Summary of Research: Govind S. Krishnaswami

Chennai Mathematical Institute, 10 March, 2022

My work in theoretical and mathematical physics with various collaborators and students spans topics in quantum field theory, particle physics, integrable systems, fluid, plasma and nonlinear dynamics.

1. **Collinear QCD:** Understanding the behavior of baryons and mesons from QCD is a challenge. One aims to explain how quarks and gluons bind to form hadrons. We have worked on $1+1$ dimensional QCD in 't Hooft's multi-color large-N limit with V John and S G Rajeev. Rajeev showed that the baryon arises as a topological soliton of a gauge-invariant bi-local reformulation of the theory. We found its ground state form-factor and mass in the chiral limit and used it to model the non-perturbative Bjorken-x dependence of nucleon structure functions [1, 2, 3, 4]. Earlier work [5] on a neutrino-nucleon deep inelastic scattering experiment had focused on measuring structure functions. Our results were shown to agree with experimental data [6, 7]. The geometry of the phase space was used to deduce a relativistic interacting parton model as a variational approximation to the bi-local solitonic picture [8, 9]. Generalizing 't Hooft's equation for mesons, an approximate equation for the spectrum of excited baryons was obtained in [3].

2. **Large-N matrix models and non-commutative probability:** Matrix models may be regarded as toy-models for the dynamics of gluons. In our work with L Akant and S G Rajeev [10, 1] we found a variational principle for the large-N equations of Euclidean multi-matrix models, circumventing a cohomological obstruction involving Voiculescu's entropy of operator-valued random variables. With A Agarwal, L Akant and S G Rajeev, this was extended to a gauge-invariant formulation of Hamiltonian multi-matrix models, by relating non-commutative Fisher information to the collective potential of Jevicki and Sakita [11].

3. **Loop equations of large-N matrix models:** It is a long-standing problem to understand and solve the loop equations or factorized Schwinger-Dyson equations for correlations of multi-matrix models in the large-N limit. We formulated them as quadratic difference equations in concatenation of gluon correlations [12]. They aren't differential equations as they involve left annihilation, which does not satisfy the Leibniz rule for concatenation. But left annihilation is a derivation of the commutative shuffle product. An approximation method was proposed by expanding concatenation around the shuffle product. At zeroth order for the Gaussian, Chern-Simons and Yang-Mills matrix models, the resulting quadratic PDEs were shown to linearize by passage to the shuffle reciprocal of correlations. However, the equations are under-determined in general, as are the loop equations themselves. With L Akant [13] we identified Ward identities related to symmetries of action and measure to supplement the loop equations. New variational and other approximation schemes for the loop equations were also found [12, 14]. In [15, 16] we found a matrix model analogue of the group of loops on space-time that plays an important role in Yang-Mills theory. It is the spectrum of a commutative shuffle-deconcatenation Hopf algebra associated to gluon correlations. The generating series for large-N correlations is

a function on this group and satisfies quadratic loop equations in convolution. The associated Schwinger-Dyson operators for Yang-Mills, Chern-Simons and Gaussian models were shown to be right-invariant vector fields on this group.

4. **Phase transition in a matrix model for gluons in baryons:** Inspired by the reduction of QCD to $1+1$ dimensions, in [17] we studied a large-N matrix model coupled to quarks. Though quarks are usually suppressed by $1/N$, their contribution rivals that of gluons in a baryon containing N quarks. After some truncations, quarks were integrated out and a 1-matrix model for gluons with polynomial + logarithmic potential obtained. We found a 3^{rd} order phase transition as the ratio of quark mass to gauge coupling is increased, separating a 2-cut chiral limit where light quarks are strongly coupled to gluons from a 1-cut phase where heavy quarks are weakly coupled to gluons.

5. **Abelian 'gauge theory' inspired by Eulerian hydrodynamics:** Our earlier work indicated the importance of non-commutative analogues of diffeomorphism groups in gauge-invariant formulations of large-N matrix models. A simpler occurrence of diffeomorphism groups is found in fluid mechanics: the configuration space of Eulerian hydrodynamics is the group of volume-preserving diffeomorphisms of the flow domain. In [18] Eulerian hydrodynamics on a surface M with volume form μ and metric was formulated as an abelian gauge theory using a duality between volume preserving vector fields and 1-forms modulo exact 1-forms. Choosing the L^2 norm of velocity as the Hamiltonian leads to standard Eulerian hydrodynamics. Interestingly, choosing magnetic energy as Hamiltonian leads to another geodesic flow on SDiff(M,μ) which we showed admits an infinite sequence of local conserved charges in involution.

6. **Non-trivial fixed points for 4D $O(N)$ scalar fields:** In [19] we studied four dimensional scalar fields in an attempt to circumvent the UV and naturalness problems in the Higgs sector of the standard model of particle physics. We found a line of UV fixed points in 4D $O(N)$ scalar field theory in the large-N limit. A mass deformation from such a fixed point would have naturally light scalar excitations on account of scale invariance. In 3D, our construction reduces to the line of large-N fixed-points in $|\phi|^6$ theory.

7. **Scale-invariant cousin of KdV:** In work with D Ahalpara, A Sen and A Thyagaraja that combined genetic programming and analytical methods, we found a remarkable nonlinear scale-invariant advection-dispersion equation for one dimensional flow $u_t + (2u_{xx}/u)u_x = \epsilon u_{xxx}$ [20]. This cousin (SIdV) of the KdV equation admits plane, solitary and cnoidal waves. It is a bridge between nonlinear dispersive advection and diffusion. For some special values of the coefficient of dispersion ϵ we could find a Hamiltonian formulation and transform it into the integrable mKdV equation or a linear equation. Numerical simulations show that SIdV displays recurrence in bounded domains. We have also shown that it is a member of an infinite dimensional family of equations sharing the KdV solitary wave.

8. **Spin quantum plasmas:** With R Nityananda, A Sen and A Thyagaraja, we critically examined [21, 22] recent semi-classical theories of spin-half quantum plasmas and claims of spin-gradient-driven light amplification in quantum plasmas [23, 24]. We showed that

some of the derivations and results contradict principles of quantum and statistical mechanics especially in their treatment of fermions and spin. Claims of large semi-classical effects of spin magnetic moments that could dominate the plasma dynamics were found to be invalid both for single-particles and collectively. Larmor moments dominate at high temperature while spin moments cancel due to Pauli blocking at low temperatures. Numerical estimates from a variety of plasmas were provided to demonstrate that spin effects are much smaller than many neglected classical effects.

9. **Fluid analogy for the Higgs mechanism:** With S S Phatak we found a novel correspondence between the Higgs mechanism and the added-mass effect in inviscid hydrodynamics [25, 26]. A rigid body accelerated through inviscid, incompressible and irrotational flow feels an added mass. The added force is linear in its acceleration but can point in a direction determined by the 'added mass tensor', which depends on the shape of the body. In our analogy, the gauge Lie algebra corresponds to the space of directions in which a rigid body can move in a fluid. The vector boson mass matrix corresponds to the added-mass tensor. The pattern of gauge symmetry breaking is encoded in the shape of an associated rigid body. Symmetries of the body are related to those of the scalar vacuum manifold. For instance, an SO(3) gauge theory spontaneously broken to SO(2) corresponds to a hollow cylindrical shell with one zero and two equal added-mass eigenvalues. The vacuum expectation value of the scalar field is analogous to the (constant) density of fluid while quantum fluctuations around the Higgs vacuum are like density fluctuations around incompressible flow. A long wavelength wave around an accelerated body should play the role of the Higgs particle.

10. **Algebra and geometry of Hamilton's quaternions.** Inspired by the relation between the algebra of complex numbers and plane geometry, William Rowan Hamilton sought an algebra of triples for application to three dimensional geometry. Unable to multiply and divide triples, he invented a non-commutative division algebra of quadruples, in what he considered his most significant work, generalizing the real and complex number systems. In this expository article with S Sachdev [27], we give a motivated introduction to quaternions and discuss how they are related to Pauli matrices, rotations in three dimensions, the three sphere, the group $\mathrm{SU}(2)$ and the celebrated Hopf fibrations.

11. **Conservative regularization of ideal fluids and plasmas.** Ideal neutral and charged fluid flows can develop singularities that may be regularized either by dissipative or conservative mechanisms. For instance, the one dimensional Hopf equation $u_t + uu_x = 0$ used to model kinematic advection and traffic flow can develop shock-like singularities with u_x diverging and u becoming multi-valued in finite time, for a large class of initial data [28]. These singularities are physically regularized either by introducing viscosity as in the Burgers equation ($u_t + uu_x = \nu u_{xx}$) or through dispersion as in the KdV equation ($u_t + uu_x = \alpha u_{xxx}$) with applications to water waves, ion acoustic waves etc. More generally, in three dimensions, compressible Eulerian flow ($\rho(\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p$) can develop shock-like and vortical singularities: enstrophy $\int \mathbf{w}^2 \, d^3r$ can become very large due to the phenomenon of vortex stretching. Here $\mathbf{w} = \nabla \times \mathbf{v}$ is the vorticity of the velocity field $\mathbf{v}$. While the celebrated Navier-Stokes equation (whose regularity is the subject of a Clay millenium problem [29]) provides a viscous regularization, a physically well-motivated

local conservative regularization was not available (the α-Euler equation of Holm, Marsden and Ratiu [30] is non-local). With S Sachdev and A Thyagaraja, we developed a new local conservative 'twirl' regularization of compressible flow, magnetohydrodynamics and two-fluid plasmas, preserving Galilean and discrete symmetries [31, 32, 33, 34]. The twirl term $-\lambda^2 \mathbf{w} \times (\nabla \times \mathbf{w})$ in the regularized Euler equation is a 3D inviscid counterpart of Navier-Stokes viscosity $\nu \nabla^2 \mathbf{v}$ and corresponds to the inclusion of a vortical energy $\int \lambda^2 \rho\, \mathbf{w}^2\, d^3r$ in the Hamiltonian. Here λ is a dynamical short-distance cutoff subject to the constitutive law $\lambda^2 \rho =$ constant, and could be taken as the Debye length or skin depth in plasmas, both of which vary inversely with the square-root of density. The regularization is minimal in nonlinearity and derivatives and could be important in flows with significant vorticity, it could also help regulate numerical simulations. It implies an a priori upper bound on enstrophy, thus regularizing vortical singularities. Lagrangian as well as Hamiltonian-Poisson bracket formulations have been found. Steady solutions of the regularized equations modelling a columnar vortex (tornado), MHD pinch, vortex sheet, channel flow and spherical/cylindrical vortices have been found.

12. **Geometrical approach to the three-body problem:** With H Senapati, we have worked on a geometrical approach to the classical planar 3-body problem [35, 36]. Trajectories are reparametrized geodesics of the Jacobi-Maupertuis metric on configuration space, which possesses translation and rotation isometries for any central potential as well as scaling isometries for the zero-energy $1/r^2$ potential. Techniques of Riemannian submersions are used to quotient the full six dimensional configuration space $\mathbb{R}^6$ by these symmetries to arrive at geodesic dynamics on the three-sphere, shape space $\mathbb{R}^3$ and the shape sphere $\mathbb{S}^2$ with collision configurations removed. We extend work of R Montgomery to show that the scalar curvature is strictly negative on these quotients and find sectional curvatures to be largely negative, indicating widespread geodesic instabilities. The qualitative dynamical consequences of this partial negativity in curvature remain to be understood. While the Jacobi-Maupertuis metric for the Newtonian potential is shown to be geodesically incomplete, it is complete for the $1/r^2$ potential, so that pairwise and triple collisions are, in a sense, regularized.

13. **Hamiltonian formulation and integrability of the Rajeev-Ranken model:** With T R Vishnu, we have been studying a 1+1 dimensional scalar field theory dual to the principal chiral model and its reduction to a mechanical system [38]. The integrable 1+1-dimensional SU(2) principal chiral model (PCM) serves as a toy-model for the theory of strong interactions (3+1-dimensional Yang-Mills theory and QCD) as it is asymptotically free and displays a mass gap [39]. Interestingly, the PCM is 'pseudo-dual' to a scalar field theory introduced by Zakharov and Mikhailov [40] and Nappi [41] that is strongly coupled in the ultraviolet and could serve as a toy-model for non-perturbative properties of theories with a Landau pole (such as $\lambda\phi^4$ in the Higgs sector of the standard model). In particular, one wishes to identify degrees of freedom appropriate to the description of the dynamics of such models at high energies. Unlike the semi-direct product of $\mathfrak{su}(2)$ and abelian current algebras of the PCM, its pseudo-dual is based on a nilpotent current algebra. Theories that admit a formulation in terms of quadratic Hamiltonians and nilpotent Lie algebras are particularly interesting: they include the harmonic and anharmonic

oscillators as well as field theories such as Maxwell, $\lambda\phi^4$ and Yang-Mills. Recently, Rajeev and Ranken [42] obtained a mechanical reduction by restricting the nilpotent scalar field theory to a class of nonlinear classical waves expressible in terms of elliptic functions, whose quantization survives the passage to the strong-coupling limit. We study the Hamiltonian and Lagrangian formulations of this model and its classical integrability, identifying Darboux coordinates, Lax pairs [43, 44, 45], classical r-matrices and a degenerate Poisson pencil. We identify Casimirs as well as a complete set of conserved quantities in involution and the canonical transformations they generate. They are related to Noether charges of the field theory and are shown to be generically functionally independent, implying Liouville integrability. We also find an interesting relation between this model and the Neumann model [46] allowing us to discover a new Hamiltonian formulation of the latter.

14. **Stability and chaos in the classical three rotor problem:** The classical three body problem [47], which arose in trying to understand the effect of the Sun on the Moon's Keplerian orbit around the Earth, has been a rich source of phenomena and a context for developing techniques [36]. Euler and Lagrange found periodic solutions while Poincaré discovered chaos in this problem. Its study catalyzed the development of perturbation theory and canonical transformations and has shed light on the nature of collisional and non-collisional singularities [48]. With H Senapati [49, 50], we have investigated a simpler variant: the equal-mass classical three rotor problem, where particles move without collisions on a circle, subject to inter-particle cosine potentials of strength g. The quantum version of the N-rotor problem models chains of coupled Josephson junctions and also arises via a partial continuum limit of a Wick rotated version of the 2d XY model of statistical mechanics. Classically, while the two rotor problem reduces to the simple pendulum, the limit of infinitely many rotors is related to the sine-Gordon field. Away from these extremes, we find that the three rotor problem displays rich dynamics with novel signatures of the transition to chaos as the relative energy E in units of g is varied. We find new periodic 'pendulum' and 'breather' orbits at all E and choreographies up to moderate E. Loosely, they furnish analogs of the Euler-Lagrange and figure-8 [51] solutions of the planar gravitational three body problem. Integrability at very low energies gives way to a rather marked transition to chaos around $E_c \approx 4g$, followed by a regime of global chaos and a gradual return to regularity as E diverges. The transition to chaos is accompanied by several striking phenomena: (a) the fraction of the area of Poincaré surfaces occupied by chaotic sections rises sharply around E_c, (b) discrete symmetries visible on Poincaré sections at lower energies are spontaneously broken around E_c, (c) $E = 4g$ is an accumulation point of a geometric cascade of stability transitions in periodic pendulum solutions and (d) the Jacobi-Maupertuis curvature [35] goes from being positive to having both signs above $E = 4g$ indicating a remarkable correlation between geodesic instabilities and the onset of chaos. Intriguingly, this also coincides with a change in topology of the Hill region of configuration space. The rather sharp transition to chaos seen in this system is somewhat uncommon among KAM systems where, typically, invariant tori break gradually as one moves away from the integrable limit. What is more, we conjecture ergodic behavior in a band of energies slightly above $E = 4g$. Thus, the three rotor system furnishes a physically interesting autonomous Hamiltonian system

potentially displaying global chaos and ergodicity.

15. **Invariant tori, action-angle variables and phase space structure of the Rajeev-Ranken model:** The Rajeev-Ranken model [42] introduced in 2016, is a Hamiltonian system with 3 degrees of freedom describing nonlinear waves in a 1+1-dimensional scalar field theory dual to the SU(2) principal chiral model (PCM). These novel 'continuous' waves could play a role similar to solitary waves in other field theories. Unlike the PCM, the scalar field theory is strongly coupled in the ultraviolet and serves as a laboratory to study non-perturbative features of theories with a perturbative Landau pole. In previous work [38], we showed that the Rajeev-Ranken model is related, but not equivalent, to the Neumann model [46]. In the present work with T R Vishnu [52], we give its equations a new interpretation as Euler equations for a centrally extended Euclidean algebra with a quadratic Hamiltonian. Thus, they bear a kinship to Kirchhoff's equations for a rigid body moving in a perfect fluid [53, 54]. Solutions of the Rajeev-Ranken model may also be interpreted as a special family of flat $\mathfrak{su}(2)$ connections in 1+1 dimensions. Though analytic solutions in terms of elliptic functions had been found [42], deeper questions about the model's structure and integrability were open. In [38], a degenerate Poisson pencil, Lax pair, r-matrix and four conserved quantities in involution were found. In this paper, we use the Casimirs of the Poisson algebra to find all symplectic leaves and Darboux coordinates on them. The system is Liouville integrable on each leaf and the generic common level sets of conserved quantities are shown to be 2-tori. Going beyond the generic cases, we find three more types of common level sets: horn-tori, circles and points. They are related to places where the conserved quantities develop relations and to the degeneration of solutions from elliptic to hyperbolic and circular. An elegant geometric construction allows us to realize each common level set as a fibre bundle with base determined by the roots of a cubic polynomial. By contrast with the dynamics on tori and circles, which is Hamiltonian, that on horn tori is shown to be a gradient flow. In fact, horn tori behave like separatrices and are also associated to a transition in the topology of energy level sets. Finally, by a careful use of the Poisson structure and elliptic function solutions, we discover a family of action-angle variables for the model. Our new results significantly improve our understanding of the classical Rajeev-Ranken model and should also be useful in understanding it quantization.

16. **Ergodicity, mixing and recurrence in the three rotor problem:** In the classical three rotor problem, three equal point masses move on a circle subject to attractive cosine potentials of strength g. In the center of mass frame, energy E is the only known conserved quantity. In earlier work [49, 50], an order-chaos-order transition was discovered in this system along with a band of global chaos for $5.33g < E < 5.6g$. In the present work with H Senapati [55], we provide numerical evidence for ergodicity and mixing in this band. The Liouville measure ensemble-average distribution functions of relative angles and angular momenta are shown to agree with the corresponding time-average distributions with a power-law approach in time. Moreover, trajectories emanating from a small volume are shown to become uniformly distributed over the energy hypersurface indicating that the dynamics is mixing. Outside this band, ergodicity and mixing fail, though the ensemble-average distributions of momenta show interesting phase transitions from

Wignerian to bimodal with increasing energy. Finally, in the band of global chaos, the distribution of recurrence times to finite size cells is found to follow an exponential law with the mean recurrence time satisfying a scaling law involving an exponent consistent with global chaos and ergodicity.

17. **Nonlinear dispersive regularization of inviscid gas dynamics:** Ideal gas dynamics can develop shock-like singularities with discontinuous density. Viscosity typically regularizes such singularities and leads to a shock structure. On the other hand, in 1d, singularities in the Hopf equation can be non-dissipatively smoothed via KdV dispersion. In [56], with S S Phatak, S Sachdev and A Thyagaraja, we develop a minimal conservative regularization of 3d ideal adiabatic flow of a gas with polytropic exponent γ. It is achieved by augmenting the Hamiltonian by a capillarity energy $\beta(\rho)(\nabla\rho)^2$. The simplest capillarity coefficient leading to local conservation laws for mass, momentum, energy and entropy using the standard Poisson brackets is $\beta(\rho) = \beta_*/\rho$ for constant β_*. This leads to a Korteweg-like stress and nonlinear terms in the momentum equation with third derivatives of ρ, which are related to the Bohm potential and Gross quantum pressure. Just like KdV, our equations admit sound waves with a leading cubic dispersion relation, solitary waves and periodic traveling waves. As with KdV, there are no steady continuous shock-like solutions satisfying the Rankine-Hugoniot conditions. Nevertheless, in 1d, for $\gamma = 2$, numerical solutions show that the gradient catastrophe is averted through the formation of pairs of solitary waves which can display approximate phase-shift scattering. Numerics also indicate recurrent behavior in periodic domains. These observations are related to an equivalence between our regularized equations (in the special case of constant specific entropy potential flow in any dimension) and the defocussing nonlinear Schrödinger equation (cubically nonlinear for $\gamma = 2$), with β_* playing the role of $\hbar^2$. Thus, our regularization of gas dynamics may be viewed as a generalization of both the single field KdV and nonlinear Schrödinger equations to include the adiabatic dynamics of density, velocity, pressure and entropy in any dimension.

18. **An introduction to Lax pairs and the zero curvature representation.** Lax pairs are a useful tool in finding conserved quantities of some dynamical systems. In the expository articles [43, 44, 45] with T R Vishnu, we give a motivated introduction to the idea of a Lax pair of matrices (L, A), first for mechanical systems such as the linear harmonic oscillator, Toda chain, Eulerian rigid body and the Rajeev-Ranken model. This is then extended to Lax operators for one-dimensional field theories such as the linear wave and KdV equations and reformulated as a zero curvature representation via a (U, V) pair which is illustrated using the nonlinear Schrödinger equation. The key idea is that of realizing a (possibly) nonlinear evolution equation as a compatibility condition between a pair of linear equations. The latter could be an eigenvalue problem for the Lax operator L and a linear evolution equation generated by A, for the corresponding eigenfunction. Alternatively, they could be the first order linear system stating the covariant constancy of an arbitrary vector with respect to the 1+1 dimensional gauge potential (V, U). The compatibility conditions are then either the Lax equation $\dot{L} = [L, A]$ or the flatness condition $U_t - V_x + [U, V] = 0$ for the corresponding gauge potential. The conserved quantities then follow from the isospectrality of the Lax and monodromy matrices.

19. **Quantum Rajeev-Ranken model as an anharmonic oscillator.** The Rajeev-Ranken (RR) model [42] is a Hamiltonian system describing screw-type nonlinear waves of wavenumber k in a scalar field theory pseudodual to the 1+1D SU(2) principal chiral model. Classically, the RR model is Liouville integrable [38, 52]. With T R Vishnu in [57], we interpret the model as a novel 3D cylindrically symmetric quartic oscillator with an additional rotational energy. The quantum theory has two dimensionless parameters. Upon separating variables in the Schrödinger equation, we find that the radial equation has a four-term recurrence relation. It is of type $[0, 1, 1_6]$ and lies beyond the ellipsoidal Lamé and Heun equations in Ince's classification. At strong coupling λ, the energies of highly excited states are shown to depend on the scaling variable λk. The energy spectrum at weak coupling and its dependence on wavenumber k in a double-scaling strong coupling limit are obtained. The semi-classical WKB quantization condition is expressed in terms of elliptic integrals. Numerical inversion enables us to establish a $(\lambda k)^{2/3}$ dispersion relation for highly energetic quantized 'screwons' at moderate and strong coupling. We also suggest a mapping between our radial equation and one of Zinn-Justin and Jentschura [58, 59] that could facilitate a resurgent WKB expansion for energy levels. In another direction, we show that the equations of motion of the RR model can also be viewed as Euler equations for a step-3 nilpotent Lie algebra. We use our canonical quantization to uncover an infinite dimensional reducible unitary representation of this nilpotent algebra, which is then decomposed using its Casimir operators.

References

[1] G. S. Krishnaswami, *Large-N Limit as a Classical Limit: Baryon in Two-dimensional QCD and Multi-Matrix Models*, PhD Thesis, U. Rochester 2004 [arXiv:hep-th/0409279].

[2] G. S. Krishnaswami, *A model of interacting partons for hadronic structure functions*, Undergraduate Thesis, University of Rochester 1999 [arXiv:hep-th/9911538].

[3] G. S. Krishnaswami, *On lightest baryon and its excitations in large-N $1+1$ dimensional QCD*, J. Phys. A: Math. Theor. **43**, 395401 (2010) arXiv:1005.4942.

[4] V. John, G.S. Krishnaswami and S.G. Rajeev, *The anti-quark distribution function of the baryon*, (1999) [arXiv:hep-ph/9908285].

[5] D. A. Harris *et al.* [NuTeV Collaboration], *Precision calibration of the NuTeV calorimeter*, Nucl. Instrum. Meth. A **447**, 377 (2000) [arXiv:hep-ex/9908056].

[6] G.S. Krishnaswami and S.G. Rajeev, *The x dependence of parton distributions compared with neutrino data*, (1999) [arXiv:hep-ph/9908279].

[7] V. John, G. S. Krishnaswami and S. G. Rajeev, *An interacting parton model for quark and anti-quark distributions in the baryon*, Phys. Lett. B **487**, 125 (2000) [arXiv:hep-ph/0310027].

[8] V. John, G. S. Krishnaswami and S. G. Rajeev, *Parton model from bi-local solitonic picture of the baryon in two-dimensions*, Phys. Lett. B **492**, 63 (2000) [arXiv:hep-th/0310014].

[9] G. S. Krishnaswami and S. G. Rajeev, *A model of interacting partons for hadronic structure functions*, Phys. Lett. B **441**, 429 (1998) [arXiv:hep-ph/9807345].

[10] L. Akant, G. S. Krishnaswami and S. G. Rajeev, *Entropy of operator-valued random variables: A variational principle for large N matrix models*, Int. J. Mod. Phys. A **17**, 2413 (2002) [arXiv:hep-th/0111263].

[11] A. Agarwal, L. Akant, G. S. Krishnaswami and S. G. Rajeev, *Collective potential for large-N Hamiltonian matrix models and free Fisher information*, Int. J. Mod. Phys. A **18**, 917 (2003) [arXiv:hep-th/0207200].

[12] G. S. Krishnaswami, *Multi-matrix loop equations: algebraic & differential structures and an approximation based on deformation quantization*, JHEP **08**, 035 (2006) [arXiv:hep-th/0606224].

[13] L. Akant and G. S. Krishnaswami, *Non-anomalous 'Ward' identities to supplement large-N multi-matrix loop equations for correlations*, JHEP **02**, 073 (2007) [arXiv:hep-th/0611350].

[14] G. S. Krishnaswami, *Variational ansatz for gaussian + Yang-Mills two matrix model compared with Monte-Carlo simulations in 't Hooft limit*, (2003) [arXiv:hep-th/0310110].

[15] G. S. Krishnaswami, *Schwinger-Dyson operators as invariant vector fields on a matrix-model analogue of the group of loops*, J. Math. Phys. **49**, 062303 (2008) [arXiv:0803.0487].

[16] G. S. Krishnaswami, *Schwinger-Dyson operator of Yang-Mills matrix models with ghosts and derivations of the graded shuffle algebra*, J. Phys. A: Math. Theor. **41**, 145402 (2008) [arXiv:0708.3056].

[17] G. S. Krishnaswami, *Phase transition in matrix model with logarithmic action: Toy-model for gluons in baryons*, JHEP **03**, 067 (2006) [arXiv:hep-th/0601216].

[18] G. S. Krishnaswami, $2+1$ *Abelian 'Gauge Theory' Inspired by Ideal Hydrodynamics*, Int. J. Mod. Phys. A **21**, 3771 (2006) [arXiv:hep-th/0507283].

[19] G. S. Krishnaswami, *Possible large-N fixed-points and naturalness for $O(N)$ scalar fields*, J. Phys. A: Math. Theor. **42**, 345403 (2009) [arXiv:0904.4799].

[20] A. Sen, D. Ahalpara, A. Thyagaraja and G. S. Krishnaswami, *A KdV-like advection-dispersion equation with some remarkable properties*, Communications in Nonlinear Science and Numerical Simulation **17**, 4115 (2012) [arXiv:1109.3745].

[21] G. S. Krishnaswami, R. Nityananda, A. Sen, A. Thyagaraja, *Comment on "Spin-Gradient-Driven Light Amplification in a Quantum Plasma"*, Phys. Rev. Lett. **112**, 129501 (2014) [arXiv:1403.0228].

[22] G. S. Krishnaswami, R. Nityananda, A. Sen, A. Thyagaraja, *A critique of recent semi-classical spin-half quantum plasma theories*, Contrib. Plasma Phys. **55**(1), 3 (2015) [arXiv:1407.6865].

[23] M. Marklund and G. Brodin, *Dynamics of spin 1/2 quantum plasmas*, Phys. Rev. Lett., **98**, 025001 (2007).

[24] S. Braun, F. A. Asenjo and S. M. Mahajan, *Spin-Gradient-Driven Light Amplification in a Quantum Plasma*, Phys. Rev. Lett., **109**, 175003 (2012).

[25] G. S. Krishnaswami and S. S. Phatak, *Higgs Mechanism and the Added-Mass Effect*, Proc. R. Soc. A **471**, 20140803 (2015) [arXiv:1407.2689, hep-th].

[26] G. S. Krishnaswami and S. S. Phatak, *The Added Mass Effect and the Higgs Mechanism: How Accelerated Bodies and Elementary Particles Can Gain Inertia*, Resonance **25**(2), 191 (2020); arXiv:2005.04620 [physics.flu-dyn,hep-ph].

[27] G. S. Krishnaswami and S. Sachdev, *Algebra and geometry of Hamilton's quaternions*, Resonance 21(6), 529–544 June (2016); arXiv:1606.03315.

[28] G. B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York (1999).

[29] C. L. Fefferman, *Existence and smoothness of the Navier-Stokes equation*, Official problem description, Millenium Prize Problem, Clay Mathematics Institute (2000).

[30] D. D. Holm, J. E. Marsden and T. S. Ratiu, *Euler-Poincare equations and semidirect products with applications to continuum theories*, Adv. in Math. **137**, 1 (1998).

[31] A. Thyagaraja, *Conservative regularization of ideal hydrodynamics and magnetohydrodynamics*, Physics of Plasmas **17**, 032503 (2010)

[32] G. S. Krishnaswami, S. Sachdev and A. Thyagaraja, *Local conservative regularizations of compressible magnetohydrodynamic and neutral flows*, Phys. Plasmas **23**, 022308 (2016) [arXiv:1602.04323, physics.plasm-ph].

[33] G. S. Krishnaswami, S. Sachdev and A. Thyagaraja *Conservative regularization of compressible flow and ideal magnetohydrodynamics*, (Nov 2016) [arXiv:1510.01606v2, physics.flu-dyn].

[34] G. S. Krishnaswami, S. Sachdev, A. Thyagaraja, *Conservative regularization of compressible dissipationless two-fluid plasmas*, Phys. Plasmas **25**, 022306 (2018), [arXiv:1711.05236, physics.plasm-ph].

[35] G. S. Krishnaswami and H. Senapati, *Curvature and geodesic instabilities in a geometrical approach to the planar three-body problem*, J. Math. Phys. **57**, 102901 (2016) [arXiv:1606.05091, math-ph].

[36] G. S. Krishnaswami and H. Senapati, *An introduction to the classical three-body problem: From periodic solutions to instabilities and chaos*, Resonance, **24**, 1, 87–114 January (2019); arXiv:1901.07289 [nlin-CD].

[37] R. Montgomery, *Hyperbolic pants fit a three-body problem*, Ergodic Theory Dyn. Syst. **25**, 921 (2005).

[38] G. S. Krishnaswami and T. R. Vishnu, *On the Hamiltonian formulation, integrability and algebraic structures of the Rajeev-Ranken model*, J. Phys. Commun. **3**, 025005 (2019), DOI: 10.1088/2399-6528/ab02a9; arXiv:1804.02859 [hep-th].

[39] A. M. Polyakov, *Gauge fields and strings*, Harwood Academic Publishers, Chur (1987).

[40] V. E. Zakharov and A. V. Mikhailov, *Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method*, Zh. Eksp. Teor. Fiz. **74**, 1953 (1978).

[41] C. R. Nappi, *Some properties of an analog of the chiral model*, Phys. Rev. D **21**, 418 (1980).

[42] S. G. Rajeev and E. Ranken, *Highly nonlinear wave solutions in a dual to the chiral model*, Phys. Rev. D **93**, 105016 (2016).

[43] G. S. Krishnaswami and T. R. Vishnu, *An introduction to Lax pairs and the zero curvature representation*, arXiv:2004.05791 [nlin.SI].

[44] G. S. Krishnaswami and T. R. Vishnu, *The idea of a Lax pair - Part I: Conserved quantities for a dynamical system*, Resonance, 25(12), 1705, (2020).

[45] G. S. Krishnaswami and T. R. Vishnu, *The idea of a Lax pair - Part II: Continuum wave equations*, Resonance, 26(2), 257, (2021).

[46] O. Babelon, D. Bernard and M. Talon, *Introduction to classical integrable systems*, Cambridge University Press, Cambridge (2003).

[47] Gutzwiller, M. C., *Moon-Earth-Sun: The oldest three-body problem*, Reviews of Modern Physics, **70**, 589 (1998).

[48] Saari, D G and Xia, Z, *Off to infinity in finite time*, Notices of the AMS, **42**, 538 (1993).

[49] G. S. Krishnaswami and H. Senapati, *Stability and chaos in the classical three rotor problem*, Indian Academy of Sciences Conference Series **2**(1), 139 (2019), arXiv:1810.01317 [nlin-CD].

[50] G. S. Krishnaswami and H. Senapati, *Classical three rotor problem: periodic solutions, stability and chaos*, Chaos **29**(12), 123121 (2019) [Editor's Pick]; DOI: 10.1063/1.5110032; arXiv:1811.05807v2 [nlin-CD].

[51] A. Chenciner and R. Montgomery, *A remarkable periodic solution of the three-body problem in the case of equal masses*, Ann. of Math. Second Series, **152**(3), 881-901 (2000).

[52] G. S. Krishnaswami and T. R. Vishnu, *Invariant tori, action-angle variables, and phase space structure of the Rajeev-Ranken model*, J. Math. Phys. 60, 082902 (2019); arXiv:1906.03141 [nlin.SI].

[53] L. M. Milne-Thomson, *Theoretical hydrodynamics*, 4^{th} Ed., Macmillan, London (1962); Chapt. XVII, p. 528.

[54] B. A. Dubrovin, I. M. Krichever and S. P. Novikov, *Integrable systems. I*, Encyclopaedia of mathematical sciences, Vol 4, Dynamical systems IV, Symplectic geometry and its applications, V. I. Arnold and S. P. Novikov (Eds.), Springer-Verlag, Heidelberg, 173, (1990).

[55] G. S. Krishnaswami and H. Senapati, *Ergodicity, mixing and recurrence in the three rotor problem*, Chaos **30**(4), 043112 (2020) [Editor's Pick]; arXiv:1910.04455 [nlin-CD] (Oct 2019).

[56] G. S. Krishnaswami, S. S. Phatak, S. Sachdev and A. Thyagaraja, *Nonlinear dispersive regularization of inviscid gas dynamics*, AIP Advances **10**(2), 025303 (2020), arXiv:1910.07836 [physics.flu-dyn].

[57] G. S. Krishnaswami and T. R. Vishnu, *Quantum Rajeev-Ranken model as an anharmonic oscillator*, J. Math. Phys. **63**, 032101 (2022), arXiv:2111.03858 [math-ph, hep-th].

[58] J. Zinn-Justin and U. D. Jentschura, *Multi-Instantons and Exact Results-I: Conjectures, WKB Expansions and Instanton interactions*, Ann. Phys. **313**, 197 (2005).

[59] J. Zinn-Justin and U. D. Jentschura, *Multi-Instantons and Exact Results-II: Specific Cases, Higher-Order Effects and Numerical Calculations*, Ann. Phys. **313**, 269 (2005).