Path Integrals in Quantum Mechanics
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2 Formulations of quantum mechanics and time evolution operator

2.1 Schrödinger picture

• In the Schrödinger formulation of QM, the state of the system is represented by a vector in a complex vector space (wavefunction) $|\psi\rangle$ that evolves in time via the Schrödinger equation

(SE). For a particle in a potential V, we have

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle \quad \text{where} \quad H = -\nabla^2 + V(\mathbf{r}) \tag{1}$$

is the Hamiltonian operator. However, the wave function itself is not measured. Rather, when we measure an observable A in a normalized state ψ, we get one of its eigenvalues a with a probability given by the square of the inner product (projection) $|\langle\phi_a|\psi\rangle|^2$. The system collapses to a (normalized) eigenstate ϕ_a where $A\phi_a = a\phi_a$. If we make several copies of the system in the same state $\psi(t)$, and measure A in each of the copies, the average value obtained is $\langle\psi(t)|A|\psi(t)\rangle$

2.2 Time evolution operator

• Given an initial state $\psi(t_0)$, the SE determines the state $\psi(t)$ at any later time. For a time-independent Hamiltonian we may write down the solution by inspection

$$\psi(t) = e^{-iH(t-t_0)/\hbar}\psi(0). \tag{2}$$

For such a Hamiltonian, the operator $U(t,t_0) = e^{-iH(t-t_0)/\hbar}$ is called the time-evolution operator as it evolves the state forward in time. It is unitary $(UU^\dagger = U^\dagger U = I)$ since H is hermitian and U may be expanded in an exponential series:

$$U(t,t_0) = \sum_{n=0}^{\infty}\left(\frac{1}{i\hbar}\right)^n \frac{(t-t_0)^n H^n}{n!}. \tag{3}$$

Interestingly, U satisfies the SE. In fact, even if H is time-dependent, we may use the linearity of the SE to write

$$\psi(t) = U(t,t_0)\psi(t_0), \tag{4}$$

where for consistency U must satisfy the initial condition $U(t_0,t_0) = I$. Putting this in the SE, and requiring it to hold for any initial state leads us to an evolution equation for U:

$$i\hbar\,\partial_t U(t,t_0) = HU(t,t_0) \tag{5}$$

• **Reproducing property/composition law** $U(t,t_0)$ satisfies a composition law which is simply the statement that one can evolve directly from t_0 to $t_2 > t_0$ or in two steps from $t_0 \to t_1$ and then from $t_1 \to t_2$. Thus, composing two time evolution operators *whose initial and final times coincide*, reproduces another time evolution operator:

$$U(t_2,t_0) = U(t_2,t_1)U(t_1,t_0) \quad \text{for} \quad t_2 \geq t_1 \geq t_0. \tag{6}$$

• In the case of a time independent hamiltonian the composition law says that

$$e^{-iH(t-t_0)/\hbar} = e^{-iH(t-t_1)/\hbar}e^{-iH(t_1-t_0)/\hbar} \tag{7}$$

In this case, $U(t,t_0)$ is only a function of the time difference and we may denote the reduced time-evolution operator $\tilde{U}(t) = e^{-iHt/\hbar}$. Then we have $\tilde{U}(t+s) = \tilde{U}(t)\tilde{U}(s)$. In this case, the composition law is commutative $\tilde{U}(t)\tilde{U}(s) = \tilde{U}(s)\tilde{U}(t)$. $\tilde{U}(0) = I$ and $\tilde{U}(-t)\tilde{U}(t) = I$ defines the inverse with $U(-t) = U(t)^\dagger$. Thus the reduced time evolution operators of a system with time-independent Hamiltonian may be used to obtain a unitary representation of the abelian group of rotations $\mathrm{U}(1)$.

• For a time-dependent Hamiltonian, $U(t,t')$ is generally not just a function of the time difference as time-translation invariance is broken, so we cannot write $U(t,t') = \tilde{U}(t-t')$. The $U(t,t')$ do not form a group since in general we cannot compose two of them to produce another time-evolution operator, this is possible only if the final time of the right factor matches the initial time of the left factor.

• The reproducing property is important. It can be used to define the time-evolution operator, and thereby serve as an alternative to the Schrödinger equation. This is similar to how we can define the exponential function of calculus by the functional equation $f(t+s) = f(t)f(s)$ subject to the initial condition $f(0) = 1$. There is a one parameter family of solutions to this functional equation $f(t) = e^{ht}$ parametrized by a 'constant of integration/hamiltonian' $h = f'(0)$.

2.3 Heisenberg picture

We may use the time evolution operator to go from the Schrödinger to the Heisenberg picture. For simplicity, let us suppose that the Hamiltonian is time independent and focus on the physically measurable expectation value $\psi(t)|A|\psi(t)\rangle$. We can express this expectation value at time t in terms of the expectation value (of a different operator) at a reference time (say $t = 0$) using the time-evolution operator $U = e^{-iHt}$:

$$\langle\psi(t)|A|\psi(t)\rangle = \langle\psi(0)|e^{iHt/\hbar}Ae^{-iHt/\hbar}|\psi(0)\rangle \tag{8}$$

The operator $A_h(t) = e^{iHt/\hbar}Ae^{-iHt/\hbar}$ is called the operator A in the Heisenberg picture. The original operator A (sometimes called A_s) is said to be in the Schrodinger picture. Operators in the Heisenberg picture are related to those in the Schrodinger picture via a unitary transformation $A_h(t) = U^\dagger A U$.

• Thus, to calculate the expected value of an observable A at time t in the Heisenberg picture, we must evaluate $\langle\psi(0)|A_h(t)|\psi(0)\rangle$. Since we only need $\psi(0)$, we will say that the state of the system in the Heisenberg picture is $\psi_h = \psi(0)$. We can of course also write $\psi_h = U^\dagger\psi(t)$.

• Thus, in the Heisenberg formulation, states do not change in time, but the operators change with time. In the Schrödinger formulation, the state of the system evolves in time, while operators do not change with time (except if they are explicitly time-dependent).

• The Hamiltonian is the same in both pictures $H_h = U^\dagger H U = H$ since the time evolution operator U commutes with H. Irrespective of whether we work in the Schrödinger or Heisenberg pictures, physically measurable quantities are the same. We have already seen that expectation values are the same in both pictures $\langle\psi(t)|A|\psi(t)\rangle = \langle\psi_h|A_h|\psi_h\rangle$. More generally, the eigenvalues of operators are the same in both pictures. This is because $A_s = A$ and $A_h = U^\dagger A U$ being related by a unitary transformation, share the same spectrum.

• In addition, inner products (projections, whose squares give probabilities of measurements) $\langle\phi(t)|\psi(t)\rangle = \langle\phi_h|\psi_h\rangle$ are also the same in both pictures. Here the system is in the Schrödinger state $\psi(t)$; we measure an observable, and get an eigenvalue corresponding to the eigenfunction $\phi(t)$.

• States do not evolve in time, so what replaces the Schrödinger equation in the Heisenberg picture? It is replaced by the Heisenberg equation of motion, which tells us how operators in the

Heisenberg picture evolve. Suppose $A_h(t) = U^\dagger A U$ where A is a Schrödinger picture operator (that may have some explicit time dependence), then

$$i\hbar \frac{dA_h}{dt} = i\hbar \dot{U}^\dagger A U + i\hbar U^\dagger \frac{\partial A}{\partial t} U + i\hbar U^\dagger A \dot{U} \tag{9}$$

From $U = e^{-iHt/\hbar}$ and $U^\dagger = e^{-Ht/\hbar}$ we first observe that $H, U, U^\dagger$ all commute with each other (after all, each is a function of H and $[H, H] = 0$). We also find

$$i\hbar \dot{U} = HU, \quad \text{and} \quad i\hbar \dot{U}^\dagger = -HU. \tag{10}$$

Thus the time evolution of A_h is given by the Heisenberg equation of motion

$$i\hbar \frac{dA_h}{dt} = i\hbar \frac{\partial A_h}{\partial t} + [A_h, H]. \tag{11}$$

In particular, if A does not have any explicit time dependence, then $i\hbar \dot{A}_h = [A_h, H]$. Moreover, if $[A, H] = 0$ (which is equivalent to $[A_h, H] = 0$), the Heisenberg operator $A_h(t)$ is a constant of motion. In other words, each of its matrix elements is time-independent.

• For a free particle, the Heisenberg picture momentum is a constant of motion $\dot{p}_h = 0$, since $[p, p^2/2m] = 0$.

2.4 Relation between classical and quantum mechanical formalisms

• There is a third way of formulating quantum mechanics, Feynman's path integral approach. To see how it fits in, let us recall the various formalisms of classical dynamics and mention their quantum counterparts:

1. Time-dependent Hamilton-Jacobi equation for Hamilton's principal function $\partial_t S + H(q, \frac{\partial S}{\partial q}) = 0$ or time-independent Hamilton-Jacobi equation for Hamilton's characteristic function $H(q, \frac{\partial W}{\partial q}) = E$ were $S = W - Et$ $\leftrightarrow$ Time-dependent $i\hbar \partial_t \psi = H\psi$ and time-independent $H\psi = E\psi$ Schrödinger equations for wave function with $\psi \sim e^{iS/\hbar}$.
2. Hamilton's 1st order equations of motion expressed in terms of Poisson brackets $\dot{f} = \{f, H\}$ $\leftrightarrow$ Heisenberg equations of motion $i\hbar \dot{\hat{f}} = [\hat{f}, \hat{H}]$.
3. Euler-Lagrange equations for trajectory joining two configurations as extrema of action $\leftrightarrow$ Position space path integral representation of quantum mechanical amplitude. Similarly, the action principle for Hamilton's equations $\leftrightarrow$ phase space path integral.
4. Newtonian's second law, generally non-linear 2nd order ODE $\leftrightarrow$ Stochastic ODE with quantum fluctuations entering through stochastic term in ODE.

It is noteworthy that the later formulations of classical mechanics were generalized to the quantum theory somewhat earlier than the original Newtonian approach.

3 The propagator: time evolution operator in the position basis

• The time evolution operator $U(t,t_0)$ can be expressed in any basis. For a time-independent Hamiltonian, it is simplest in the energy basis. If $H|n\rangle = E_n|n\rangle$, then

$$\langle n|U(t,0)|m\rangle = \langle n|e^{-iHt/\hbar}|m\rangle = e^{-iE_nt/\hbar}\delta_{nm}. \tag{12}$$

• To work our way to the path integral formulation, it is instructive to consider the time evolution operator in a basis of position eigenstates $|x'\rangle$. In the position basis, if we denote $\langle x|U(t,t')|x'\rangle = U(xt;x't')$, then

$$\psi(x,t) = \langle x|\psi(t)\rangle = \langle x|U(t,t')|\psi(t')\rangle = \int dx' \; \langle x|U(t,t')|x'\rangle\langle x'|\psi(t')\rangle = \int dx' U(xt;x't')\psi(x',t'). \tag{13}$$

In a sense, $U(x't';xt)$ propagates the initial wave function to the final wave function. So the time evolution operator in the position basis is also called the *propagator*. In particular, if the initial state was delta localized at the point x_0, then $\psi(x,t) = U(xt;x_0t_0)$[1]. So the matrix elements of the propagator give the amplitude for finding the particle at x' at time t' given that it was at location x_0 at time t_0.

• The reproducing property can be expressed in any basis. For example, in the position basis we get

$$\langle x_2|U(t_2,t_0)|x_0\rangle = \int dx_1 \; \langle x_2|U(t_2,t_1)|x_1\rangle\langle x_1|U(t_1,t_0)|x_0\rangle \tag{14}$$

or $U(x''t'';x,t) = \int dx' \; U(x''t'';x't') \; U(x't';xt)$.

• Most often we do not directly know the time evolution operator in the position basis. But suppose we know the energy levels and eigenfunctions, then we can get an expression for the propagator $U(x'\,t';x\,t)$. Suppose the energy levels are discrete $H\psi_n = E_n\psi_n$ then

$$\begin{aligned}\langle x_f|U(t_f,t_i)|x_i\rangle &= \sum_{nn'}\langle x_f|n\rangle\langle n|U(t_f,t_i)|n'\rangle\langle n'|x_i\rangle = \sum_{nn'}\psi_n(x_f)\langle n|e^{-\frac{i}{\hbar}H(t_f-t_i)}|n'\rangle\psi^*_{n'}(x_i) \\ &= \sum_n \psi_n(x_f)e^{-\frac{i}{\hbar}E_n(t_f-t_i)}\psi^*_n(x_i).\end{aligned} \tag{15}$$

To better understand the propagator, we find the free particle propagator using our knowledge of free particle energies and eigenfunctions.

4 Free particle propagator

• The free particle hamiltonian $H = p^2/2m$ is diagonal in the basis of momentum eigenstates $H|k\rangle = \frac{\hbar^2k^2}{2m}|k\rangle$, and so is the time evolution operator $U(t,t') = e^{-\frac{i}{\hbar}H(t-t')}$

$$U(t,t')|k\rangle = e^{-\frac{i}{\hbar}\frac{\hbar^2k^2}{2m}(t-t')}|k\rangle \quad\Rightarrow\quad \langle k|U(t,t')|k'\rangle = 2\pi\delta(k-k')e^{-\frac{i\hbar k^2}{2m}(t-t')} \tag{16}$$

[1] Strictly, $\psi(x,t_0) = \delta(x-x_0)$ is not a good initial state, it isn't normalizable. Indeed, it is a plane wave in momentum space $\tilde\psi(k) = e^{-ikx_0}$, and we know that plane waves are orthogonal but not normalizable. In a more careful treatment, we would have to take say a gaussian wave packet for the initial state localized around x_0, instead of a delta-localized initial wave function.

In the basis of position eigenstates $\langle k|x\rangle = e^{-ikx}$ we have

$$\langle x|U(t,t')|x'\rangle = \int [dk][dk']\, \langle x|k\rangle\, \langle k|U(t,t')|k'\rangle\, \langle k'|x'\rangle = \int [dk]\; e^{-\frac{i\hbar k^2(t-t')}{2m}+ik(x-x')} \tag{17}$$

The 'gaussian integral'[2] is done by completing the square $-ak^2+bk = -a(k-b/2a)^2+b^2/4a$ where

$$a = \frac{i\hbar}{2m}(t-t'), \quad b = i(x-x') \;\;\Rightarrow\;\; U = \int [dk] e^{-ak^2+bk} = \frac{1}{2\pi} e^{b^2/4a}\sqrt{\frac{\pi}{a}} \tag{18}$$

Thus the propagator is

$$U(x\,t; x'\,t') = \left(\frac{m}{ih(t-t')}\right)^{\frac{1}{2}} \exp\left[\frac{i}{\hbar}\frac{m}{2}\frac{(x-x')^2}{(t-t')}\right]. \tag{19}$$

Similarly in three dimensions we have

$$U(\vec r\,t; \vec r'\,t') = \left(\frac{m}{ih(t-t')}\right)^{\frac{3}{2}} \exp\left[\frac{i}{\hbar}\frac{m}{2}\frac{|\vec r-\vec r'|^2}{(t-t')}\right]. \tag{20}$$

Since H is time-independent, U depends only on the difference $t-t'$. As H is translation invariant, U only depends on the difference $\vec r - \vec r'$ and furthermore only on the magnitude of the difference on account of rotation invariance.

• The propagator is a gaussian in $(x-x')$ with a (complex) standard deviation σ

$$U(x\,t; x'\,t') = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-x')^2}{2\sigma^2}} \quad \text{where} \quad \sigma = \sqrt{\frac{i\hbar(t-t')}{m}} \tag{21}$$

Since $\sigma \propto \sqrt{t-t'}$, the 'width' $|\sigma|$ of the gaussian grows with time. This is an indication of the dispersive broadening of the probability amplitude as time passes. To properly understand this phenomenon, we must use this propagator to evolve, say, a gaussian wave packet forward in time and see it broaden out. The advantage of having an explicit formula for the propagator is that it can be used to evolve *any* state forward in time, not just a gaussian wave packet.

• Since the limit of gaussians as the width tends to zero is

$$\lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} = \delta(x), \tag{22}$$

the propagator satisfies the unit initial condition representing a particle initially localized at x'

$$\lim_{t\to t'} U(x,t;x',t') = \delta(x-x'). \tag{23}$$

• The free particle propagator (19) may be written in terms of the classical action of the straight line path $x(t) = x_i + \frac{x_f-x_i}{t_f-t_i}(t-t_i)$ traversed by a classical particle in going from $x_i \to x_f$ as time runs from $t_i \to t_f$. The velocity is constant, and so is the Lagrangian $L(t) = \frac{1}{2}m\dot x^2 = \frac{1}{2}m\frac{(x_f-x_i)^2}{(t_f-t_i)^2}$ along such a straight line trajectory, so the classical action for this trajectory is

$$S(x_f(t_f), x_i(t_i)) = \int_{t_i}^{t_f} L\, dt = \frac{m}{2}\frac{(x_f-x_i)^2}{(t_f-t_i)}. \tag{24}$$

[2]This is not an ordinary real gaussian integral, but an oscillatory integral as the exponent is imaginary. More care is needed to justify the answer obtained below than we provide here.

Thus (U is dimensionless, but its matrix elements in the position basis have dimension $1/\text{length}$)

$$U(x_f\, t_f; x_i\, t_i) = \left(\frac{m}{ih(t_f - t_i)}\right)^{\frac{1}{2}} \exp\left[\frac{i}{\hbar} S\left(x_f(t_f), x_i(t_i)\right)\right]. \tag{25}$$

Thus the amplitude for the free particle to be found at x_f at t_f given that it was at x_i at t_i is proportional to the exponential of $(i/\hbar)\times$ the action for the classical trajectory between those two points. A similar formula holds in 3d with the exponent of the pre-factor $1/2$ replaced by $3/2$. We emphasize that this formula for the paopagator is special to a free particle and does not generally hold for a particle in a potential.

5 Feynman path integral for a free particle

• Since the time evolution operator satisfies the reproducing property, we can write the free particle propagator as a product of time evolution operators. Let us divide the time interval $[t_i, t_f]$ into n subintervals $t_i = t_0 < t_1 < \cdots < t_{n-1} < t_n = t_f$ (say equally spaced $t_{j+1} - t_j = \Delta t$, for simplicity). Then

$$U(t_f; t_i) = U(t_n, t_{n-1})U(t_{n-1}, t_{n-2}) \cdots U(t_1, t_0). \tag{26}$$

The amplitude for the free particle to go from $x_0 = x_i(t_i)$ to $x_n = x_f(t_f)$ is

$$\langle x_f | U(t_f, t_i) | x_i \rangle = \int dx_{n-1} \cdots dx_1\, \langle x_n | U(t_n, t_{n-1}) | x_{n-1} \rangle \cdots \langle x_1 | U(t_1, t_0) | x_0 \rangle. \tag{27}$$

Written in terms of the classical action, we have an exact formula for each n:

$$U(x_n, t_n; x_0, t_0) = \left(\frac{m}{ih\Delta t}\right)^{\frac{n}{2}} \int dx_1 \cdots dx_{n-1}\, e^{(i/\hbar)\{S[x(t_n), x(t_{n-1})] + S[x(t_{n-1}), x(t_{n-2})] + \cdots + S[x(t_1), x(t_0)]\}}. \tag{28}$$

So the propagator is an integral over all piecewise straight line paths going from $x_0(t_0) \to x_n(t_n)$, each comprising n segments. This is best illustrated by a figure. Each segment is a classical trajectory and contributes a phase factor equal to $(i/\hbar)\times$ its classical action. Though each segment is a classical trajectory, when joined together, the resulting piecewise linear paths are typically not classical trajectories. Now if we let $n \to \infty$, formally we find that the free particle propagator is proportional to an integral over all the paths connecting the initial and final locations, each weighted by a phase proportional to the classical action for the path. Absorbing the pre-factor (and its dimensions) into a pre-factor C and formally denoting the integration element on the space of paths by $\mathrm{D}[x]$,

$$\langle x_f | U(t_f, t_i) | x_i \rangle = C \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathrm{D}[x]\; e^{\frac{i}{\hbar} S[x]} = C \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathrm{D}[x]\; e^{\frac{i}{\hbar} \int_{t_i}^{t_f} \frac{1}{2} m \dot{x}^2\, dt}. \tag{29}$$

This representation of the free particle propagator is called the Feynman path integral. The integral is over the space of paths connecting x_i and x_f. This is an infinite dimensional space, and it is not easy to define integration over such an infinite dimensional space. However (28) is a completely well-defined and exact formula that involves integration over a finite $(n-1)$

dimensional space[3]. A similar path integral representation is available for a particle moving in a potential $V(x)$. In that case, the weights for the individual paths are given by the exponential of the classical action $S[x] = \int_{t_i}^{t_f} dt \, \left(\frac{1}{2}m\dot{x}^2 - V(x)\right)$. Though we do not have an explicit formula like (25) for the propagator of a particle moving in an arbitrary potential $V(x)$, it can be shown that the above path integral representation for the time-evolution operator continues to hold.

• Classically the particle follows a trajectory that solves Newton's equation. The principle of stationary action says that a classical trajectory between $x_i(t_i)$ and $x_f(t_f)$ is one for which the classical action functional is extremal. Quantum mechanically, the above formula says that one way to compute the propagator, is to evaluate a sum over paths. This does not mean that the particle travels along all these paths, nor does it imply that the particle has *any* well-defined trajectory. However, we sometimes loosely say that in QM, the particle samples all paths including the classical trajectory.

• The Feynman path integral reformulates the problem of solving the SE for the time evolution operator on a Hilbert space. Rather than work with operators and Hilbert spaces, it says that we may compute the sum of phases contributed by various paths, each weighted by its classical action. So the problem of quantum evolution is couched in terms of some classical concepts. However, QM has *not* been reduced to classical mechanics. No where in CM do we admit paths for particles that are not classical trajectories.

• We can recover the principle of extremal action from the Feynman path integral by appropriately considering the limit $\hbar \to 0$. Each path $x(t)$ contributes a phase $(i/\hbar)S[x]$ to the sum over paths. Now consider two adjacent paths $x(t)$ and $x(t)+\delta x(t)$ with $\delta x(t_i) = \delta x(t_f) = 0$. Suppose further that $S'[x] \neq 0$. In the semi-classical limit, the difference in their actions $S[x] - S[x+\delta x]$ will typically be quite large compared to $\hbar$. So they contribute with rather different phases $e^{\frac{i}{\hbar}S}$. In this manner, the amplitudes of nearby paths contribute 'random' (i.e. not all correlated and pointing in one direction) phases which destructively interfere and cancel out. Thus these paths do not contribute significantly to the propagator in the semi-classical approximation. However, there is occasionally a path $x_{cl}(t)$ in whose vicinity all paths contribute constructively to the sum. This happens if the action is stationary, which is precisely the case for the classical trajectory $S'[x_{cl}] = 0$. In other words, paths in the neighborhood of the classical trajectory have roughly the same classical action and therefore contribute roughly the same phase $e^{(i/\hbar)S[x]}$ to the sum over paths. This constructive interference in the neighborhood of the classical trajectory explains why we may approximate quantum dynamics by motion along the classical trajectory in the classical limit. All the other paths in the Feynman path integral contribute negligibly to the propagator when $\hbar \to 0$.

• Note that we are not taking the $\hbar \to 0$ limit of the propagator $U(t, t')$. We are only discussing the relative contributions of various paths to the path integral in the $\hbar \to 0$ limit. We already know from the semiclassical WKB analysis that the wave function does not have a good $\hbar \to 0$ limit, as it has an essential singularity at $\hbar = 0$. Similarly, the time-evolution operator does not have a good classical limit. However, as is evident from (25), the logarithm of the time evolution operator (times $-i\hbar$) has a good classical limit, indeed, it is the classical action of the classical trajectory in that case[4].

[3] So we use integration over a finite dimensional space to approximate integration over an infinite dimensional space. This is analogous to how we use finite Riemann sums to approximate integration over the infinite set of points in an interval.

[4] Though the limits $t \to t'$ and $\hbar \to 0$ look formally the same, a more careful treatment of (19) shows that

• We can interpret interference and diffraction phenomena for matter waves (e.g. electrons) in terms of the Feynman path integral. In the absence of any obstacles, the amplitude for the particle to go from $x_i(t_i)$ to $x_f(t_f)$ is given by a sum over all paths connecting these locations. If an obstacle is introduced, certain paths are forbidden, but there are still many paths that 'go around' the obstacle, though they are not classical trajectories. These are the paths that contribute to 'diffraction around an obstacle'. In double slit interference, the amplitude at a point $x_f(t_f)$ on the screen is given by a sum over paths. These include piecewise straight line paths ('classical trajectories') that go through either one of the slits S_1 or S_2. But there are other paths that go through S_1, come out of S_2 and go back out through S_1 before reaching the screen. We must sum over all these paths. The contributions of most of these paths cancel out due to destructive interference with nearby paths since the action is not stationary around them. In the semi-classical limit, it is the two piecewise straight-line paths around which the action is stationary, that contribute maximally to the amplitude. Thus it is sufficient to consider the interference between these two paths to get the interference pattern on the screen to first approximation.

6 Path integral for a particle in a potential

• Consider a particle in a potential with hamiltonian $H = \frac{\hat{p}^2}{2m} + V(\hat{x})$. We wish to find a path integral representation for the propagator. Here, unlike for the free particle we do not have an explicit formula for U since we do not know the energy levels and eigenfunctions of H. Nevertheless, we wish to write the propagator in terms of classical quantities like the Lagrangian/Hamiltonian/action. Recall that $U(t,t') = e^{-\frac{i}{\hbar}H(t-t')}$. Let us begin by expressing the matrix elements of H in term of the classical hamiltonian. $\hat{H}$ in the position basis is a differential operator and in the momentum basis is also a differential operator. But interestingly, the mixed matrix elements $\langle p|\hat{H}|x\rangle$ are directly related to the classical hamiltonian

$$\langle k|\hat{H}|x\rangle = \langle k|x\rangle H(x,p) = e^{-ikx} H(x,p) \tag{30}$$

To see this, note that $\hat{x}|x\rangle = x|x\rangle$, $\langle k|\hat{p} = \langle k|p$ where $p = \hbar k$ and x, p are real numbers, not operators. So $\langle k|\hat{x}|x\rangle = x\langle k|x\rangle$ and $\langle k|\hat{p}|x\rangle = p\langle k|x\rangle$. Thus

$$\begin{aligned}\hat{H}|x\rangle &= \left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)|x\rangle = \left(\frac{\hat{p}^2}{2m} + V(x)\right)|x\rangle \quad \Rightarrow \\ \langle k|\hat{H}|x\rangle &= \langle k|\left(\frac{p^2}{2m} + V(x)\right)|x\rangle = \left(\frac{p^2}{2m} + V(x)\right)\langle k|x\rangle = e^{-ikx}\left(\frac{p^2}{2m} + V(x)\right) = e^{-ikx}H(x,p)\end{aligned}$$

However, since $\hat{x}$ and $\hat{p}$ do not commute, for a non-constant potential,

$$\langle k|e^{-\frac{i}{\hbar}(\frac{\hat{p}^2}{2m}+V(\hat{x}))(t-t')}|x\rangle \neq e^{-\frac{i}{\hbar}\left(\frac{p^2}{2m}+V(x)\right)(t-t')} e^{-ikx}. \tag{31}$$

Nevertheless, they are approximately equal if $t - t' = \Delta t$ is small. As a consequence, the mixed matrix elements of the infinitesimal time evolution operator may also be expressed in terms of

they are not the same. In a sense the limit $t \to t'$ needs to be taken via real gaussians while the limit $\hbar \to 0$ is the naive one. This is to be expected on physical grounds, the propagator must tend to the identity at $t = t'$ and must have an essential singularity as $\hbar \to 0$

the classical hamiltonian. For small Δt, $U(\Delta t) \approx I - \frac{i}{\hbar} H \Delta t$, so

$$\langle k|U(\Delta t)|x\rangle \approx \langle k|I - \frac{i}{\hbar}\hat{H}\Delta t|x\rangle = \left(1 - \frac{i}{\hbar}H(x,p)\Delta t\right) e^{-ikx} \approx e^{-\frac{i}{\hbar}H(x,p)\Delta t} e^{-ikx} \qquad (32)$$

Unlike the case of a free particle where we had an exact formula (25) for $\langle x|U(\Delta t)|x'\rangle$ in terms of the classical action, here we only have an approximate formula for $\langle k|U(\Delta t)|x\rangle$ in terms of the classical hamiltonian. However, this is adequate to derive a Feynman path integral representation, since we are going to let $\Delta t \to 0$ eventually.

• We can use these mixed matrix elements to evaluate the propagator in the position basis. As before, we sub-divide the time $t_f - t_i = n\Delta t$ into n equal steps $t_j = t_i + j\Delta t$ for $0 \le j \le n$ and denote $x_i = x_0, x_f = x_n$. Using the reproducing property we have

$$\langle x_f|U(t_f,t_i)|x_i\rangle = \int dx_1 \cdots dx_{n-1} \langle x_n|U(t_n,t_{n-1})|x_{n-1}\rangle\langle x_{n-1}|U(t_{n-1},t_{n-2})|x_{n-2}\rangle \cdots \langle x_1|U(t_1,t_0)|x_0\rangle \qquad (33)$$

In order to exploit our formula for the mixed matrix elements of U, we insert complete sets of momentum eigenstates in n places. Thus

$$\begin{aligned}
U(x_f t_f, x_i t_i) &= \int dx_1 \cdots dx_{n-1} [dk_0 \cdots dk_{n-1}] \; \langle x_n|k_{n-1}\rangle\langle k_{n-1}|U(t_n,t_{n-1})|x_{n-1}\rangle\langle x_{n-1}|k_{n-2}\rangle \\
&\quad \langle k_{n-2}|U(t_{n-1},t_{n-2})|x_{n-2}\rangle \cdots \langle k_1|U(t_2,t_1)|x_1\rangle\langle x_1|k_0\rangle\langle k_0|U(t_1,t_0)|x_0\rangle \\
&\approx \int dx_1 \cdots dx_{n-1} [dk_0 \cdots dk_{n-1}] \; \exp\left[i\sum_{j=0}^{n-1} k_j (x_{j+1} - x_j) - \frac{i}{\hbar}\sum_{j=0}^{n-1} H(x_j,p_j)\Delta t \right] \\
&= \int dx_1 \cdots dx_{n-1} [dk_0 \cdots dk_{n-1}] \; \exp\left[\frac{i}{\hbar}\sum_{j=0}^{n-1} \Delta t \left(p_j \frac{(x_{j+1}-x_j)}{\Delta t} - H(x_j,p_j) \right) \right]
\end{aligned}$$

Now as $n \to \infty$ the exponent tends to $\frac{i}{\hbar}\int_{t_i}^{t_f} (p\dot{x} - H(x,p))\, dt$, a formula familiar from classical mechanics. The first term is the abbreviated action we came across in the the semiclassical approximation. We write formally (absorbing the numerical factors of $1/2\pi\hbar$ into a pre-factor C)

$$U(x_f t_f, x_i t_i) = C \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathrm{D}[x]\, \mathrm{D}[p]\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} (p\dot{x} - H(x,p))\, dt} \qquad (34)$$

This is called a phase space path integral, as we integrate over paths in phase space $(x(t),p(t))$. Notice however, that the initial and final momenta are unconstrained, unlike the initial and final positions. To get the configuration space path integral, we perform the gaussian integral over the momenta. This is possible since $H(x,p) = p^2/2m + V(x)$ is quadratic in the momenta. Returning to the finite n formula, let us consider one of the p integrals

$$I_j = \int \frac{dp_j}{h} e^{\frac{i}{\hbar}\left(p_j(x_{j+1}-x_j) - \frac{p_j^2}{2m}\right)\Delta t} = \sqrt{\frac{m}{ih\Delta t}} \exp\left[\frac{i}{\hbar}\frac{m}{2}\frac{(x_{j+1}-x_j)^2}{\Delta t} \right] \qquad (35)$$

Thus we have an expression for $U(x_f t_f, x_i t_i)$ which becomes increasingly accurate as $n \to \infty$:

$$U(x_f t_f, x_i t_i) \approx \left(\frac{m}{ih\Delta t}\right)^{n/2} \int dx_1 \cdots dx_{n-1} \exp\left[\frac{i}{\hbar}\sum_{j=0}^{n-1} \left(\frac{1}{2} m \frac{(x_{j+1}-x_j)^2}{(\Delta t)^2} - V(x_j) \right) \Delta t \right] \qquad (36)$$

In the limit $n \to \infty$ we see that the exponent becomes the classical action for the path $x(t)$. We write the propagator formally as a path integral

$$U(x_f t_f, x_i t_i) = C \int \mathrm{D}[x] \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f}\left[\frac{1}{2}m\dot{x}^2 - V(x)\right]dt} \tag{37}$$

where some (dimensional) factors have been absorbed into the pre-factor C. We are not in a position to give a direct mathematically precise definition for such a path integral. What is more, C, $D[x]$, $D[p]$ in all likelihood cannot be given a meaning in isolation. However, it is likely that the integral as a whole can be given a mathematically precise meaning. In any case, the finite n version above gives a sequence of calculable approximants which can be improved by making n larger, just as we can improve our calculation of the area of a region of the plane by using a finer square grid. We may also profitably regard the path integral as a short-hand notation for the previous multiple integral as n is made large. This is similar to the way we regard the expression $\int_a^b f(x)dx$ as a short-hand notation for the process of taking Riemann sums. Just as we first learned to integrate polynomials and trigonometric functions before attempting to define the integral of an arbitrary function, it is necessary to understand the path integral and its physical implications for simple quantum mechanical systems before attempting to give a mathematically precise definition of the path integral.

7 Linear harmonic oscillator via path integrals

7.1 Harmonic oscillator propagator by path integral

• It was reasonably easy to find the energy levels of the SHO by solving the Schrödinger eigenvalue problem using creation-annihilation operators. We could use the spectrum of energies E_n and eigenfunctions $\psi_n(x) = \langle x|n\rangle$ (Hermite polynomial times gaussian) to find the propagator by summing the series

$$\langle x_f|U(t_f,t_i)|x_i\rangle = \sum_n e^{-iE_n(t_f-t_i)/\hbar}\psi_n(x_f)\psi_n^*(x_i). \tag{38}$$

The path integral gives a different way of finding the SHO propagator. In fact, we can even find the energy spectrum from the propagator. Recall the path integral representation

$$U(x_f,t_f;x_i,t_i) = C\int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathrm{D}[x] \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f}\left[\frac{1}{2}m\dot{x}(t)^2 - \frac{1}{2}m\omega^2 x(t)^2\right]dt}. \tag{39}$$

The main problem is to give a meaning to this path integral by defining it as the limit of appropriate multi-dimensional integrals. In particular, we haven't tried to define the integration element on paths $\mathrm{D}[x]$ by taking a limit of time-sliced integrals $dx_1 \cdots dx_{N-1}$ nor the factor C by taking the limit of $C_N = (m/ih\Delta t)^{N/2}$ since these may not be individually meaningful. Indeed C_N tends to infinity. However, the pre-factor C_N will be multiplied by certain other factors arising from evaluation of the integral $\int \mathrm{D}[x]\cdots$, these other factors will tend to zero, so that the product has a finite limit. Moreover, all these factors will be seen to be independent of ω, x_i and x_f and can be fixed by requiring that the SHO propagator reduce to the free particle propagator in the limit $\omega \to 0$.

• Since U is only a function of the difference $t_f - t_i$, we may without loss of generality take $t_i = 0$ and write $T = t_f$. We will evaluate this integral over paths in the generic case where ωT

isn't an integer multiple of π. We have already seen that there is a unique classical trajectory joining x_i, t_i and x_f, t_f in this case. The exceptional cases $\omega T = n\pi$ are more subtle since there are either none or infinitely many classical trajectories joining x_i to x_f if $T\omega$ is an integer multiple of π.

• The Lagrangian is quadratic in $x(t)$, so the above path integral looks like an (infinite dimensional) gaussian integral. To exploit this feature, let x_{cl} be the unique classical trajectory satisfying the above boundary conditions and let us write the path $x(t)$ as $x(t) = x_{cl}(t) + \delta x(t)$ where $\delta x(t)$ is an arbitrary (not necessarily small) variation in the path satisfying $\delta x(0) = \delta x(T) = 0$. Then S is extremal at x_{cl} and we may write the path integral as an integral over the variations δx

$$\begin{aligned} U &= Ce^{\frac{i}{\hbar}S[x_{cl}]} \int_{\delta x(t_i)=0}^{\delta x(t_f)=0} \mathrm{D}[\delta x] \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} \left[\frac{1}{2} m\, \delta\dot{x}^2 - \frac{1}{2} m\omega^2 \delta x^2 \right] dt \right\} \\ &= Ce^{\frac{i}{\hbar}S[x_{cl}]} \int_{\delta x(t_i)=0}^{\delta x(t_f)=0} \mathrm{D}[\delta x] \exp \frac{i}{\hbar} \int_0^T \delta x(t)\, A\, \delta x(t)\, dt \;\; \text{where} \;\; A = -\frac{m}{2}\left(\frac{d^2}{dt^2} + \omega^2\right) \end{aligned} \quad (40)$$

The exponent is quadratic in the variables of integration $\delta x(t)$, so this is an infinite dimensional analogue of a gaussian integral[5]. We give meaning to it as a limit of finite dimensional discretized integrals. There are many ways of discretizing the integral. Rather than time-slice the interval (which is how we arrived at the path integral in the first place), let us follow the somewhat more elegant method of Fourier monomials, which are an eigenbasis for the hessian operator A (second variation of the action). $\delta x(t)$ is a function that vanishes at the end points of the interval $[0, T]$. The Fourier sine monomials $\phi_n(t) = \sin \frac{n\pi t}{T}$ are a complete orthogonal set of eigenfunctions of A in the Hilbert space of square-integrable functions on $[0, T]$ vanishing at the end points. ϕ_n have non-zero eigenvalues as long as $\omega T \neq n\pi$:

$$\text{for } n = 1, 2, \ldots, \quad A\phi_n = -\frac{1}{2} m \left(\frac{d^2}{dt^2} + \omega^2\right) \sin \frac{n\pi t}{T} = \lambda_n \sin \frac{n\pi t}{T} \quad \text{where} \quad \lambda_n = \frac{m}{2}\left(\frac{n^2\pi^2}{T^2} - \omega^2\right) \neq 0.$$

So we should expect the gaussian integral to simplify in the Fourier sine basis, which in effect is a convenient basis to compute the determinant of A. We may expand $\delta x(t)$ in a Fourier sine series

$$\delta x(t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi t}{T}, \quad \text{where} \quad c_n \in \mathbb{R}. \quad (41)$$

The information in $\delta x(t)$ is contained in the Fourier coefficients, so an integration over all paths may be replaced by an integration over all possible Fourier coefficients, any Jacobian from the change in integration element will be absorbed into the pre factor C. To make it a finite dimensional integral, we restrict to Fourier polynomials of degree N and eventually let $N \to \infty$. First we write the integrand in terms of the Fourier coefficients. Using orthogonality of $\sin \frac{n\pi t}{T}$, we have

$$\int_0^T \left[\frac{1}{2} m\delta\dot{x}^2 - \frac{1}{2} m\omega^2 \delta x^2 \right] = \frac{m}{2}\frac{T}{2} \sum_1^{\infty} c_n^2 \left(\frac{n^2\pi^2}{T^2} - \omega^2\right) \quad (42)$$

There are no cross terms $c_n c_m$ for $n \neq m$ as A is diagonal in the Fourier sine basis. Thus the

[5] E.g. A finite dimensional gaussian integral over $x_1, \cdots, x_N$: $I = \int e^{-x^t A x} \prod_i dx_i$ for a real symmetric matrix A. It may be evaluated by going to a basis in which A is diagonal, one gets $I = \int e^{-\sum_n a_n y_n^2} \prod_n dy_n = \pi^{N/2}(a_1 \cdots a_N)^{-1/2} = \pi^{N/2}/\sqrt{\det A}$

$N^{\rm th}$ approximant to the propagator is a product of gaussian integrals

$$\begin{aligned} U_N(x_f,T;x_i,0) &= C_N e^{\frac{i}{\hbar}S[x_{cl}]} \int_{-\infty}^{\infty} \left(\prod_{n=1}^N dc_n\right) \exp\left\{-\frac{mT}{4i\hbar}\sum_1^N c_n^2 \left(\frac{n^2\pi^2}{T^2}-\omega^2\right)\right\} \\ &= C_N e^{\frac{i}{\hbar}S[x_{cl}]} \prod_1^N \left[\int_{-\infty}^{\infty} e^{-\alpha_n c_n^2}\, dc_n\right] \\ \text{where } \alpha_n &= \frac{mT}{4i\hbar}\left(\frac{n^2\pi^2}{T^2}-\omega^2\right) = \frac{m\pi^2 n^2}{4i\hbar T}\left(1-\frac{\omega^2T^2}{n^2\pi^2}\right). \end{aligned}$$

The gaussian integrals are evaluated and one gets

$$U_N = \frac{C_N \pi^{N/2}}{\sqrt{\prod_1^N \frac{m\pi^2n^2}{4i\hbar T}}} e^{\frac{i}{\hbar}S[x_{cl}]} \prod_1^N \left(1-\frac{\omega^2T^2}{n^2\pi^2}\right)^{-1/2} = \tilde{C}_N(h,m,T)\, e^{\frac{i}{\hbar}S[x_{cl}]} \left[\prod_{n=1}^N \left(1-\frac{\omega^2T^2}{n^2\pi^2}\right)\right]^{-1/2}. \tag{43}$$

Both $C_N\pi^{N/2}$ as well as the denominator $\sqrt{\prod_1^N \frac{m\pi^2n^2}{4i\hbar T}}$ are divergent as $N\to\infty$, but the limit is taken in such a way that the quotient $\tilde{C}_N$ has a finite limit. Note that the new pre-factor $\tilde{C}_N$ is independent of ω, x_i and x_f. So let us denote $\tilde{C}(h,m,T) = \lim_{N\to\infty}\tilde{C}_N$. As for the ω-dependent product (see homework),

$$\lim_{N\to\infty}\prod_{n=1}^N \left(1-\frac{\omega^2T^2}{n^2\pi^2}\right) = \frac{\sin\omega T}{\omega T}. \tag{44}$$

Thus the SHO propagator is

$$U^{\rm SHO}(x_f,T;x_i,0) = \tilde{C}(h,m,T)\sqrt{\frac{\omega T}{\sin\omega T}}\exp\left\{\frac{i}{\hbar}S[x_{cl}]\right\} \tag{45}$$

The factor $\tilde{C}$ is fixed by comparing with the free particle propagator

$$\lim_{\omega\to 0} U^{\rm SHO} = U^{\text{free particle}} = \sqrt{\frac{m}{ihT}} e^{\frac{i}{\hbar}S[x_{cl}]} \quad\Rightarrow\quad \tilde{C} = \sqrt{\frac{m}{ihT}} \tag{46}$$

So the SHO propagator (also known as the Mehler kernel) when $\omega T \neq n\pi$ is

$$U(x_f,T;x_i,0) = \sqrt{\frac{m\omega}{ih\sin\omega T}}\exp\left\{\frac{i}{\hbar}S[x_{cl}]\right\}. \tag{47}$$

where $x_{cl}(t)$ is the unique classical trajectory satisfying $x(0)=x_i$ and $x(T)=x_f$ and

$$S[x_{cl}] = \frac{m\omega}{2\sin\omega T}\left[(x_i^2+x_f^2)\cos\omega T - 2x_ix_f\right]. \tag{48}$$

7.2 Harmonic oscillator spectrum from path integral

Now that we have evaluated the SHO propagator by path integrals, we put it to use to obtain the SHO energy levels. This provides an alternate route to the SHO spectrum without any need to solve the Schrödinger eigenvalue problem. The main idea is to exploit the relation

$$U(T) = U(T,0) = e^{-iHT/\hbar} = \sum_{nn'} |n\rangle\langle n|e^{-iHT/\hbar}|n'\rangle\langle n'| = \sum_n e^{-iE_nT/\hbar}|n\rangle\langle n| \tag{49}$$

To concentrate on the energy spectrum, we evaluate the trace of U. U is diagonal in the energy basis, so its trace is easily expressed in terms of the energy levels.

$$\text{tr}\, U(T) = \sum_n e^{-iE_n T/\hbar} \tag{50}$$

The trace is basis independent, so we will compute it in the position basis using our formula (47) for the propagator, assuming $\omega T \neq n\pi$. By comparing the answer with the previous expression, we aim to extract the energy levels. Recall that the SHO propagator is

$$U(x_f, T; x_i, 0) = \sqrt{\frac{m\omega}{ih \sin \omega T}} \exp\left\{\frac{i}{\hbar} S[x_{cl}]\right\} \tag{51}$$

For $t_f - t_i = T \neq n\pi/\omega$, the unique classical trajectory joining x_i to x_f is (denote $\cos \omega t_i = c_i$ etc)

$$x(t) = a\cos\omega t + b \sin \omega t \quad \text{where} \quad a = \frac{s_f x_i - s_i x_f}{s_{f-i}} \quad \text{and} \quad b = \frac{c_i x_f - c_f x_i}{s_{f-i}}. \tag{52}$$

The Lagrangian for this trajectory $L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2$ is (denote $s = \sin \omega t, c = \cos \omega t$)

$$L = \frac{1}{2} m\omega^2 \left[a^2 \left(s^2 - c^2\right) + b^2 \left(c^2 - s^2\right) - 4absc\right] = \frac{1}{2} m\omega^2 \left[\left(b^2 - a^2\right) \cos 2\omega t - 2ab \sin 2\omega t\right]. \tag{53}$$

Without loss of generality, we take $t_i = 0, t_f = T$ and the action for this path is

$$S[x] = \int_0^T L\, dt = \frac{m\omega}{2} \left[\frac{\left(b^2 - a^2\right)}{2} \sin 2\omega T - 2\, ab \sin^2 \omega T\right] = \frac{m\omega}{2 \sin \omega T} \left[(x_i^2 + x_f^2) \cos \omega T - 2 x_i x_f\right]$$

where $a = x_i$ and $b = \frac{x_f - x_i c_f}{s_f}$. To evaluate the trace of the propagator,

$$\text{tr}\, U(T) = \int_{-\infty}^{\infty} \langle x \mid U(T) \mid x \rangle dx, \tag{54}$$

we need the action of the classical trajectory $x(t)$ with $x_i = x_f = x$ for $0 \leq t \leq T$. In this case, one finds[6]

$$a = x, \quad b = x \tan \frac{\omega T}{2}, \quad x(t) = x[\cos \omega t + \tan \frac{\omega T}{2} \sin \omega t] \;\text{ and }\; S[x, T; x, 0] = -m\omega x^2 \tan\left(\frac{\omega T}{2}\right)$$

[6]This trajectory makes sense as long as $T \neq (2n+1)\pi/\omega$ which is ensured by our assumption that $\omega T \neq n\pi$. It is checked that $x(0) = x(T) = x$. Note that for this trajectory, the particle returns to the point x after a time T that has nothing to do with the period of oscillation $T^* = 2\pi/\omega$. This is because, when the particle returns to x, its velocity is reversed in sign. The time it takes for this need not be a half period, nor have any relation to the period. To visualize this, imagine a point x near the maximal extension of an oscillating spring. The tip of the spring passes through x on its way out and returns to x on its way in, and the time elapsed T has nothing to do with the period of oscillation. To drive home this point, move x closer to the maximal extension point. Then T will decrease, while the period of oscillation T^* is unaltered. Of course, there are exceptional points x to which a particle can return only after a time equal to a multiple of a half-period $T^*/2$. These are the points of maximal extension and the point $x = 0$. These exceptional cases are mostly omitted via the assumption $T \neq n\pi/\omega$. The exceptional case $x = 0$ is included via the trajectory $x(t) \equiv 0$, which returns to $x = 0$ after *any* time T and has zero action.

Thus the trace of the propagator has been reduced to a gaussian integral which we evaluate

$$\text{tr}\, U(T) = \int_{-\infty}^{\infty} U(x,T;x,0)dx = \sqrt{\frac{m\omega}{ih\sin\omega T}} \int_{-\infty}^{\infty} \exp\left[-\frac{i}{\hbar} m\omega x^2 \tan\frac{\omega T}{2}\right] \, dx = \left(2i\sin\frac{\omega T}{2}\right)^{-1} \tag{55}$$

We can now use this remarkably simple formula for the trace of the propagator to recover the SHO spectrum. We wish to write $\text{tr}\, U$ as a sum of phases each proportional to the time T, and compare with the expression $\text{tr}\, U = \sum_n e^{-iE_nT/\hbar}$ to read off the energies

$$\text{tr}\, U = \frac{1}{e^{i\omega T/2} - e^{-i\omega T/2}} = \frac{e^{-i\omega T/2}}{1 - e^{-i\omega T}} = \sum_0^\infty \exp\left[-\frac{iT}{\hbar}\hbar\omega\left(n + \frac{1}{2}\right)\right] \tag{56}$$

From this we infer the spectrum of energies of the SHO, $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$.