (\tilde{Q}_{g\circ f}, \tilde{P}_{g\circ f})$ when g follows f. Similarly we have $(\tilde{Q}_{f\circ g}, \tilde{P}_{f\circ g})$.

   (a) $\langle \mathbf{5} \rangle$ Give expressions for $\tilde{Q}^i_{g\circ f}$ and $\tilde{Q}^i_{f\circ g}$ correct to *quadratic* order in infinitesimals f and g which are the infinitesimal generators of the corresponding CTs.

   (b) $\langle \mathbf{5} \rangle$ Calculate the difference $\tilde{Q}^i_{g\circ f} - \tilde{Q}^i_{f\circ g}$ to leading non-trivial order in infinitesimals and show that it is given by a p.b. (of what?) with $\{f, g\}$. In effect, this shows that the p.b. is the Lie bracket in the Lie algebra of infinitesimal CTs.

3. $\langle \mathbf{10} \rangle$ Generators for infinitesimal canonical transformations on phase plane.

   (a) $\langle \mathbf{2} \rangle$ Argue why the infinitesimal translation on phase space, $Q = q + \epsilon_1$, $P = p + \epsilon_2$ is canonical.

   (b) $\langle \mathbf{2} \rangle$ Find an infinitesimal generator $g(q,p)$ for the above infinitesimal translation.

   (c) $\langle \mathbf{2} \rangle$ Consider the infinitesimal generator $f(q,p) = -\frac{1}{2}\theta(q^2+p^2)$. Find the infinitesimal canonical transformation it generates.

   (d) $\langle \mathbf{2} \rangle$ Plot on the $q-p$ phase plane the effect of this infinitesimal canonical transformation on a typical phase point (q,p), i.e., indicate q, p and Q, P for small θ.

   (e) $\langle \mathbf{2} \rangle$ State in words what infinitesimal canonical transformation f generates.