Classical Mechanics (PG), Autumn 2013 CMI
Problem set 10
Due at the beginning of lecture on Wednesday Sept 25, 2013
Invariant Lagrangians, Poisson brackets

1. $\langle \mathbf{8} \rangle$ Show that an infinitesimal transformation $q \to q + \tilde{\delta} q$ that is a symmetry of the Lagrangian (leaves it unchanged to linear order in infinitesimals), is automatically a symmetry of the equations of motion (leaves them invariant to linear order in infinitesimals and so takes solutions to solutions). This does not require detailed calculation, but carefully presented reasoning and a diagram. Hints: To set things up, consider the action of a path joining $q_1 = q(t_1)$ to $q_2 = q(t_2)$. Under what circumstances is the path a trajectory? Explain in words and draw a picture. Then consider the effect of the symmetry transformation $\tilde{\delta} q$ on the original path. You must distinguish between the variations of a path δq that are considered in deriving the equations of motion and the transformation $q \to q + \tilde{\delta} q$ that is a symmetry of the Lagrangian. In what respects are they different?

2. $\langle \mathbf{21} \rangle$ Consider a particle moving in a central potential on the plane $L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - V(\sqrt{x^2 + y^2})$. x, y, p_x, p_y are regarded as old coordinates and momenta. Define new plane polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. We wish to compare the Poisson brackets among the new coordinates and momenta to those among the old coordinates and momenta.

   (a) $\langle \mathbf{2} \rangle$ Find the partial derivatives (denoted by subscripts) $r_x, r_y, \theta_x, \theta_y$. Show that you get
   $$r_x = \frac{x}{r}, \quad r_y = \frac{y}{r}, \quad \theta_x = -\frac{y}{r^2}, \quad \theta_y = \frac{x}{r^2}. \tag{1}$$

   (b) $\langle \mathbf{2} \rangle$ Show that the new velocities may be expressed in terms of the old velocities and coordinates (here we use $r = \sqrt{x^2 + y^2}$ for brevity)
   $$\dot{r} = \frac{x\dot{x}}{r} + \frac{y\dot{y}}{r} \quad \text{and} \quad \dot{\theta} = \frac{x}{r^2}\dot{y} - \frac{y}{r^2}\dot{x}. \tag{2}$$

   (c) $\langle \mathbf{2} \rangle$ Use these formulae to verify that the new Lagrangian is
   $$\tilde{L}(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{3}$$

   (d) $\langle \mathbf{2} \rangle$ Find the new momenta from the Lagrangian in the new variables and express them in terms of the old coordinates and momenta. Show that you get
   $$p_r = \frac{x}{r} p_x + \frac{y}{r} p_y \quad \text{and} \quad p_\theta = x p_y - y p_x. \tag{4}$$

   (e) $\langle \mathbf{1} \rangle$ Find the partial derivatives of the new coordinates with respect to the old momenta $r_{p_x}, r_{p_y}, \theta_{p_x}, \theta_{py}$.

   (f) $\langle \mathbf{5} \rangle$ Find the partial derivatives of the new momenta with respect to the old coordinates and momenta. You must give 8 formulae, two of which must be shown to be
   $$(p_r)_x = \frac{p_x}{r} - \frac{x^2}{r^3} p_x - \frac{xy}{r^3} p_y \quad \text{and} \quad (p_r)_y = \frac{p_y}{r} - \frac{y^2}{r^3} p_y - \frac{xy}{r^3} p_x. \tag{5}$$

   (g) $\langle \mathbf{6} \rangle$ Working from the definition $\{f, g\} = f_x g_{p_x} - f_{p_x} g_x + f_y g_{p_y} - f_{p_y} g_y$, find the 6 (after accounting for anti-symmetry) p.b. among the new coordinates and momenta (a) $\{r, \theta\}$, (b) $\{r, p_r\}$, (c) $\{r, p_\theta\}$, (d) $\{\theta, p_\theta\}$, (e) $\{\theta, p_r\}$ and (f) $\{p_r, p_\theta\}$.

   (h) $\langle \mathbf{1} \rangle$ Compare the p.b. relations among the old variables to those among the new variables.