Classical Mechanics (PG), Autumn 2013 CMI
Problem set 1
Due at the beginning of lecture on Wednesday August 7, 2013
2 body central force problem

1. Consider the transformation to center of mass and relative coordinates

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \quad \text{and} \quad \mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2) \tag{1}$$

Here m_1, m_2 are masses of two particles located at $\mathbf{r}_1, \mathbf{r}_2$ and $M = m_1 + m_2$.

(a) $\langle \mathbf{6} \rangle$ Express the derivatives with respect to $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of derivatives with respect to $\mathbf{r}$ and $\mathbf{R}$. Show that you get

$$\nabla_1 = -\nabla_{\mathbf{r}} + \frac{m_1}{M}\nabla_{\mathbf{R}} \quad \text{and} \quad \nabla_2 = \nabla_{\mathbf{r}} + \frac{m_2}{M}\nabla_{\mathbf{R}} \tag{2}$$

Hint: The fact that these are all 3D vectors is not essential. As a warm up, you could treat them as one dimensional and replace vector gradients by partial derivatives.
ANS: We are making a change of variables from (r_1, r_2) to (R, r). $r^i = r_2^i - r_1^i$, $R^i = \frac{m_1}{M} r_1^i + \frac{m_2}{M} r_2^i$. Then

$$\frac{\partial r^j}{\partial r_1^i} = -\delta_i^j, \quad \frac{\partial R^j}{\partial r_1^i} = \frac{m_1}{M}\delta_i^j, \quad \frac{\partial r^j}{\partial r_2^i} = \delta_i^j, \quad \frac{\partial R^j}{\partial r_2^i} = \frac{m_2}{M}\delta_i^j \tag{3}$$

Thus

$$\frac{\partial f}{\partial r_1^i} = \frac{\partial r^j}{\partial r_1^i}\frac{\partial f}{\partial r^j} + \frac{\partial R^j}{\partial r_1^i}\frac{\partial f}{\partial R^j} = -\frac{\partial f}{\partial r^i} + \frac{m_1}{M}\frac{\partial f}{\partial R^i} \quad \Rightarrow \quad \nabla_1 = -\nabla_r + \frac{m_1}{M}\nabla_R. \tag{4}$$

(b) $\langle \mathbf{2} \rangle$ Show that when acting on functions of $r = |\mathbf{r}|$ alone,

$$\nabla_1 V(r) = -\nabla_r V(r) \quad \text{and} \quad \nabla_2 V(r) = +\nabla_r V(r) \tag{5}$$

Here $\nabla_1 = \nabla_{\mathbf{r}_1}$ etc.

(c) $\langle \mathbf{5} \rangle$ Start with Newton's second law for the masses m_1, m_2 in their mutual gravitational potential

$$m_1\ddot{\mathbf{r}}_1 = -\nabla_1 V(r) \quad \text{and} \quad m_2\ddot{\mathbf{r}}_2 = -\nabla_2 V(r) \quad \text{where} \quad V(r) = -\frac{\alpha}{r}. \tag{6}$$

Using appropriate integrating factors and the above results, derive a conserved total energy

$$E = \frac{1}{2}(m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2) + V(r). \tag{7}$$

(d) $\langle \mathbf{3} \rangle$ Find $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of the center of mass and relative coordinates.