

Non-anomalous ‘Ward’ identities to supplement large- N multi-matrix loop equations for correlations

Levent Akant

Feza Gürsey Institute,

Emek Mah.Rasathane Yolu No:68, 34684, Çengelköy, Istanbul, Turkey

E-mail: akant@gursey.gov.tr

Govind S. Krishnaswami

Institute for Theoretical Physics & Spinoza Institute

Utrecht University, Postbus 80.195, 3508 TD, Utrecht, The Netherlands

E-mail: g.s.krishnaswami@phys.uu.nl

ABSTRACT: This work concerns single-trace correlations of Euclidean multi-matrix models. In the large- N limit we show that Schwinger-Dyson equations (SDE) imply loop equations (LE) and non-anomalous Ward identities (WI). LE are associated to generic infinitesimal changes of matrix variables (vector fields). WI correspond to vector fields preserving measure and action. The former are analogous to Makeenko-Migdal equations and the latter to Slavnov-Taylor identities. LE correspond to leading large- N SDE. WI correspond to $1/N^2$ suppressed SDE. But they become leading equations since LE for non-anomalous vector fields are vacuous. We show that symmetries at $N = \infty$ persist at finite N , preventing mixing with multi-trace correlations. For 1 matrix, there are no non-anomalous infinitesimal symmetries. For 2 or more matrices, measure preserving vector fields form an infinite dimensional graded Lie algebra, and non-anomalous action preserving ones a subalgebra. For Gaussian, Chern-Simons and Yang-Mills models we identify up to cubic non-anomalous vector fields, though they can be arbitrarily non-linear. WI are homogeneous linear equations. We use them with the LE to determine some correlations of these models. WI alleviate the underdeterminacy of LE. Non-anomalous symmetries give a naturalness-type explanation for why several linear combinations of correlations in these models vanish.

KEYWORDS: $1/N$ Expansion, BRST Symmetry, M(atr)ix Theories, Matrix Models.

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1. Introduction

Hermitian multi-matrix models are quantum systems where the dynamical variables are a set of $N \times N$ hermitian matrices. Observables must be basis independent, i.e. invariant under the global adjoint action of $U(N)$ on the matrices. Expectation values of observables are determined by an average over all matrix elements with respect to a Boltzmann weight specified by an action. Matrix models simplify in 't Hooft's large- N limit, since fluctuations in $U(N)$ -invariant observables are small in this limit.

Multi-matrix models are simplified models for the dynamics of gauge fields in Yang-Mills theory. It is a fundamental and challenging problem to determine the free energy and correlation functions of multi-matrix models, and elucidate the mathematical framework needed to study them. Multi-matrix models are much harder to understand than single-matrix models, but also have a much richer structure. Large- N matrix models and more generally large- N gauge theories have been studied ever since their relevance as an approximation to the theory of strong interactions was pointed out by 't Hooft in the mid 1970s [1–3]. Important progress in obtaining the loop equations of Yang-Mills theory and study of the large- N limit was made in the late 1970s and early 1980s by Migdal and Makeenko [4], Cvitanovic [5], Yaffe [6], Jevicki and Sakita [7] and others [8]. The subject was applied to random surface theory, 2d string theory and the matrix approach to M-theory in the 1990s. Meanwhile, there has been a steady stream of developments in matrix models of which we cite a few examples. These include their connections to non-commutative probability theory [9–14], the study of multi-matrix symmetry algebras and their connections to spin chains [15, 16], exact solutions [17] and their relation to CFT [18] and algebraic geometry and detailed studies of the loop equations [19, 20]. Much of the existing literature deals with 1-matrix models or exact solutions for specific observables of carefully chosen multi-matrix models. We hope to complement this by developing a framework and methods that apply to general multi-matrix models.

Throughout physics, we exploit symmetries to simplify dynamical equations by reducing the number of unknowns. The quantum dynamical equations of a large- N multi-matrix model are the loop equations¹ for single-trace correlations. These correlations are analogs of gluon and ghost correlation functions of Yang-Mills theory. Here, we develop a general framework to find non-anomalous infinitesimal symmetries of multi-matrix models (i.e. those that preserve both action and measure). These symmetries are used to infer Ward identities, which supplement the loop equations to determine correlations. These

¹The name loop equations is used because these equations are analogous to the Makeenko-Migdal equations of Yang-Mills theory, which were formulated for the Wilson loops. The word loop has nothing to do with loops in Feynman diagrams. Another name for these equations is factorized Schwinger-Dyson equations.

non-anomalous symmetries and Ward identities can be regarded as finite dimensional analogues of BRST invariance and Slavnov-Taylor identities of Yang-Mills theory. The ideas are illustrated with examples from 2 and 3-matrix models.

To motivate this work, we explain how we came to think along these lines. We were trying to solve the loop equations (LE) to determine large- N single trace correlations of some specific multi-matrix models [20]. Single-trace correlations are the basic objects of interest, since multi-trace correlations factorize into products of single-trace ones in the large- N limit. The LE state the invariance of the partition function under infinitesimal but non-linear changes of integration variables, in the large- N limit. They relate a change in action to a change in measure. Such infinitesimal changes of integration variables can be regarded as vector fields. A priori, there is one LE for each such vector field. In many non-gaussian cases, we found the LE were underdetermined. Moreover, this underdeterminacy seemed related to the fact that for several changes of variable, the LE were vacuous. In other words, for some vector fields, both the change in action and change in measure simultaneously vanished in the large- N limit. We were looking for additional equations to supplement the LE and alleviate their underdeterminacy.

Now, the LE can be regarded as the large- N limit of the finite- N Schwinger-Dyson equations (SDE). The SDE are conditions for the invariance of matrix integrals for multi-trace correlations under infinitesimal non-linear changes of integration variables. They relate a change in action to a sum of a change in measure and change in observable. While the first two are usually of order N^0 , the latter² is usually of order N^{-2} . So naively, in the large- N limit, the latter drops out and using factorization, we get back the LE.³ However, in the special case where the vector field is a symmetry of both action and measure in the large- N limit, the naive large- N limit of the SDE is vacuous and one must go to the next order in $1/N^2$ to get the leading condition. For generic vector fields, this $\mathcal{O}(1/N^2)$ SDE would not be an equation for the single-trace correlations alone, since it involves $\mathcal{O}(1/N^2)$ corrections to the factorized result for multi-trace correlations. However, remarkably, we show that if a vector field is a symmetry of the action in the large- N limit, then it is also a symmetry⁴ at each order in $1/N^2$. The same also holds for symmetries of measure. The simple reason is that there are more independent variables as N grows, and so more conditions on a vector field to be a symmetry as N increases. Non-anomalous vector fields define simultaneous symmetries of action and measure in the large- N limit. Thus, for non-anomalous vector fields, the change in action and measure terms drop out of the $\mathcal{O}(1/N^2)$ SDE, which then becomes a condition for invariance of the expectation value of the observable in the large- N limit, schematically $L_v G_{i_1 \dots i_n} = 0$. This latter condition only involves single-trace correlations $G_{i_1 \dots i_n}$, and is what we call a non-anomalous Ward identity (WI). WI are associated to vector fields v that leave both action and measure invariant in the large- N limit. L_v is the Lie derivative along v . It is precisely for such vector fields that

²There is no $\mathcal{O}(1/N)$ contribution in a matrix model, i.e. in the absence of quarks or N -vectors.

³This is why LE are also called factorized Schwinger-Dyson equations. The name Virasoro constraints is also used, especially in applications to string theory.

⁴There are potentially more symmetries at finite- N than at large- N , so the converse is almost certainly false.

the LE are vacuous. We show that such vector fields form an infinite dimensional graded Lie algebra. Thus, as $N \rightarrow \infty$, the SDE imply not just the LE, which are associated to generic vector fields, but also WI, which are associated to non-anomalous vector fields. The latter are easily overlooked in a naive passage to the large- N limit. Naively, the WI appear to be ‘universal’, i.e. independent of the matrix model action, but this is not really true. Whether or not a WI holds is determined by whether the corresponding vector field is a simultaneous symmetry of both action and measure.

In retrospect, these non-anomalous symmetries and WI are not unexpected. In Yang-Mills theory, non-anomalous symmetries include Poincare and BRST invariance. WI for the latter are Slavnov-Taylor identities. Our non-anomalous WI share with the Slavnov-Taylor identities the structural similarity of being homogeneous linear equations for correlations. Just like our WI, the Slavnov-Taylor identities seem independent of the gauge-fixed Yang-Mills action, until one realizes they hold only because the action and measure are BRST invariant. In Yang-Mills theory, while Poincare transformations act linearly on the fields, BRST transformations are quadratically non-linear. For specific matrix models, we find non-anomalous symmetries that are linear, quadratic ($n = 2$) and cubic ($n = 3$); there is no limit to the possible non-linearity of such symmetries. Moreover, for $n > 1$, some rank- $n+1$ non-anomalous symmetries can be obtained via the Lie brackets of rank- n symmetries. This is reminiscent of how Poisson brackets of conserved charges (if non-vanishing), give higher conserved charges in integrable models.

We find a significant difference between 1-matrix models and multi-matrix models. The measure for a single matrix in the large- N limit admits only one continuous symmetry, i.e. translations of the matrix. Translations, however, are not a symmetry of any non-trivial 1-matrix action. Thus, non-trivial 1-matrix models have no non-anomalous WI. Interestingly, we find that the measures for multi-matrix models allow for large classes of symmetries, some of which may also be symmetries of a given action.

In practice, once a non-anomalous symmetry of a model is known, it is easier to first solve the resulting WI and then consider the LE. The WI, being homogeneous linear equations, force several correlations or linear combinations thereof to vanish. This simplifies analysis of the LE, which are mildly non-linear. However, we caution that some WI may contain the same information as contained in the LE, while others may provide new conditions.

We emphasize that the techniques and results of this paper are exact. They do not involve any approximation beyond the passage to $N = \infty$. Our methods apply to single-trace correlations of generic hermitian multi-matrix models with polynomial actions. They are not special to any subclass of actions or correlations. Of course, the non-anomalous symmetries and WI will depend on which model we consider. One lesson we learned is that though it is a bit laborious, it is possible to solve the LE and WI of large- N multi-matrix models to determine exact correlations, starting from the lowest rank ones.

Finally, our derivation of the SDE, LE and WI makes use of the matrix integral representation for correlations. In cases where these integrals converge, we expect the equations to be rigorously valid. When the matrix integrals diverge, the SDE, LE and WI are only formal statements and their consistency is not guaranteed by our work. Indeed, we seem

to find an example where formal use of these equations for a model whose matrix integrals diverge, leads to inconsistencies. We are yet to understand the deeper significance of this.

We can give another interpretation of our results. Suppose one were to calculate single trace correlations of a large- N multi-matrix model. Then in many cases one would find there are several linear combinations of correlations that vanish. One might look for a naturalness-type explanation for this i.e., a non-anomalous symmetry that forces those linear combinations to vanish. In some cases, there is a discrete symmetry (such as $A \rightarrow -A$ for correlations of odd order for an even action) that does the job. The results of this paper may be regarded as the discovery of several new continuous non-anomalous symmetries of multi-matrix models. For example, $\delta A_1 = a[A_1, [A_2, A_1]]$ and $\delta A_2 = a[[A_1, A_2], A_2]$ is a non-anomalous symmetry of the gaussian+YM 2-matrix model for all real a . Such symmetries lead to WI, which ensure that the quantities in question vanish.

Organization and summary of results. In section 2 we determine the SDE of hermitian multi-matrix models. We show that in the large- N limit, they lead to LE supplemented by WI. It is asserted that the WI are to be imposed for every vector field that is a simultaneous symmetry of action and measure in the large- N limit. The proof of validity of the WI is completed in sections 3.3 and 4.1. In section 2.1 we explain why the WI trivialize for a 1-matrix model. Section 2.2 exhibits that multi-matrix LE are often underdetermined and this motivates the need for additional equations to determine correlations. WI potentially alleviate the underdeterminacy of LE. In section 3 we characterize measure preserving vector fields of multi-matrix models in the large- N limit. We show that they form an infinite dimensional Lie algebra (section 3.4). Measure preserving transformations of 2- and 3-matrix models are given in section 3.5. We work out the linear and quadratic non-anomalous symmetries of Gaussian, Chern-Simons, Yang-Mills and Gaussian+Yang-Mills multi-matrix models in sections 4.3 and 4.4 and also construct some cubic symmetries via Lie brackets of quadratic ones. In section 5 we explicitly give the LE and non-anomalous WI for the Gaussian, Gaussian+YM and Chern-Simons models. We show that several correlations vanish, determine some non-vanishing correlations, and also obtain non-trivial relations among other non-vanishing correlations. In section 5.4 we show that formal use of LE and WI for a model whose matrix integrals do not converge potentially leads to inconsistencies. Some outstanding questions are collected in 6. In appendix A we give an alternate derivation of the SDE that preserves hermiticity of matrices. In appendix A.1 we consider some other possible changes of variables in an unsuccessful search for equations satisfied by the $N = \infty$ single trace correlations, over and above the LE and WI. In appendix C we argue that the WI by themselves (without use of LE) cannot determine all correlations of a non-trivial model. In appendix D we quote a useful formula for the number of cyclically symmetric tensors of rank- n in a Λ -matrix model.

2. Schwinger-Dyson equations and Ward identities

We consider a bosonic⁵ Euclidean matrix model with Λ random hermitian matrices $A_i, i =$

⁵It is possible to extend these methods to models with gluon and ghost fields, such as gauge fixed Yang-

$1, 2, \dots, \Lambda$. The action $\text{tr } S(A) = \text{tr} \sum_{|J| \leq m} S^J A_J$ is taken to be a polynomial. Due to the trace, only the cyclic projections of the coupling tensors S^I contribute to the action. Multi-indices are denoted by capital letters, for example, $I = i_1 i_2 \dots i_n$. Repeated lower and upper indices as summed and $|I|$ denotes the length of the multi-index.

Observables are functions of A_i that are invariant under the global adjoint action $A_i \rightarrow U A_i U^\dagger$ of $U \in U(N)$. An important class of such functions are the trace invariants $\Phi_I = \frac{1}{N} \text{tr } A_I$. The partition function and multi-trace correlations are defined as

$$Z = \int \prod_{j=1}^{\Lambda} dA_j e^{-N \text{tr } S(A)} \quad \text{and} \quad \langle \Phi_{K_1} \dots \Phi_{K_n} \rangle = \frac{1}{Z} \int \prod_j dA_j e^{-N \text{tr } S(A)} \Phi_{K_1} \dots \Phi_{K_n}. \quad (2.1)$$

$\langle \Phi_{K_1} \dots \Phi_{K_n} \rangle$ is symmetric under interchange of any pair from K_1, \dots, K_n . It is cyclically symmetric in each K_i separately. $\langle \Phi_{K_1} \dots \Phi_{K_n} \rangle$ may be expanded in inverse powers of N^2

$$\langle \Phi_{K_1} \dots \Phi_{K_n} \rangle = G_{K_1; K_2; \dots; K_n}^{(0)} + \frac{1}{N^2} G_{K_1; K_2; \dots; K_n}^{(2)} + \frac{1}{N^4} G_{K_1; K_2; \dots; K_n}^{(4)} + \dots \quad (2.2)$$

The coefficient of N^{-2h} can be regarded as a sum of Feynman diagrams that can be drawn on a Riemann surface with h handles and n disks cut out.⁶ The perimeter of each disk is associated to one of the inserted K_i 's. In particular, for $h = 0$, these are planar diagrams. Each of the $G_{K_1; \dots; K_n}^{(2h)}$ is symmetric in the multi-indices K_1, \dots, K_n . Factorization of multi-trace correlations in the large- N limit [22] means that $G^{(0)}$ can be written as a product of single trace correlations

$$\lim_{N \rightarrow \infty} \langle \Phi_{K_1} \dots \Phi_{K_n} \rangle = G_{K_1; K_2; \dots; K_n}^{(0)} = G_{K_1} \dots G_{K_n}, \quad \text{where } G_K = \lim_{N \rightarrow \infty} \langle \frac{\text{tr}}{N} \Phi_K \rangle. \quad (2.3)$$

The single-trace gluon correlations G_K are cyclically symmetric in K and satisfy the hermiticity condition $G_K^* = G_{\bar{K}}$ provided S^I also satisfy this property. Here \bar{K} is the word K with order of indices reversed. G_K will also be referred to as moments, they are the moments of a non-commutative probability distribution [13] when $\Lambda > 1$. The rank of G_K is defined as $|K|$.

To determine correlations, we derive Schwinger-Dyson equations (SDE), conditions for invariance of matrix integrals for $\langle \Phi_{K_1} \dots \Phi_{K_n} \rangle$ under infinitesimal non-linear changes of variables⁷

$$[A_i]_b^a \rightarrow [A'_i]_b^a = [A_i]_b^a + v_i^I [A_I]_b^a, \quad v_i^I \text{ infinitesimal real parameters for } |I| \geq 0. \quad (2.4)$$

These include the BRST-type of transformations used to derive the Slavnov-Taylor identities of gauge-fixed Yang-Mills theory. For example, in Lorentz gauge the BRST transformations are infinitesimal quadratic transformations (λ is an infinitesimal anti-commuting

Mills theory. In this case the A_i would include hermitian matrices as well as matrices with grassmann entries.

⁶A pictorial representation of (2.2) for a four-point correlation would resemble figure 1.8 on page 31 of ref. [21]

⁷These changes of variable don't always preserve hermiticity of A_i . Under a change of integration variable, the integrand, measure as well as domain of integration may change, but the value of the integral is unchanged. It is possible to derive the SDE by making hermitian changes of variable, see appendix A.

parameter),

$$A_\mu \rightarrow A_\mu + \lambda \partial_\mu c + \lambda [A_\mu, c], \quad c \rightarrow c + \lambda [c, c]_+, \quad \bar{c} \rightarrow \bar{c} + \lambda \partial^\mu A_\mu. \quad (2.5)$$

To calculate the effect of (2.4) on the integral (2.1) defining $\langle \Phi_{K_1} \cdots \Phi_{K_n} \rangle$ we need⁸ the infinitesimal change in action, measure and the inserted observable Φ_K

$$\begin{aligned} e^{-N \text{tr } S^J A_J} &\mapsto e^{-N \text{tr } S^J A_J} (1 - N^2 v_i^I S^{J_1 i J_2} \Phi_{J_1 I J_2}) + \mathcal{O}(v^2), \\ \det \left(\frac{\partial [A_i]_b^a}{\partial [A_j]_d^c} \right) &= 1 + N^2 v_i^I \delta_I^{I_1 i I_2} \Phi_{I_1} \Phi_{I_2} + \mathcal{O}(v^2) \\ \Phi_K &\mapsto \Phi_K + \delta_K^{L i M} v_i^I \Phi_{L I M}. \end{aligned} \quad (2.6)$$

The conditions for invariance of $\langle \Phi_{K_1} \cdots \Phi_{K_n} \rangle$ to linear order in v_i^I are the finite N SDE⁹

$$\begin{aligned} v_i^I S^{J_1 i J_2} \langle \Phi_{J_1 I J_2} \Phi_{K_1} \cdots \Phi_{K_n} \rangle &= v_i^I \delta_I^{I_1 i I_2} \langle \Phi_{I_1} \Phi_{I_2} \Phi_{K_1} \cdots \Phi_{K_n} \rangle \\ &+ \frac{1}{N^2} \sum_{p=1}^n \delta_{K_p}^{L p i M_p} v_i^I \langle \Phi_{K_1} \cdots \Phi_{K_{p-1}} \Phi_{L p I M_p} \Phi_{K_{p+1}} \cdots \Phi_{K_n} \rangle. \end{aligned} \quad (2.7)$$

There is a priori one such SDE for each vector field v_i^I and each $n = 0, 1, 2, \dots$, where n is the number of insertions. The l.h.s. is the expectation value of the change in action (along with Φ_K insertions). The first term on the r.h.s. is the expectation value of the change in measure (with Φ_K insertions) and the second term on the r.h.s. is the expectation value of the change in insertions Φ_K . So far, we have not made any approximations. Let us now expand the multi-trace correlations according to (2.2) and the factorization formula (2.3). The SDE at order $1/N^0$ are the large- N factorized SDE(fSDE) or loop equations(LE). They only involve the large- N limits of single trace correlations G_J

$$v_i^I S^{J_1 i J_2} G_{J_1 I J_2} = v_i^I \delta_I^{I_1 i I_2} G_{I_1} G_{I_2} = v_i^I \eta_I^i. \quad (2.8)$$

Using the notation $L_v = v_i^I L_I^i$ for the vector fields associated to the infinitesimal changes (2.4), these LE may be written $L_v S^J G_J = v_i^I \eta_I^i$ where $\eta_I^i = \delta_I^{I_1 i I_2} G_{I_1} G_{I_2}$. Here the action of the vector fields on the moments is $L_I^i G_J = \delta_J^{J_1 i J_2} G_{J_1 I J_2}$ and extends by linearity and the Leibnitz rule to polynomials in the G_J . Moreover, the Lie bracket of two such vector fields is

$$[L_I^i, L_J^j] = \delta_J^{J_1 i J_2} L_{J_1 I J_2}^j - \delta_I^{I_1 j I_2} L_{I_1 J I_2}^i \quad (2.9)$$

or $[L_u, L_v] = L_w$ where $L_w = w_k^K L_K^k$ and

$$w_k^K = \sum_{K=K_1 I K_2} (u_i^I v_k^{K_1 i K_2} - u_k^{K_1 i K_2} v_i^I). \quad (2.10)$$

⁸ $\langle \Phi_{K_1} \cdots \Phi_{K_n} \rangle$ is the quotient of two integrals. Here we make a change of variable in the numerator but not the integral for Z in the denominator. We could change variables in each, but this doesn't give new equations.

⁹For the purpose of deriving the loop equations and Ward identities, it is adequate to start with SDE for single trace correlations, so the reader could set $n = 1$ in a first reading.

At $\mathcal{O}(1/N^2)$, the SDE (one for each v and $n \geq 0$) involve the G_J as well as the $G_K^{(2)}$'s

$$v_i^I S^{J_1 i J_2} G_{J_1 I J_2; K_1; \dots; K_n}^{(2)} = v_i^I \delta_I^{J_1 i J_2} G_{J_1; J_2; K_1; \dots; K_n}^{(2)} + v_i^I \sum_{p=1}^n \delta_{K_p}^{L_p i M_p} G_{K_1} \cdots G_{K_{p-1}} G_{L_p I M_p} G_{K_{p+1}} \cdots G_{K_n}. \quad (2.11)$$

We could continue listing the SDE at each order in $1/N^2$, but we refrain from doing so since they no longer involve the single trace correlations G_J . G_J are the primary objects of interest in the large- N limit and our goal is to determine them for a given action $S(A)$. It would be ideal if we could uniquely determine them by solving the LE (2.8). Unfortunately, as was demonstrated in [20] (and reviewed in 2.2), this is not possible for many interesting actions $S(A)$, since the LE are underdetermined. One source of this problem was that there are vector fields v for which both l.h.s. and r.h.s. of (2.8) identically vanish for all¹⁰ G_J , so that the LE for those v are vacuous. Such vector fields are associated to simultaneous symmetries of the action and measure in the large- N limit. We will call such symmetries *non-anomalous symmetries* of the large- N limit.¹¹ Of course, in general, v need not be a symmetry of either action or measure.

We would like to use the $\mathcal{O}(1/N^2)$ SDE (2.11) to determine the G_J that the LE do not fix. In principle, (2.11) are always valid. However, (2.11) involve the $G^{(2)}$'s which we do not wish to determine (and most likely cannot, without also involving the $\mathcal{O}(\frac{1}{N^4})$ SDE and so on). Thus, we would like to use the subleading SDE (2.11) only for those v for which the LE (2.8) are vacuous. But even these equations would seem to involve the pesky $G^{(2)}$'s. Fortunately, a remarkable stroke of good fortune comes to our rescue. Suppose a vector field v is such that it is a simultaneous symmetry of the action and measure at $N = \infty$, i.e. $v_i^I S^{J_1 i J_2} G_{J_1 I J_2} = 0 = v_i^I \eta_I^i$ for all G_J . Then we will show (sections 3.3 and 4.1) that v is also a simultaneous symmetry at finite N , and thence a symmetry at each order in $1/N^2$:

$$v_i^I \delta_I^{J_1 i J_2} \langle \Phi_{J_1} \Phi_{J_2} \Phi_{K_1} \cdots \Phi_{K_n} \rangle = 0 = v_i^I S^{J_1 i J_2} \langle \Phi_{J_1 I J_2} \Phi_{K_1} \cdots \Phi_{K_n} \rangle \quad \forall \quad n = 0, 1, 2, \dots$$

$$\text{if} \quad v_i^I S^{J_1 i J_2} G_{J_1 I J_2} = 0 = v_i^I \eta_I^i \quad \forall \quad G_J. \quad (2.12)$$

Thus, the terms involving the $G^{(2)}$'s in (2.11) would identically vanish for such v and (2.11) would reduce to a set of ‘Ward’ identities

$$v_i^I \sum_{p=1}^n \delta_{K_p}^{L_p i M_p} G_{K_1} \cdots G_{K_{p-1}} G_{L_p I M_p} G_{K_{p+1}} \cdots G_{K_n} = 0 \quad \forall \quad K_j, \quad n \geq 1. \quad (2.13)$$

We call these ‘Ward’ identities (WI) since they are analogues of the Ward-Takahashi-Slavnov-Taylor identities of Yang-Mills theory. The latter are a consequence of BRST changes of variable (2.5) in functional integrals. Recall that the BRST transformations are also non-anomalous in the sense that they leave both the gauge fixed Yang-Mills action and measure invariant.

¹⁰When we say ‘for all G_J ’, we really mean ‘for all cyclic and hermitian G_J ’.

¹¹Anomalous symmetries leave the action invariant but not the measure.

These Ward identities can be written more compactly as

$$L_v(G_{K_1} \cdots G_{K_n}) = 0 \quad \forall K_p, n \geq 1. \quad (2.14)$$

Those for $n > 1$ follow from those for $n = 1$ and the Leibnitz rule. So the WI may be taken as

$$L_v G_K = 0 \quad \forall K \quad \text{provided } v \text{ is such that } v_i^I S^{J_1 i J_2} G_{J_1 I J_2} = 0 = v_i^I \eta_I^i \quad \forall G_J. \quad (2.15)$$

It is satisfying to see that WI, which arise as a consequence of non-anomalous symmetries, may be regarded as a special case of the more general concept of Schwinger-Dyson equations. This is really a statement about quantum field theory in general, though we are discussing matrix models here. Traditionally [23] Ward-like identities are not regarded as related to Schwinger-Dyson equations in this manner. As pointed out in [20] and reviewed in section 2.2, the factorized large- N SDE are often insufficient to determine the correlations of a matrix model. However, when the fSDE are supplemented by the above WI, it becomes possible to determine many (and possibly all) the correlations, as we will see in later sections.

2.1 The case of a single matrix

For a 1-matrix model with action $\text{tr } S(A) = \text{tr } \sum_{1 \leq n \leq m} S_n A^n$, we use the changes of variable $L_v : A \rightarrow A + \sum_{n \geq -1} v_n A^{n+1}$. A convenient basis is $L_n A = A^{n+1}$, $n = -1, 0, 1, \dots$. These are familiar from the Lie algebra of polynomial vector fields on the real line, $L_n = x^{n+1} \frac{\partial}{\partial x}$, $n = -1, 0, 1, \dots$. Their Lie bracket is $[L_m, L_n] = (n - m)L_{m+n}$. Equivalently,

$$L_u = \sum_{n \geq -1} u_n L_n \quad \text{with} \quad [L_u, L_v] = L_w \quad \text{where} \quad w_k = \sum_{\substack{m+n=k \\ m, n \geq -1}} (n - m) u_m v_n. \quad (2.16)$$

Their action on the moments $G_n = \lim_{N \rightarrow \infty} \langle \Phi_n \rangle$ is $L_k G_n = n G_{k+n}$. Here $\Phi_n = \frac{\text{tr}}{N} A^n$. The moments are real by hermiticity of A . The LE are

$$\sum_{k \geq -1} v_k \sum_{n=0}^m n S_n G_{k+n} = \sum_{k \geq -1} v_k \eta_k = \sum_{k \geq -1} v_k \sum_{\substack{p+q=k \\ p, q \geq 0}} G_p G_q. \quad (2.17)$$

The l.h.s. is the expectation value of change in action $\sum_{k \geq -1} v_k L_k \sum_{1 \leq n \leq m} S_n G_n$ while the r.h.s. is the expectation value of the infinitesimal change in measure in the large- N limit.

The only vector fields for which the $N = \infty$ expectation value of the change in measure vanishes, are translations $A \rightarrow A + v_{-1} \mathbf{1}$. To see this note that the change in measure term is

$$\sum_{k \geq -1} v_k \sum_{\substack{p+q=k \\ p, q \geq 0}} G_p G_q = v_0 + 2v_1 G_1 + v_2 (2G_2 + G_1^2) + \dots \quad (2.18)$$

If this is to vanish for arbitrary¹² real G_1, G_2, \dots , then we must have $v_0 = v_1 = v_2 = \dots = 0$. Thus only v_{-1} can be non-vanishing, which corresponds to a translation.

The only action for which translations are a symmetry in the large- N limit is the trivial action, $S(A) = \text{constant}$: the expectation value of change in action under a translation is

$$\sum_{k \geq -1} v_k \sum_{n=0}^m n S_n G_{n+k} = v_{-1} \sum_{n=0}^m n S_n G_{k+n} = v_{-1} (S_1 + 2S_2 G_1 + 3S_3 G_2 + \dots + m S_m G_{m-1}). \tag{2.19}$$

If this must vanish $\forall G_n$, we must have $S_1 = S_2 = \dots = S_m = 0$, i.e. a trivial action. Thus, for a 1-matrix model, we have no infinitesimal simultaneous symmetries of action and measure in the large- N limit. Consequently, there are no WI to supplement the LE with.

This leaves a small mystery for 1-matrix models. As discussed in [20], the LE (2.17) of a 1-matrix model are underdetermined. They do not fix G_1, G_2, \dots, G_{m-2} . The higher moments are fixed in terms of these by the LE. How are the first few moments to be determined if there are no WI to supplement the LE? Of course, for a 1-matrix model, there are alternative techniques such as solving the integral equation for the eigenvalue density [2]. For multi-matrix models, the LE are often more severely underdetermined (there are an infinite number of moments that are not fixed). Remarkably, for multi-matrix models, where no alternative systematic method of solution exists, the WI *do* alleviate the underdeterminacy of the LE (section 5).

2.2 Underdeterminacy of multi-matrix loop equations

The multi-matrix LE (2.8), can also be written as $S^{J_1 i J_2} G_{J_1 I J_2} = \delta_I^{I_1 I_2} G_{I_1} G_{I_2}$ for each i and I . This form, where we take the monomial basis L_I^i for vector fields L_v is convenient for our current discussion. In general, these LE are underdetermined, as found in section 2.2 of ref. [20]. Part of the reason for this underdeterminacy is the presence of non-anomalous symmetries of action and measure. First, we establish that the LE for given I, i can be regarded as a system of inhomogeneous linear equations for higher rank correlations with lower rank ones possibly appearing non-linearly. From the LE, it is clear that if there are any correlations appearing on the l.h.s., they will be of a higher rank than the ones on the r.h.s. . More precisely, suppose the action is an m^{th} order polynomial (i.e. there is a non-vanishing coupling tensor S^K with $|K| = m$). l.h.s. of the LE for given I, i (if it is non-trivial¹³), involves correlations only linearly and with a rank between $|I|$ and $|I| + m - 1$, while the highest rank correlation on the r.h.s. has rank $|I| - 1$. Even if the l.h.s. vanishes, the highest rank correlation in the LE still appears linearly, but now on the r.h.s., and

¹²The only constraints on the moments are that they be real and satisfy the moment inequalities, i.e. that the Hankel matrix $g_{i,j} = G_{i+j}$ be a positive matrix.

¹³Even if there is an $S^K \neq 0$ with $|K| = m$, it may still happen that for some choice of I and i , the coefficients of all correlations of rank $|I| + m - 1$ on the l.h.s. of the LE vanish. An example is the gaussian+YM 2-matrix model LE with $m = 4$ and empty I , given later in this section. Thus, it is not true in general that the maximal rank correlation appearing in a LE has rank $|I| + m - 1$. This possibility, which is special to multi-matrix models and has no analogue for 1-matrix models was overlooked in ref. [20].

has rank $|I| - 1$. However, in many cases, we find that this system of linear equations is inadequate to determine all G_I .

Let us illustrate this with a Gaussian + Yang-Mills 2-matrix model $\text{tr } S(A) = \text{tr} [\frac{m}{2}(A_1^2 + A_2^2) - \frac{1}{2\alpha}[A_1, A_2]^2]$. The matrix integrals for this model converge, and the correlations make rigorous sense and could for instance be measured numerically. So it makes sense to try to find them by solving the LE. In this case, the cyclically symmetric coupling tensors are

$$S^{11} = S^{22} = \frac{m}{2}, S^{1122} = S^{2112} = S^{2211} = S^{1221} = 1/(4\alpha) \text{ and } S^{1212} = S^{2121} = -1/(2\alpha). \quad (2.20)$$

The LE are

$$\begin{aligned} i = 1 : mG_{I1} - \alpha^{-1}(2G_{I212} - G_{I221} - G_{I122}) &= \delta_I^{I_1 I_2} G_{I_1} G_{I_2} \text{ and} \\ i = 2 : mG_{I2} - \alpha^{-1}(2G_{I121} - G_{I112} - G_{I211}) &= \delta_I^{I_1 I_2} G_{I_1} G_{I_2}. \end{aligned} \quad (2.21)$$

For $I = \emptyset$, the LE say¹⁴ $G_1 = G_2 = 0$. The LE for $|I| = 1$ relate 2- and 4-point correlations:

$$\begin{aligned} I = 1 : mG_{11} - \frac{1}{\alpha}(2G_{1212} - 2G_{1221}) &= 1 \text{ \& } mG_{12} - \frac{1}{\alpha}(2G_{1121} - G_{1112} - G_{1211}) = 0 \\ I = 2 : mG_{21} - \frac{1}{\alpha}(2G_{2212} - G_{2221} - G_{2122}) &= 0 \text{ \& } mG_{22} - \frac{1}{\alpha}(2G_{1212} - 2G_{1221}) = 1. \end{aligned} \quad (2.22)$$

They give the conditions $G_{12} = G_{21} = 0, G_{11} = G_{22}$ and $mG_{11} = 1 + \frac{1}{\alpha}(2G_{1212} - 2G_{1122})$. They do not determine G_{11} and give only one relation among the 6 independent rank-4 moments. LE with $|I| = 2$ relate 3- and 5-point correlations (since we already found $G_i = 0$.)

$$\begin{aligned} I = 11 : mG_{111} - \frac{1}{\alpha}(2G_{11212} - 2G_{11122}) &= 0 \text{ \& } mG_{112} = 0. \\ I = 12 : mG_{121} - \frac{1}{\alpha}(G_{12122} - G_{11222}) &= 0 \text{ \& } mG_{122} - \frac{1}{\alpha}(G_{11212} - G_{11122}) = 0 \\ I = 21 : mG_{211} - \frac{1}{\alpha}(G_{12122} - G_{11222}) &= 0 \text{ \& } mG_{212} - \frac{1}{\alpha}(G_{11212} - G_{11122}) = 0 \\ I = 22 : mG_{221} = 0 \text{ \& } mG_{222} - \frac{1}{\alpha}(2G_{12122} - 2G_{11222}) &= 0. \end{aligned} \quad (2.23)$$

The $|I| = 2$ LE imply that all G_{ijk} vanish and give two relations among the 8 independent¹⁵ 5th rank moments, $G_{12122} = G_{11222}$ and $G_{11212} = G_{11122}$. The $|I| = 3$ LE relate 2- 4- and

¹⁴This example illustrates that even in a quartic model ($m=4$), the LE may determine correlations of rank $m - 2 = 2$ or less.

¹⁵ $c(n, \Lambda)$ denotes the dimension of the space of cyclically symmetric hermitian tensors of rank n in a Λ matrix model. A formula for $c(n, \Lambda)$ is given in appendix D.

6-point moments (we omit those equations that contain no new information)

$$\begin{aligned}
 I = 111 : \quad & mG_{1111} - \frac{1}{\alpha}(2G_{111212} - 2G_{111122}) = 2G_{11} \quad \text{and} \quad mG_{1112} = 0 \\
 I = 112 : \quad & 2G_{112212} = G_{111222} + G_{112122} \\
 & mG_{1122} - \frac{1}{\alpha}(2G_{111212} - G_{112112} - G_{111122}) = G_{11} \\
 I = 121 : \quad & 2G_{121212} = G_{112122} + G_{112212} \\
 & mG_{1212} = \frac{2}{\alpha}(G_{112112} - G_{111212}) \\
 I = 211 : \quad & 2G_{112122} = G_{112212} + G_{111222} \\
 I = 222 : \quad & mG_{1222} = 0 \quad \text{and} \quad mG_{2222} - \frac{1}{\alpha}(2G_{121222} - 2G_{112222}) = 2G_{22} \\
 I = 122 : \quad & mG_{1122} - \frac{1}{\alpha}(2G_{121222} - G_{112222} - G_{122122}) = G_{22} \\
 I = 212 : \quad & mG_{1212} = \frac{2}{\alpha}(G_{122122} - G_{121222}). \tag{2.24}
 \end{aligned}$$

However, these 11 equations (even if all are independent), are inadequate to find the $c(n = 6, \Lambda = 2) = 14$ independent rank-6 correlations, let alone the unknown 2 & 4-point correlations.

Similarly, consider a Yang-Mills 2-matrix model $\text{tr } S(A) = -\frac{1}{2\alpha} \text{tr } [A_1, A_2]^2$. The matrix integrals do not converge here due to the flat directions in the commutator squared action. So our derivation of the LE and WI are not strictly valid in the case, though they can be considered formally. In particular, it is not clear that the LE and WI form a consistent system of equations for this model. Nor is it clear how one could check an answer for a particular correlation, say by Monte Carlo integration, since the matrix integrals do not converge. Nevertheless, we can consider the LE formally here in order to show that they are underdetermined. The LE are

$$\frac{1}{\alpha}v_1^I(G_{I122} + G_{I221} - G_{I212}) + \frac{1}{\alpha}v_2^I(G_{I211} + G_{I112} - G_{I121}) = [v_1^I\delta_I^{I_1I_2} + v_2^I\delta_I^{I_12I_2}]G_{I_1}G_{I_2}. \tag{2.25}$$

Since v_i^I are arbitrary, we get a pair of LE (for each word I with $|I| \geq 0$)

$$G_{I221} + G_{I122} - 2G_{I212} = \alpha\delta_I^{I_1I_2}G_{I_1}G_{I_2}, \quad G_{I112} + G_{I211} - 2G_{I121} = \alpha\delta_I^{I_12I_2}G_{I_1}G_{I_2}. \tag{2.26}$$

All correlations of rank 1 or 2 are undetermined. In addition, taking $I = \emptyset$ does not give any relation for third rank moments, since the l.h.s. of the LE identically vanish on account of cyclic symmetry. As for rank-4 moments, we get only one relation $2G_{1212} - 2G_{1122} = -\alpha$, from the LE, which is inadequate to fix the 6 independent 4th rank correlations.

Similarly, the LE of the Chern-Simons 3-matrix model $\text{tr } S(A) = \frac{2i\kappa}{3}\epsilon^{ijk} \text{tr } A_i A_j A_k$ and Mehta 2-matrix model [24] $\text{tr } [cA_1 A_2 + (g/4)(A_1^4 + A_2^4)]$ are underdetermined (§2.2 of [20]).

3. Measure preserving transformations

Our aim in this section is to determine the vector fields $L_v : A_i \rightarrow A_i + v_i^I A_I$ under whose action the matrix model measure is invariant. We call such transformations measure or volume preserving. These vector fields are universal in the sense that they are independent of the choice of action $S(A)$. They can only depend on the size of the matrices (N), the number of matrices (Λ) and on the ensemble from which the matrices are drawn (hermitian in our case).

The main result of this section¹⁶ is that vector fields $L_v = v_i^I L_I^i$ satisfying (3.29) are measure preserving for any N . In the large- N limit these are the only ones, but for finite N there could be more. In particular, a symmetry of the matrix model measure at $N = \infty$ is automatically a symmetry of the measure at finite N and consequently at each order in $1/N^2$. A simpler sufficient condition for a vector field to be measure preserving is

$$v_i^{(I)i(J)} + v_i^{(J)i(I)} = 0, \quad \forall I, J. \tag{3.1}$$

Here (\dots) denotes cyclic symmetrization (3.12). Measure preserving vector fields form an infinite dimensional Lie algebra for $\Lambda > 1$ with Lie bracket (2.9) (see section 3.4). For $\Lambda = 1$, it is a 1-dimensional abelian Lie algebra consisting of translations $A \rightarrow A + v_{-1} \mathbf{1}$ (see section 2.1).

3.1 Change in measure due to action of (homogeneous) vector fields

If all I appearing in the components v_i^I of the vector field $L_v = v_i^I L_I^i$ have the same length $|I|$, then we will call such a vector field homogenous of rank $|I|$. The variation of the measure under (a not necessarily homogenous) infinitesimal change of variable,

$$A_i \rightarrow A'_i = A_i + v_i^I A_I \tag{3.2}$$

is the first order term in the expansion of the determinant of the Jacobian J in powers of v_i^I

$$J = \det \left[\frac{\partial A'_{ic}}{\partial A_{ja}} \right] = \det [\delta_j^i \delta_c^a \delta_b^d + v_i^I \delta_I^{I_1 I_2} \Phi_{I_1} \Phi_{I_2} + \mathcal{O}(v^2)] = 1 + N^2 v_i^I \delta_I^{I_1 I_2} \Phi_{I_1} \Phi_{I_2} + \mathcal{O}(v^2). \tag{3.3}$$

Here $\Phi_I = \frac{tr}{N} A_I$. Thus infinitesimally, the change in the measure per N^2 is

$$\frac{\delta J}{N^2} = v_i^I \delta_I^{I_1 I_2} \Phi_{I_1} \Phi_{I_2}. \tag{3.4}$$

We want to determine those v_i^I 's for which $\delta J/N^2 = 0$ for all Φ_K which are cyclically symmetric in K and hermitian $\Phi_K^* = \Phi_{\bar{K}}$. So we should set the coefficients of independent Φ_K to zero. Unfortunately, for finite- N , the analysis is complicated by the fact that Φ_K are not all independent. Indeed, they are related by trace identities (analogues of conditions

¹⁶Section 3.2, where this is established is a bit long and can be skipped in a first reading.

from vanishing of characteristic polynomial for a single matrix). However, in the large- N limit, the Φ_I (and consequently their expectation values, $G_I = \langle \Phi_I \rangle$) are independent up to cyclic symmetry and hermiticity. The expectation value of the infinitesimal change in measure becomes

$$\lim_{N \rightarrow \infty} \left\langle \frac{\delta J}{N^2} \right\rangle = v_i^I \delta_I^{I_1 I_2} G_{I_1} G_{I_2} = v_i^I \eta_i^i \equiv v \cdot \eta. \tag{3.5}$$

Setting the coefficient of each independent monomial in the G_I to zero leads to a characterization of the measure preserving vector fields v_i^I in the large- N limit (section 3.2).

An inspection of $v \cdot \eta$ reveals that it vanishes under the sum of two homogeneous transformations $\delta A_i = \sum_{|I|=const} v_i^I A_I + \sum_{|J|=const} v_i^J A_J$ with $|I| \neq |J|$ if and only if it vanishes under each separately. So without loss of generality, we restrict to homogenous vector fields.

Notice from (2.9) that the commutator of homogenous vector fields of rank $p > 1$ and $q > 1$ is a homogenous vector field of rank $p + q - 1$. Using this, we define the grading of a homogenous vector field $\sum_{|I|=const} v_i^I L_I^i$ as $|I| - 1$. With this, the Lie algebra \mathcal{L} of all L 's¹⁷ becomes a Lie algebra $\mathcal{L} = \bigoplus_{p \geq 0} \mathcal{L}_p$ graded by the non-negative integers, where

$$\mathcal{L}_p = span \{ L_I^i : |I| = p + 1 \} \quad \text{and} \quad [\mathcal{L}_p, \mathcal{L}_q] \subset \mathcal{L}_{p+q}. \tag{3.6}$$

This can be used to generate homogeneous higher rank volume preserving vector fields from ones with lower rank, provided the latter do not form an abelian Lie algebra.

3.2 Characterization of measure preserving vector fields for $N = \infty$

Roughly, the condition that a vector field be volume preserving becomes stronger as $N \rightarrow \infty$, since the number of independent trace invariants grows in this limit. So there are potentially a lot more volume preserving vector fields at finite- N than at $N = \infty$. Fortunately, the condition that a vector field be volume preserving at $N = \infty$ will be seen to be a sufficient condition for it to be volume preserving at any finite N . In this manner, we will establish that if the change of measure term in the LE (2.8) vanishes for a given vector field v , then it also vanishes¹⁸ in the finite- N Schwinger-Dyson equations (2.8) and indeed at each order in $1/N^2$ for the same vector field v . To characterize volume preserving vector fields in the large- N limit we must solve the equations $v \cdot \eta = 0$ for v_i^I . Let us begin with homogeneous vector fields of lowest rank.

- **Constant shift:** $\delta A_i = v_i \mathbf{1}$. In this case $v \cdot \eta = 0$. So all homogeneous vector fields of rank zero are symmetries of the measure. This reflects translation invariance of the measure.
- **Linear transformation:** $\delta A_i = v_i^j A_j$ are measure preserving iff they are traceless:

$$v \cdot \eta = v_i^j \delta_j^i = v_i^i \quad \text{so} \quad v \cdot \eta = 0 \Leftrightarrow v \text{ is traceless } v_i^i = 0. \tag{3.7}$$

¹⁷Except translations $v_i^\emptyset L_\emptyset^i : A_i \rightarrow A_i + v_i^\emptyset \mathbf{1}$, which are a separate abelian algebra. \emptyset is the empty word.

¹⁸There likely exist vector fields preserving the measure at finite- N but not at $N = \infty$.

- **Quadratic:** $\delta A_i = v_i^{jk} A_j A_k$. In this case,

$$v \cdot \eta = v_i^{jk} (\delta_k^i G_j + \delta_j^i G_k) = v_i^{ji} G_j + v_i^{ik} G_k = (v_i^{ij} + v_i^{ji}) G_j \quad (3.8)$$

Thus quadratic vector fields that preserve the measure must satisfy $v_i^{ij} + v_i^{ji} = 0$.

- **Cubic:** $\delta A_i = v_i^{jkl} A_j A_k A_l$

$$v \cdot \eta = v_i^{jkl} (\delta_{jkl}^{imn} G_{mn} + \delta_{jkl}^{\min} G_m G_n + \delta_{jkl}^{mni} G_{mn}) = (v_i^{imn} + v_i^{mni}) G_{mn} + v_i^{\min} G_m G_n. \quad (3.9)$$

This must vanish for all cyclic and hermitian G_K . The linear term in G 's is cyclically symmetric in mn , so it is not necessary that the coefficient of G_{mn} and G_{nm} separately vanish. Rather, only the cyclic projection of G_{mn} 's coefficient must vanish. Similarly, the quadratic term in G 's is symmetric under $m \leftrightarrow n$ so only the symmetric projection of its coefficient must vanish. Moreover, hermiticity implies $G_j^* = G_j$ and $G_{mn}^* = G_{nm} = G_{mn}$, so all 1- and 2-point correlations are real. We need not worry about setting the coefficients of their imaginary parts to zero. Thus $v \cdot \eta$ vanishes identically if and only if

$$v_i^{imn} + v_i^{inm} + v_i^{mni} + v_i^{nmi} = 0 \quad \text{and} \quad v_i^{\min} + v_i^{nim} = 0. \quad (3.10)$$

We can write this more succinctly as

$$v_i^{i(mn)} + v_i^{(mn)i} = 0 \quad \text{and} \quad v_i^{\min} + v_i^{nim} = 0. \quad (3.11)$$

Here we introduced the cyclic symmetrization operation (\dots) which is defined as¹⁹

$$v_i^{(J)iK} = \sum_{\pi \in C_{|J|}} v_i^{\pi(J)iK}, \quad v_i^{Ji(K)} = \sum_{\sigma \in C_{|K|}} v_i^{Ji\sigma(K)}, \quad v_i^{(J)i(K)} = \sum_{\substack{\pi \in C_{|J|} \\ \sigma \in C_{|K|}}} v_i^{\pi(J)i\sigma(K)}. \quad (3.12)$$

- **Quartic:** $\delta A_i = v_i^{jklm} A_j A_k A_l A_m$

$$\begin{aligned} v \cdot \eta &= v_i^{jklm} (\delta_{jklm}^{ipqr} G_{pqr} + \delta_{jklm}^{piqr} G_p G_{qr} + \delta_{jklm}^{pqir} G_{pq} G_r + \delta_{jklm}^{pqr i} G_{pqr}) \\ &= (v_i^{ipqr} + v_i^{pqr i}) G_{pqr} + v_i^{piqr} G_p G_{qr} + v_i^{pqir} G_{pq} G_r \\ &= (v_i^{ipqr} + v_i^{pqr i}) G_{pqr} + (v_i^{piqr} + v_i^{qr ip}) G_p G_{qr}. \end{aligned} \quad (3.13)$$

Here, the quadratic term is cyclically symmetric in qr , so its coefficient must be cyclically symmetrized in qr . Similarly, the linear term is cyclically symmetric in pqr so we must cyclically symmetrize its coefficient in pqr . However, there is a further subtlety that we must address: G_K are complex numbers, but their real and imaginary parts are related to those of $G_{\bar{K}}$ via $G_K^* = G_{\bar{K}}$. Hermiticity implies that G_p and G_{qr} are real, so for the quadratic term, it is necessary and sufficient that $(v_i^I$ are real)

$$v_i^{pi(qr)} + v_i^{(qr)ip} = 0. \quad (3.14)$$

¹⁹ $C_{|J|}$ is the cyclic group of order $|J|$. Note that we do not divide by the number of terms.

On the other hand, $G_{pqr}^* = G_{rqp}$. So $\Re G_{pqr} = \Re G_{rqp}$ and $\Im G_{pqr} = -\Im G_{rqp}$. So it is necessary and sufficient to set the coefficients of $\Re G_{pqr} + \Re G_{rqp}$ and $\Im G_{pqr} - \Im G_{rqp}$ to zero separately:

$$v_i^{i(pqr)} + v_i^{(pqr)i} + v_i^{i(rqp)} + v_i^{(rqp)i} = 0 \text{ and } v_i^{i(pqr)} + v_i^{(pqr)i} - v_i^{i(rqp)} - v_i^{(rqp)i} = 0 \quad \forall pqr. \quad (3.15)$$

However, these two conditions are equivalent (by adding and subtracting) to the single condition

$$v_i^{i(pqr)} + v_i^{(pqr)i} = 0 \quad \forall pqr. \quad (3.16)$$

Therefore $v \cdot \eta$ vanishes if and only if

$$v_i^{i(pqr)} + v_i^{(pqr)i} = 0 \quad \text{and} \quad v_i^{pi(qr)} + v_i^{(qr)ip} = 0. \quad (3.17)$$

- **Quintic:** For a rank 5 vector field $A_i \rightarrow A_i + v_i^{pqrst} A_{pqrst}$ to be volume preserving we need

$$v \cdot \eta = \sum_{pqrs} (v_i^{ipqrs} + v_i^{pqr si}) G_{pqrs} + (v_i^{piqrs} + v_i^{qrs ip}) G_p G_{qrs} + (v_i^{pqirs} + v_i^{rsipq}) G_{pq} G_{rs} = 0. \quad (3.18)$$

Since moments of different ranks are independent, $v \cdot \eta = 0$ iff the following three equations are satisfied (we have cyclically symmetrized as in previous cases in order to reduce to a sum over equivalence classes under cyclic symmetry, which is denoted \sim)

$$\begin{aligned} \sum_{pqrs/\sim} (v_i^{i(pqrs)} + v_i^{(pqrs)i}) G_{pqrs} = 0, \quad \sum_{p,q,rs/\sim} (v_i^{pi(qrs)} + v_i^{(qrs)ip}) G_p G_{qrs} = 0, \\ \text{and} \quad \sum_{pq/\sim, rs/\sim} (v_i^{(pq)i(rs)} + v_i^{(rs)i(pq)}) G_{pq} G_{rs} = 0. \end{aligned} \quad (3.19)$$

It remains to take care of the relations imposed by hermiticity to select the independent monomials. Consider the first equation in (3.19). By hermiticity $\Re G_{pqrs} = \Re G_{srpq}$ and $\Im G_{pqrs} = -\Im G_{srqp}$. So we further restrict the sum to equivalence classes under reversal of order of indices. We will denote the combination of the quotient by cyclic symmetrization and reversal of order of indices by the symbol \sim' . Then the first condition in (3.19) becomes the pair

$$\begin{aligned} \sum_{pqrs/\sim'} \left[v_i^{i(pqrs)} + v_i^{(pqrs)i} + v_i^{i(\overline{pqrs})} + v_i^{(\overline{pqrs})i} \right] \Re G_{pqrs} = 0 \quad \text{and} \\ \sum_{pqrs/\sim'} \left[v_i^{i(pqrs)} + v_i^{(pqrs)i} - v_i^{i(\overline{pqrs})} - v_i^{(\overline{pqrs})i} \right] \Im G_{pqrs} = 0. \end{aligned} \quad (3.20)$$

Since the sum is over independent moments, we set the coefficients to zero and get

$$\begin{aligned} v_i^{i(pqrs)} + v_i^{(pqrs)i} + v_i^{i(\overline{pqrs})} + v_i^{(\overline{pqrs})i} = 0 \quad \text{and} \\ v_i^{i(pqrs)} + v_i^{(pqrs)i} - v_i^{i(\overline{pqrs})} - v_i^{(\overline{pqrs})i} = 0, \end{aligned} \quad (3.21)$$

for each equivalence class $pqrs$ under the relation \sim' . However, this pair is equivalent to

$$v_i^{i(pqrs)} + v_i^{(pqrs)i} = 0 \quad \forall \text{ cyclic equivalence classes } pqrs / \sim. \quad (3.22)$$

A similar analysis of the last two conditions in (3.19) using hermiticity ($G_p^* = G_p$ and $\Re G_{qrs} = \Re G_{\overline{qrs}}$ and $\Im G_{qrs} = -\Im G_{\overline{qrs}}$ and $G_{pq}^* = G_{pq}$) allows us to identify the coefficients of independent moments and set them to zero. When the dust settles, the necessary and sufficient conditions for a 5th rank tensor to be measure preserving are

$$v_i^{i(pqrs)} + v_i^{(pqrs)i} = 0; \quad v_i^{pi(qrs)} + v_i^{(qrs)ip} = 0; \quad v_i^{(pq)i(rs)} + v_i^{(rs)i(pq)} = 0 \quad \forall p, q, r, s. \quad (3.23)$$

We see that so far, the hermiticity relations between the G_K , though taken into account, did not make their presence felt in the final answer. This simplification is due to G_i and G_{ij} being real. The hermiticity relations will play a role in the necessary and sufficient conditions for rank 7 and higher vector fields to be measure preserving. This is because it is the first case where η_i^j involves quadratic monomials in moments where both factors can be complex, e.g. $G_{pqr}G_{stu}$. This leads to complications which we now deal with in the general case.

- **Rank n :** In the general case, $\delta A_i = v_i^{j_1 \dots j_n} A_{j_1} \dots A_{j_n}$ and $v \cdot \eta = v_i^{I_1 I_2} G_{I_1} G_{I_2}$ is a quadratic polynomial in moments. The necessary and sufficient conditions on v for $v \cdot \eta = 0$ are got by selecting the independent monomials and setting their coefficients to zero. We use three relations: (a) commutativity of products $G_I G_J = G_J G_I$, (b) cyclicity $G_I = G_J$ if I, J are cyclically related and (c) hermiticity $G_I = G_{\bar{I}}^*$ or $\Re G_I = \Re G_{\bar{I}}$ and $\Im G_I = -\Im G_{\bar{I}}$. (a) \Rightarrow we must symmetrize the coefficients in I_1 and I_2 and restrict the sum over I_1 and I_2 to include only (say) $G_{I_1} G_{I_2}$ and not $G_{I_2} G_{I_1}$ (this is denoted \sum').

$$v \cdot \eta = 0 \Leftrightarrow \sum'_{I_1, I_2} (v_i^{I_1 I_2} + v_i^{I_2 I_1}) G_{I_1} G_{I_2} = 0. \quad (3.24)$$

Relation (b) means we must cyclically symmetrize coefficients in I_1 and I_2 and further restrict the sum to cyclic equivalence classes of I_1 and I_2 (denoted I_1 / \sim)

$$v \cdot \eta = 0 \Leftrightarrow \sum'_{\substack{I_1 / \sim \\ I_2 / \sim}} \left[v_i^{(I_1)i(I_2)} + v_i^{(I_2)i(I_1)} \right] G_{I_1} G_{I_2} = 0. \quad (3.25)$$

Implementing (c) is more tricky. We must identify monomials that are independent after accounting for hermiticity. Taking \Re & \Im parts ($v_i^I \in \mathbf{R}$), we write $v \cdot \eta = 0$ as

the pair

$$\begin{aligned} \sum'_{\substack{I/\sim \\ J/\sim}} \left[v_i^{(I)i(J)} + v_i^{(J)i(I)} \right] (\Re G_I \Re G_J - \Im G_I \Im G_J) &= 0 \quad \text{and} \\ \sum'_{\substack{I/\sim \\ J/\sim}} \left[v_i^{(I)i(J)} + v_i^{(J)i(I)} \right] (\Re G_I \Im G_J + \Re G_J \Im G_I) &= 0. \end{aligned} \quad (3.26)$$

The last two terms can be combined. Hermiticity $\Rightarrow \Re G_I = \Re G_{\bar{I}}, \Im G_I = -\Im G_{\bar{I}}$. So $\Re G_I \Re G_J$ is independent of $\Im G_I \Im G_J$ and we can set each part to zero separately. Thus $v \cdot \eta = 0$ iff

$$\begin{aligned} \sum'_{\substack{I/\sim \\ J/\sim}} \left[v_i^{(I)i(J)} + v_i^{(J)i(I)} \right] \Re G_I \Re G_J &= 0, \quad \sum'_{\substack{I/\sim \\ J/\sim}} \left[v_i^{(I)i(J)} + v_i^{(J)i(I)} \right] \Im G_I \Im G_J = 0, \\ \text{and} \quad \sum'_{\substack{I/\sim \\ J/\sim}} \left[v_i^{(I)i(J)} + v_i^{(J)i(I)} \right] \Re G_I \Im G_J &= 0. \end{aligned} \quad (3.27)$$

Here the sums include words I as well as their mirror images \bar{I} , so the monomials such as $\Re G_I \Re G_J$ are not all independent on account of the hermiticity relations. We must further restrict the sums to equivalence classes under reversal of order of letters in a word to get a truly independent basis for quadratic polynomials. Once this is done we set the coefficients to zero and find that $v \cdot \eta = 0$ iff (the signs are determined by the hermiticity relations)

$$\begin{aligned} v_i^{(I)i(J)} + v_i^{(\bar{I})i(J)} + v_i^{(I)i(\bar{J})} + v_i^{(\bar{I})i(\bar{J})} + I \leftrightarrow J &= 0, \\ v_i^{(I)i(J)} - v_i^{(\bar{I})i(J)} - v_i^{(I)i(\bar{J})} + v_i^{(\bar{I})i(\bar{J})} + I \leftrightarrow J &= 0 \quad \text{and} \\ v_i^{(I)i(J)} + v_i^{(\bar{I})i(J)} - v_i^{(I)i(\bar{J})} - v_i^{(\bar{I})i(\bar{J})} + I \leftrightarrow J &= 0. \end{aligned} \quad (3.28)$$

These can be slightly simplified to the following three conditions

$$\begin{aligned} v_i^{(I)i(J)} + v_i^{(\bar{I})i(\bar{J})} + I \leftrightarrow J &= 0, \\ v_i^{(\bar{I})i(J)} + v_i^{(I)i(\bar{J})} + I \leftrightarrow J &= 0, \\ v_i^{(I)i(J)} - v_i^{(I)i(\bar{J})} + I \leftrightarrow J &= 0. \end{aligned} \quad (3.29)$$

Thus, a homogeneous vector field v_i^I of rank n is volume preserving at $N = \infty$ ($v \cdot \eta = 0$), iff conditions (3.29) are satisfied for each multi-index I and J such that $|I| + |J| = n - 1$. Since conditions (3.29) are somewhat lengthy (though easy to remember), it is pertinent to add that a sufficient (but in general not necessary) condition for v to be measure preserving is

$$v_i^{(I)i(J)} + v_i^{(J)i(I)} = 0 \quad \forall I, J \quad \text{with} \quad |I| + |J| = n - 1. \quad (3.30)$$

More explicitly, this sufficient condition may be written as ($[n]$ is the greatest integer part of n)

$$\begin{aligned}
 v_i^{i(j_1 \dots j_{n-1})} + v_i^{(j_1 \dots j_{n-1})i} &= 0, \\
 v_i^{j_1 i(j_2 \dots j_{n-1})} + v_i^{(j_2 \dots j_{n-1})i j_1} &= 0, \\
 v_i^{(j_1 j_2) i(j_3 \dots j_{n-1})} + v_i^{(j_3 \dots j_{n-1}) i(j_1 j_2)} &= 0, \\
 &\vdots \\
 v_i^{(j_1 j_2 \dots j_{[\frac{n-1}{2}]}) i(j_{[\frac{n+1}{2}] \dots j_{n-1})} + v_i^{(j_{[\frac{n+1}{2}] \dots j_{n-1})} i(j_1 j_2 \dots j_{[\frac{n-1}{2}]})} &= 0.
 \end{aligned} \tag{3.31}$$

In fact, (3.30) is both necessary and sufficient for vector fields of rank ≤ 6 .

Now, it is easy to see that a volume preserving vector field for $N = \infty$ is automatically volume preserving for finite N . Suppose v_i^I is such that $v \cdot \eta = v_i^I \delta_I^{I_1 I_2} G_{I_1} G_{I_2} = 0$, or equivalently, such that conditions (3.29) are satisfied. Then it will automatically satisfy $\frac{\delta J}{N^2} = v_i^I \delta_I^{I_1 I_2} \Phi_{I_1} \Phi_{I_2} = 0$. For, all we needed for $v \cdot \eta = 0$ was commutativity of the product of two G_I 's, cyclic symmetry and hermiticity of the G_I . All these properties are also satisfied by the Φ_I 's. Of course, there are likely to be vector fields other than those satisfying (3.29) (i.e. $v \cdot \eta \neq 0$) for which $\frac{\delta J}{N^2} = 0$.

3.3 Volume preserving vector fields annihilate measure terms in LE and SDE

Thus far, we have shown that vector fields characterized in (3.29) leave the measure invariant $\delta J/N^2 = v_i^I \delta_I^{I_1 I_2} \Phi_{I_1} \Phi_{I_2} = 0$ both for finite and infinite N . Moreover, they were all the vector fields that left the measure invariant for $N = \infty$: $v_i^I \delta_I^{I_1 I_2} G_{I_1} G_{I_2} = v_i^I \eta_I^i = 0$. In other words, the r.h.s. of the LE (2.8) identically vanish iff the vector field v satisfies (3.29).

On the other hand, multiplying (3.4) by $\Phi_{K_1} \dots \Phi_{K_n}$ and taking expectation values we get

$$v_i^I \delta_I^{I_1 I_2} \langle \Phi_{I_1} \Phi_{I_2} \Phi_{K_1} \dots \Phi_{K_n} \rangle = 0 \quad \forall \quad \Phi_K \quad \text{and} \quad n = 0, 1, 2, \dots \tag{3.32}$$

provided v satisfy (3.29). Combining with the result of the previous paragraph, we see that vector fields for which the r.h.s. of the LE (2.8) vanish, also annihilate the change of measure term on the r.h.s. of the finite- N Schwinger-Dyson equations (2.8). Furthermore, multiplying by N^2 and letting $N \rightarrow \infty$ we see that the same class of vector fields also annihilate the change of measure term on the r.h.s. of the $\mathcal{O}(1/N^2)$ SDE (2.11)

$$v_i^I \delta_I^{I_1 I_2} G_{I_1; I_2; K_1; \dots; K_n}^{(2)} = 0. \tag{3.33}$$

This is a part of the result we needed in section 2 to establish the WI (2.15). The other part involves identifying which of these volume preserving vector fields also leaves the action of a specific matrix model invariant, a task we will undertake in section 4.

3.4 Volume preserving vector fields form an infinite dimensional Lie algebra

It should be possible, but laborious, to check that the Lie bracket of two vector fields of the form (3.29) is again of the same form (we have checked this for vector fields of some low

ranks). But there is a simpler argument (which uses a much deeper result from [13]) that shows they form a Lie algebra. In [13] it was shown that there is an entropy function²⁰ χ such that $L_I^i \chi = \eta_I^i = \delta_I^{I_1 I_2} G_{I_1} G_{I_2}$. Suppose L_u, L_v are volume preserving. From our results in sections 3.2 and 3.3, this means $u_i^I \eta_I^i = 0$ and $v_i^I \eta_I^i = 0$. Then we have $L_u \chi = 0$ and $L_v \chi = 0$. It follows therefore that $[L_u, L_v] \chi = (L_u L_v - L_v L_u) \chi = 0$. Thus $L_w = [L_u, L_v]$ is also volume preserving. We conclude that volume preserving vector fields form a Lie algebra.

Example. The measure preserving vector fields corresponding to linear transformations, $L_u = u_j^i L_i^j$ form the $sl_\Lambda(\mathbf{R})$ Lie algebra for a Λ -matrix model. We already found (section 3.2) that measure preserving linear transformations are the traceless ones. Here, we check that their Lie bracket implied by (2.9) is the same as the $sl_\Lambda(\mathbf{R})$ Lie algebra.

$$[L_u, L_v] = u_j^i v_l^k [L_i^j, L_k^l] = u_j^i v_l^k (\delta_k^j L_i^l - \delta_i^l L_k^j) = u_k^i v_l^k L_i^l - u_j^i v_i^k L_k^j = w_i^i L_i^i, \quad (3.34)$$

where $w_i^i = u_k^i v_i^k - v_k^i u_i^k = ([u, v])_i^i$ is just the matrix commutator. Thus $[L_u, L_v] = L_{[u, v]}$ and the linear symmetries form the Lie algebra $sl_\Lambda(\mathbf{R})$.

Moreover, for $\Lambda > 1$ we can show that the space of measure preserving vector fields is infinite dimensional. It is sufficient to consider each rank separately. First, the space of rank- n vector fields $v_i^{i_1 i_2 \dots i_n}$ is Λ^{n+1} dimensional. For a rank- n vector field to be measure preserving it is sufficient (though not necessary) that it satisfy equations (3.30). There are at most Λ^{n-1} such linear equations (if they were not linearly independent or necessary, there would be even fewer). Thus, the space of solutions is at least $\Lambda^{n+1} - \Lambda^{n-1}$ dimensional. This grows exponentially with rank, so measure preserving vector fields are an infinite dimensional Lie algebra for $\Lambda > 1$.

3.5 Explicit examples for 2 and 3 matrix models

From sections 3.2 and 3.4, we know that linear volume preserving vector fields are traceless matrices v_i^j , i.e. elements of $sl_\Lambda(\mathbf{R})$. This is a $\Lambda^2 - 1$ dimensional space (3 dimensional for a 2-matrix model and 8 dimensional for a 3-matrix model).

A generic quadratic vector field v_i^{jk} in a Λ -matrix model is specified by Λ^3 parameters. But volume preserving vector fields obey relations given in section 3.2, which restrict the number of independent coefficients. Let us work out volume preserving v_i^{jk} for 2 and 3-matrix models and determine the dimension of the space of such vector fields. The condition for v_i^{jk} to be volume preserving is $v_i^{ij} + v_i^{ji} = 0$. In a 2-matrix model this is the pair of equations

$$2v_1^{11} + v_2^{21} + v_2^{12} = 0 \quad \text{and} \quad 2v_2^{22} + v_1^{12} + v_1^{21} = 0. \quad (3.35)$$

So quadratic volume preserving vector fields are the $2^3 - 2 = 6$ parameter family

$$\begin{aligned} L_v = v_k^{ij} L_{ij}^k &= v_2^{11} L_{11}^2 + v_1^{22} L_{22}^1 + v_1^{12} \left[L_{12}^1 - \frac{1}{2} L_{22}^2 \right] + v_1^{21} \left[L_{21}^1 - \frac{1}{2} L_{22}^2 \right] \\ &\quad + v_2^{12} \left[L_{12}^2 - \frac{1}{2} L_{11}^1 \right] + v_2^{21} \left[L_{21}^2 - \frac{1}{2} L_{11}^1 \right]. \end{aligned} \quad (3.36)$$

²⁰However, χ cannot be expressed as a formal power series in G_I .

In a 3-matrix model there are three independent conditions

$$\begin{aligned} 2v_1^{11} + v_2^{21} + v_3^{31} + v_2^{12} + v_3^{13} &= 0, & 2v_2^{22} + v_1^{12} + v_3^{32} + v_1^{21} + v_3^{23} &= 0 \\ \text{and } 2v_3^{33} + v_1^{13} + v_2^{23} + v_1^{31} + v_2^{32} &= 0. \end{aligned} \tag{3.37}$$

So quadratic volume preserving vector fields are a $3^3 - 3 = 24$ parameter family for $\Lambda = 3$.

4. Transformations that also preserve action

4.1 Establishing validity of Ward identities: last step

So far, we have identified the vector fields $L_v = v_i^I L_I^i$ which leave the measure invariant in the large- N limit and observed that they continue to be measure preserving even at finite N . In order to obtain the WI (2.15), we need to determine which among these L_v are also symmetries of the action. These are the non-anomalous infinitesimal symmetries. The answer will, of course, depend on the action of the matrix model being studied. For the infinitesimal change in $(\frac{1}{N} \times)$ the action $S(A) = \text{tr } S^J A_J$ to vanish under $A_i \rightarrow A_i + v_i^I A_I$, we need

$$L_v S^J \Phi_J = v_i^I S^{J_i i J_2} \Phi_{J_1 I J_2} = 0. \tag{4.1}$$

However, for finite N , not all the Φ_I are independent even after accounting for cyclicity and hermiticity, due to the trace identities and other such constraints satisfied by the Φ_I . So it is *not* straightforward to identify the *necessary* conditions on v_i^I . But in the large- N limit we may treat the Φ_I as independent variables (up to cyclicity and hermiticity). Taking expectation values, we must solve for v_i^I in the equations

$$v_i^I S^{J_1 i J_2} G_{J_1 I J_2} = 0 \quad \forall \text{ cyclic and hermitian } G_I. \tag{4.2}$$

For such vector fields, the l.h.s. of the LE (2.8) identically vanish. Moreover, a vector field that solves (4.2) will automatically solve the finite- N equation (4.1), though the converse need not be true. This is because all we use is cyclicity and hermiticity of G_I , which is also true of the Φ_I . Now multiplying (4.1) by $\Phi_{K_1} \cdots \Phi_{K_n}$, the same vector fields also satisfy

$$v_i^I S^{J_i i J_2} \Phi_{J_1 I J_2} \Phi_{K_1} \cdots \Phi_{K_n} = 0. \tag{4.3}$$

Taking expectation values, we see that symmetries of the action in the large- N limit automatically annihilate the change in action term (with insertions) appearing in the finite- N SDE (2.8). In particular, multiplying by N^2 and letting $N \rightarrow \infty$, we see that the vector fields satisfying (4.2) also annihilate the change of action term on the l.h.s. of the $\mathcal{O}(1/N^2)$ SDE (2.11)

$$v_i^I S^{J_i i J_2} G_{J_1 I J_2 K_1 \cdots K_n}^{(2)} = 0. \tag{4.4}$$

Combining this with our result from section (3.3) on volume preserving vector fields, we come to the following conclusion. Suppose the vector field v is such that both l.h.s. and r.h.s. of the large- N LE identically vanish, $v_i^I S^{J_1 i J_2} G_{J_1 I J_2} = 0 = v_i^I \eta_I^i$. Then the

change in action and change in measure term in the $\mathcal{O}(1/N^2)$ SDE also vanish identically $v_i^I S^{J_i i J_2} G_{J_1 I J_2 K_1 \dots K_n}^{(2)} = 0 = v_i^I \delta_I^{I_1 i I_2} G_{I_1; I_2; K_1; \dots; K_n}^{(2)}$. As a consequence, for such vector fields (non-anomalous vector fields), the $\mathcal{O}(1/N^2)$ SDE become WI (2.15) which may be summarized as $L_v G_K = 0$ for all K . This completes the proof of validity of the WI.

4.2 Non-anomalous symmetries of specific models

It is straightforward to see that non-anomalous vector fields form a Lie sub-algebra of the infinite dimensional Lie algebra of measure preserving vector fields (section 3.4). For, $L_v S(G) = 0$ and $L_w S(G) = 0$ implies that $[L_v, L_w]S(G) = 0$. However, this Lie algebra is not necessarily infinite dimensional and depends on the action of the matrix model.

This brings us to the task of determining the non-anomalous infinitesimal symmetries of specific matrix models. In looking for measure preserving vector fields, recall (section 3.1) that we could break up the problem into finding homogeneous measure preserving vector fields of a given rank.²¹ The same strategy does not work in general for symmetries of the action. However, if the action is itself a homogeneous polynomial,²² then (4.2) does not mix vector fields of different ranks. In that case, every solution to (4.2) is a sum of homogeneous solutions. More generally, the action may not be a homogeneous polynomial, as for a Gaussian + Yang-Mills model. In such cases, not every solution of (4.2) is necessarily a sum of homogeneous solutions, though there may still be large classes of homogeneous solutions. For this reason, we begin by determining non-anomalous homogeneous action-preserving vector fields of low rank.

A priori, it is not clear that there are *any* vector fields that leave both action and measure invariant. Indeed, for a 1-matrix model (section 2.1) there are none. We were pleasantly surprised to find not just linear but also non-linear non-anomalous symmetries for several interesting multi-matrix models. We begin with linear non-anomalous symmetries in section 4.3 and give examples of non-linear non-anomalous symmetries in section 4.4.

4.3 Examples of linear non-anomalous symmetries

We determine linear symmetries of both action and measure for the Gaussian Λ -matrix model, Chern-Simons 3-matrix model, Yang-Mills and Gaussian+YM Λ -matrix models. The linear non-anomalous symmetries of the gaussian, CS 3-matrix model, YM 2-matrix model and gaussian+YM 2-matrix models form the orthogonal Lie algebra with respect to the covariance matrix, $sl_3(\mathbf{R})$, $sl_2(\mathbf{R})$ and $o(2)$ Lie algebras respectively. Not every multi-matrix model has non-trivial linear non-anomalous symmetries. The pure-quartic 2-matrix model $\text{tr } S(A) = \text{tr } (A^4 + B^4)$ or the model studied by Mehta [24], $\text{tr } S(A) = \text{tr } [cA_1 A_2 + (g/4)(A_1^4 + A_2^4)]$ have no non-trivial linear action preserving symmetries. Linear symmetries form a closed Lie algebra among themselves (section 3.4), so their Lie brackets cannot be used to generate new symmetries.

²¹A homogeneous vector field v of rank n is one whose components v_i^I are non-vanishing only for $|I| = n$. We call rank-1 vector fields linear transformations, rank-2 vector fields quadratic changes of variable and so on.

²²Examples include the Gaussian, Chern-Simons and Yang-Mills models.

4.3.1 Linear symmetries of Gaussian

The gaussian Λ -matrix model is defined by the action $\text{tr } S(A) = \text{tr } \frac{1}{2} C^{ij} A_i A_j$ where C^{ij} is a positive real symmetric ‘covariance’ matrix. We seek infinitesimal linear transformations $\delta A_i = v_i^j A_j$ that leave the action as well as measure invariant in the large N limit. In section 3.2 we found that the measure preserving transformations are the traceless ones $v_i^i = 0$ forming the Lie algebra $sl_\Lambda(\mathbf{R})$. Here we find that the vector fields v_i^j that preserve both the action and measure in the large- N limit are those that satisfy $v_m^i + v_k^j C^{ki} C_{jm} = 0$. This is the condition that v_i^j be an orthogonal transformation with respect to a metric given by the covariance. In particular, for a unit covariance $C^{ij} = \delta^{ij}$, these are the antisymmetric matrices.

For the expectation value of the Gaussian action to be invariant at $N = \infty$, we need

$$L_v S(G) = v_k^j C^{kl} G_{lj} = \frac{1}{2} (\tilde{v}^{jl} + \tilde{v}^{lj}) G_{lj} = 0 \quad \forall \text{ cyclic and hermitian } G_{lj}. \quad (4.5)$$

We used C^{kl} and its inverse C_{lm} to raise and lower indices, $\tilde{v}^{jl} = v_k^j C^{kl}$, $\tilde{v}^{jl} C_{lm} = v_m^j$, $C^{kl} C_{lm} = \delta_m^k$. The condition for a symmetry of the action is that \tilde{v} be anti-symmetric

$$\tilde{v}^{jl} + \tilde{v}^{lj} = 0. \quad (4.6)$$

If \tilde{v} is anti-symmetric, then v is automatically traceless. So action preserving linear transformations are automatically measure preserving. To see this we first rewrite antisymmetry of \tilde{v} as a condition on v by lowering an index $v_m^i + v_k^j C^{ki} C_{jm} = 0$. Taking the trace we get $v_i^i + v_k^j C^{ki} C_{ji} = 0$ which implies $v_i^i = 0$. Thus, the non-anomalous linear symmetries of the gaussian are given by vector fields $v_i^j L_j^i$ that are anti-symmetric after raising an index with C^{ik} . In other words, the orthogonal Lie algebra with respect to the metric given by the covariance matrix. In particular, the dimension of the space of linear symmetries $\frac{1}{2}(\Lambda^2 - \Lambda)$, does not change as we move around in the space of non-singular symmetric covariance matrices.

Example 1. Consider a gaussian two matrix model with diagonal covariance $C^{ij} = \text{diag}(a, b)$. Then the condition that $\delta A_i = v_i^j A_j$ be action preserving in the large- N limit is $v_1^1 = v_2^2 = 0$ and $av_1^2 + bv_2^1 = 0$. These are the infinitesimal orthogonal transformations with respect to the ‘metric’ $\text{diag}(a, b)$. Such v_i^j are traceless and thus measure preserving as well.

Example 2. If the covariance of a gaussian Λ -matrix model is a multiple of the identity, then the action preserving transformations are the anti-symmetric matrices ($v_i^j + v_j^i = 0$) which form the orthogonal Lie algebra with respect to the metric δ^{ij} . Such matrices are clearly traceless so that the non-anomalous linear symmetries form the Lie algebra $o(\Lambda)$.

4.3.2 Linear symmetries of Chern Simons model

The Chern-Simons 3-matrix model has action

$$\text{tr } S(A) = \frac{2i\kappa}{3} \epsilon^{ijk} A_i A_j A_k = 2i\kappa \text{tr } A_1 [A_2, A_3] = 2i\kappa \text{tr } (A_{123} - A_{132}). \quad (4.7)$$

So the coupling tensors are $S^{ijk} = \frac{2i\kappa}{3}\epsilon^{ijk}$. We seek all vector fields $L_v = v_i^I L_I^i$ for which the change in $(\frac{1}{N}\times)$ the action vanishes for $N = \infty$ (second equality requires relabeling of indices)

$$L_v S^J G_J = v_i^I S^{J_1 i J_2} G_{J_1 I J_2} = 2i\kappa v_i^I \epsilon^{ijk} G_{Ijk} = 0. \tag{4.8}$$

v_i^I are real and have no symmetry in I . They must satisfy $v_i^I \epsilon^{ijk} G_{Ijk} = 0$ for all cyclic and hermitian G_K . Specializing to linear transformations $A_i \rightarrow A_i + v_i^j A_j$, they must satisfy

$$\sum_{1 \leq i,j,k,l \leq 3} v_i^l \epsilon^{ijk} G_{ljk} = 0. \tag{4.9}$$

We could also arrive at this condition by making a linear change of variables in the action

$$\delta \frac{1}{N} \text{tr} S = \frac{\text{tr}}{N} (v_1^i A_{i23} + v_2^i A_{i13} + v_3^i A_{i12} - v_1^i A_{i32} - v_2^i A_{i13} - v_3^i A_{i12}) = 0. \tag{4.10}$$

Writing out all the terms and using cyclicity of G_K this condition simplifies dramatically to

$$\sum_{i=1}^3 v_i^i (G_{123} - G_{132}) = 0. \tag{4.11}$$

Taking real and imaginary parts²³ we get the single condition $v_i^i \Im G_{123} = 0$, which must be satisfied for all $\Im G_{123}$. We conclude that v_i^j preserves the CS action iff it is traceless $v_i^i = 0$. We recall (section 3.2) that traceless linear transformations also preserve the matrix model measure. Thus, the CS model has a maximal family of linear non-anomalous symmetries.

From section 3.4 we know that the space of traceless real v_i^j is the Lie algebra $sl_3(\mathbf{R})$, an 8 dimensional space. The free parameters can be chosen as $v_1^2, v_1^3, v_2^1, v_2^3, v_3^1, v_3^2, v_1^1$ and v_2^2 with $v_3^3 = -v_1^1 - v_2^2$. The corresponding symmetries are an 8-parameter family of vector fields

$$L_v = v_1^2 L_2^1 + v_1^3 L_3^1 + v_2^1 L_1^2 + v_2^3 L_3^2 + v_3^1 L_1^3 + v_3^2 L_2^3 + v_1^1 L_1^1 + v_2^2 L_2^2 - (v_1^1 + v_2^2) L_3^3. \tag{4.12}$$

4.3.3 Linear symmetries of Yang-Mills model

For 2 or more matrices and a real symmetric invertible metric g_{ij} , the YM model has action $\text{tr} S(A) = -\frac{1}{4\alpha} \text{tr} [A_i, A_j][A_k, A_l] g^{ik} g^{jl}$. The expectation value of the change in the action under a linear transformation $\delta A_i = v_i^j A_j$ in the large- N limit can be written as

$$L_v S(G) = -\frac{1}{\alpha} G_{jklm} \left(v_i^m g^{ik} g^{jl} - v_i^k g^{il} g^{jm} - v_i^j g^{im} g^{kl} + v_i^k g^{im} g^{jl} \right) \tag{4.13}$$

To identify symmetries of the action, we must select independent G_{jklm} and set their coefficients to zero. First we restrict the sum to words $jklm$ up to cyclic symmetry. Thus $\delta S = 0$ iff

$$-\frac{1}{\alpha} \sum_{jklm/cyc} G_{jklm} R^{jklm} = 0 \tag{4.14}$$

²³ $v_i^I \in \mathbf{R}$. Hermiticity & cyclicity $\Rightarrow G_{123}^* = G_{132}$ which $\Rightarrow \Re G_{123} = \Re G_{132}$ and $\Im G_{123} = -\Im G_{132}$.

where the cyclically symmetric tensor R^{jklm} is

$$\begin{aligned}
 R^{jklm} &= \left[v_i^j (2g^{il}g^{km} - g^{im}g^{kl} - g^{ik}g^{lm}) + \text{cyclic}(j \rightarrow k \rightarrow l \rightarrow m \rightarrow j) \right] \\
 &= (\tilde{v}^{(jl)}g^{km} + \tilde{v}^{(km)}g^{jl}) - \frac{1}{2} \left[\tilde{v}^{(jm)}g^{kl} + \tilde{v}^{(jk)}g^{lm} + \tilde{v}^{(kl)}g^{mj} + \tilde{v}^{(lm)}g^{jk} \right]. \quad (4.15)
 \end{aligned}$$

Here we have used the metric to raise and lower indices $\tilde{v}^{jl} = v_i^j g^{il}$ and $v_m^j = \tilde{v}^{jl} g_{lm}$ and denoted the symmetric projection by $\tilde{v}^{(jk)} = \frac{1}{2}(\tilde{v}^{jk} + \tilde{v}^{kj})$. We have still to account for the hermiticity relations $\Re G_{jklm} = \Re G_{\overline{jklm}}$, $\Im G_{jklm} = -\Im G_{\overline{jklm}}$. Now, v_i^j and g^{kl} are real and $\Re G_{jklm}$ and $\Im G_{jklm}$ are independent of each other, so we have $L_v S(G) = 0$ iff

$$\sum_{jklm/\text{cyc}} R^{jklm} \Re G_{jklm} = 0 \quad \text{and} \quad \sum_{jklm/\text{cyc}} R^{jklm} \Im G_{jklm} = 0. \quad (4.16)$$

Now we must collect the coefficients of $\Re G_{jklm}$ and $\Re G_{\overline{jklm}}$ and similarly for the imaginary parts and restrict the sum to avoid \overline{jklm} if $jklm$ has already appeared. Two possibilities arise: either \overline{jklm} may be obtained from $jklm$ via cyclic permutations or not. In the former case, $\Im G_{jklm}$ vanishes and the coefficient of $\Re G_{jklm}$ must vanish for v to be a symmetry of the action. Thus we get $R^{jklm} = 0$ if $jklm$ is cyclically related to \overline{jklm} . On the other hand, if $jklm$ is not cyclically related to \overline{jklm} , then collecting coefficients we have

$$\sum_{jklm/\text{cyc, revers}} \Re G_{jklm} (R^{jklm} + R^{\overline{jklm}}) = 0 \quad \text{and} \quad \sum_{jklm/\text{cyc, revers}} \Im G_{jklm} (R^{jklm} - R^{\overline{jklm}}) = 0 \quad (4.17)$$

Now the sums are over truly independent moments. Setting coefficients to zero we get the pair of conditions $R^{jklm} + R^{\overline{jklm}} = 0$ and $R^{jklm} - R^{\overline{jklm}} = 0$, whose simultaneous solution is again $R^{jklm} = 0$. We conclude that the necessary and sufficient conditions for v_i^j to be a symmetry of the action are $R^{jklm} = 0$. By contracting with the non-singular metric to get a scalar,

$$R^{jklm} g_{jl} g_{km} = (6\Lambda - 4)v_i^i. \quad (4.18)$$

Since $\Lambda \neq 2/3$, if v is action preserving ($R^{jklm} = 0$), then $\text{tr } v = 0$ and v is automatically measure preserving. Thus, non-anomalous linear symmetries of the Yang-Mills model in the large- N limit are characterized by those v for which the tensor R^{jklm} vanishes. It suffices to check this condition for each word $jklm$ up to cyclic permutations and order reversals. Since R^{jklm} depends only on the symmetric projection of \tilde{v}^{jk} , the anti-symmetric part of \tilde{v}^{jk} is unconstrained! Thus, a sufficient condition for \tilde{v}^{ij} to be a non-anomalous symmetry is that it be anti-symmetric. However, this is not a necessary condition; there are traceless²⁴ \tilde{v}^{ij} with non-trivial symmetric projections for which $R^{jklm} = 0$.

²⁴ $\text{tr } v = v_j^j = \tilde{v}^{jl} g_{lj}$.

Example. We will demonstrate this using the simplest non-trivial example, the 2-matrix Yang-Mills model with flat metric $g_{ij} = \delta_{ij}$. In this case, the action reads $\text{tr } S(A) = -\frac{1}{2\alpha} \text{tr } [A_1, A_2]^2$. Then $v_j^i = \tilde{v}^{ik} \delta_{kj}$. The antisymmetric part of \tilde{v} automatically satisfies $R^{jklm} = 0$, so let us suppose that \tilde{v}^{ij} is a traceless symmetric tensor, i.e. $\tilde{v}^{(ij)} = \tilde{v}^{ij}$ and $\tilde{v}^{11} + \tilde{v}^{22} = 0$. Then the six independent components of R^{jklm} are all identically zero

$$\begin{aligned} R^{1111} &= 2(\tilde{v}^{11} + \tilde{v}^{11}) - 4\tilde{v}^{11} = 0, & R^{2222} &= 2(\tilde{v}^{22} + \tilde{v}^{22}) - 4\tilde{v}^{22} = 0, \\ R^{1112} &= 2\tilde{v}^{12} - (\tilde{v}^{12} + \tilde{v}^{12}) = 0, & R^{1122} &= -(\tilde{v}^{11} + \tilde{v}^{22}) = 0, \\ R^{1212} &= 2(\tilde{v}^{11} + \tilde{v}^{22}) = 0, & R^{1222} &= 2\tilde{v}^{12} - (\tilde{v}^{12} + \tilde{v}^{12}) = 0. \end{aligned} \tag{4.19}$$

So every symmetric traceless \tilde{v}^{jk} satisfies $R^{jklm} = 0$. We conclude that for $\Lambda = 2$ and $g_{ij} = \delta_{ij}$, the Lie algebra of non-anomalous symmetries is $sl_2(\mathbf{R})$.

4.3.4 Linear symmetries of Gaussian + Yang-Mills

For $\Lambda \geq 2$ let us consider a Gaussian + Yang-Mills matrix model with action

$$\text{tr } S(A) = \frac{1}{2} C^{ij} \text{tr } A_i A_j - \frac{1}{4\alpha} \text{tr } [A_i, A_j][A_k, A_l] g^{ik} g^{jl}. \tag{4.20}$$

The simplest case which we will focus on is the two matrix model with flat metric $g_{ij} = \delta_{ij}$ and with covariance a multiple of the identity $C^{ij} = m^2 \delta^{ij}$. In this case the action reads

$$\text{tr } S(A) = \text{tr } \frac{m^2}{2} (A_1^2 + A_2^2) - \frac{1}{2\alpha} \text{tr } [A_1, A_2]^2. \tag{4.21}$$

We know (sections 4.3.1, 4.3.3) that linear non-anomalous symmetries of the gaussian and Yang-Mills parts constitute the $o(2)$ and $sl_2(\mathbf{R})$ Lie algebras respectively. Their intersection is $o(2)$, which is automatically a non-anomalous symmetry algebra of (4.21). But these must be all the linear symmetries, since there can be no cancelation between $L_v S_{gauss}(G)$ which involves two point correlations and $L_v S_{YM}(G)$ which involves 4-point correlations exclusively. The corresponding conclusion for Λ matrix models (again with C^{ij} a multiple of identity and $g_{ij} = \delta_{ij}$) is that the non-anomalous linear symmetries form the orthogonal Lie algebra $o(\Lambda)$.

4.4 Examples of non-linear non-anomalous symmetries

We exhibit homogeneous quadratic infinitesimal changes of variable $\delta A_i = v_i^{jk} A_j A_k$ which leave both action and measure invariant in the large- N limit. In particular, we consider the 2-matrix gaussian with unit covariance, the 3-matrix Chern Simons model, the 2-matrix commutator-squared Yang-Mills model and the 2-matrix Gaussian+YM model. We find a 2, 18, 6 and 2 dimensional family of quadratic non-anomalous symmetries in these cases. Moreover, we show that quadratic symmetries do not form a Lie algebra by themselves. We demonstrate how to obtain non-trivial non-anomalous *cubic* symmetries via their Lie brackets.

4.4.1 Quadratic symmetries of the Gaussian model

Under an infinitesimal quadratic change of variable $\delta A_i = v_i^{jk} A_{jk}$, the change in action of a gaussian model with unit covariance is

$$\delta S = \text{tr } \delta^{ij} \delta A_i A_j = \text{tr } \delta^{ij} v_i^{mn} A_{mnj} = \text{tr } v_i^{mn} A_{mni}. \quad (4.22)$$

Specializing to a 2-matrix model in the large- N limit and taking expectation values,

$$L_v S(G) = v_1^{11} G_{111} + (v_2^{11} + v_1^{12} + v_1^{21}) G_{112} + (v_2^{12} + v_2^{22} + v_2^{21}) G_{221} + v_2^{22} G_{222}. \quad (4.23)$$

where we have collected the coefficients of the four independent third rank moments, which are all real after accounting for cyclicity and hermiticity. Thus $L_v S = 0$ implies

$$v_1^{11} = v_2^{22} = 0, \quad v_2^{11} = -v_1^{12} - v_1^{21}, \quad v_1^{22} = -v_2^{12} - v_2^{21}. \quad (4.24)$$

To be non-anomalous, v must be volume preserving as well: $v_i^{ij} + v_i^{ji} = 0$, which implies (3.35). The solution of this system of linear equations is a two parameter family

$$v_1^{11} = v_2^{22} = v_2^{11} = v_1^{22} = 0, \quad v_1^{12} = -v_1^{21} = a, \quad v_2^{12} = -v_2^{21} = b. \quad (4.25)$$

Thus, the non-anomalous quadratic symmetries of the Gaussian model with unit covariance are

$$\delta A_1 = a [A_1, A_2], \quad \delta A_2 = b [A_1, A_2], \quad a, b \in \mathbf{R}. \quad (4.26)$$

They correspond to the vector fields $L_{u_{a,b}} = a(L_{12}^1 - L_{21}^1) + b(L_{12}^2 - L_{21}^2)$. The Lie bracket of $L_{u_{a,b}}$ and $L_{u_{c,d}}$ is not a quadratic vector field. Rather, it is a cubic non-anomalous vector field

$$\begin{aligned} [L_{u_{a,b}}, L_{u_{c,d}}] &= (ad - bc) \left\{ [L_{12}^1, L_{12}^2] - [L_{12}^1, L_{21}^2] - [L_{21}^1, L_{12}^2] + [L_{21}^1, L_{21}^2] \right\} \\ &= (ad - bc) (L_{122}^2 - L_{112}^1 - 2L_{212}^2 + 2L_{121}^1 + L_{221}^2 - L_{211}^1). \end{aligned} \quad (4.27)$$

It corresponds to the one parameter family of infinitesimal changes of variable

$$\delta A_1 = (ad - bc) [A_1, [A_2, A_1]], \quad \delta A_2 = (ad - bc) [[A_1, A_2], A_2], \quad ad - bc \in \mathbf{R}. \quad (4.28)$$

There could, of course, be more cubic non-anomalous symmetries that do not arise as Lie brackets of quadratic symmetries. It is satisfying that our point of view tells us something interesting even about the gaussian matrix model.

4.4.2 Quadratic symmetries of Chern-Simons

For a homogeneous quadratic change of variable, the change in the expectation value of the Chern-Simons action $\text{tr } S(A) = \frac{2i\kappa}{3} \epsilon^{ijk} A_{ijk}$ in the large- N limit is

$$L_v S(G) = 2i\kappa v_i^{lm} \epsilon^{jki} G_{lmjk}. \quad (4.29)$$

To account for cyclicity of G_{lmjk} we cyclically symmetrize the coefficient. Let

$$T^{lmjk} = v_i^{lm} \epsilon^{jki} + v_i^{kl} \epsilon^{mji} + v_i^{jk} \epsilon^{lmi} + v_i^{mj} \epsilon^{kli}.$$

Then,
$$L_v S(G) = (i\kappa/2) \sum_{lmjk/\sim} T^{lmjk} G_{lmjk} \quad (4.30)$$

where the sum is restricted equivalence classes of $lmjk$ under cyclic permutations. Accounting for hermiticity, we get that $L_v S(G) = 0$ iff

$$\sum_{lmjk/\sim'} (T^{lmjk} + T^{\overline{lmjk}}) \Re G_{lmjk} = 0; \quad \text{and} \quad \sum_{lmjk/\sim'} (T^{lmjk} - T^{\overline{lmjk}}) \Im G_{lmjk} = 0 \quad (4.31)$$

where now the sums are further restricted modulo order reversal. Now we may set the coefficients to zero and after adding and subtracting we find

$$L_v S(G) = 0 \Leftrightarrow T^{lmjk} = 0 \quad (4.32)$$

where the condition is imposed for all words $lmjk$ modulo cyclic permutations. There are $c(n=4, \Lambda=3) = 24$ such words for a 3-matrix model. For 9 of these words (1111), (2222), (3333), (1112), (1222), (2333), (2223), (3111) and (3331), T^{lmjk} identically vanishes. The equations $T^{lmjk} = 0$ for each of the remaining $24 - 9 = 15$ words are listed below. The words are indicated in parenthesis to the left of the equations

$$\begin{aligned} (1212) \quad v_3^{12} = v_3^{21} & \quad (1123) \quad v_1^{11} + v_3^{31} + v_2^{12} = 0 & \quad (3312) \quad v_3^{33} + v_2^{23} + v_1^{31} = 0 \\ (1122) \quad v_3^{12} = v_3^{21} & \quad (1132) \quad v_1^{11} + v_3^{13} + v_2^{21} = 0 & \quad (3321) \quad v_3^{33} + v_1^{13} + v_2^{32} = 0 \\ (2323) \quad v_1^{23} = v_1^{32} & \quad (1213) \quad v_3^{13} + v_2^{21} - v_2^{12} - v_3^{31} = 0 & \quad (3132) \quad v_2^{32} + v_1^{13} - v_1^{31} - v_2^{23} = 0 \\ (2233) \quad v_1^{23} = v_1^{32} & \quad (2231) \quad v_2^{22} + v_1^{12} + v_3^{23} = 0 & \\ (3131) \quad v_2^{31} = v_2^{13} & \quad (2213) \quad v_2^{22} + v_1^{21} + v_3^{32} = 0 & \\ (3311) \quad v_2^{31} = v_2^{13} & \quad (2321) \quad v_1^{21} + v_3^{32} - v_3^{23} - v_1^{12} = 0. & \end{aligned} \quad (4.33)$$

Equations for volume preserving vector fields are $v_i^{ij} + v_i^{ji} = 0$, or explicitly

$$\begin{aligned} v_1^{11} + v_2^{21} + v_3^{31} + v_1^{11} + v_2^{12} + v_3^{13} = 0, \quad v_1^{12} + v_2^{22} + v_3^{32} + v_1^{21} + v_2^{22} + v_3^{23} = 0 \\ \text{and} \quad v_1^{13} + v_2^{23} + v_3^{33} + v_1^{31} + v_2^{32} + v_3^{33} = 0. \end{aligned} \quad (4.34)$$

We see that (1123) + (1132), (2231) + (2213) and (3321) + (3312) are equivalent to these equations. So quadratic symmetries of the CS action are also volume preserving. Moreover

$$(1132) - (1123) = (1213); \quad (2213) - (2231) = (2321); \quad (3321) - (3312) = (3132). \quad (4.35)$$

So equations (1213), (2321), (3132) may be discarded. And equations (1122), (2233), (3311) are redundant. Thus we are left with $15 - 6 = 9$ independent equations for $3^3 = 27$ unknowns v_i^{lm} , leaving an 18 parameter family of non-anomalous quadratic symmetries of the CS model.

4.4.3 Quadratic symmetries of Yang-Mills model

The action of the two matrix Yang-Mills model may be written $S = -\frac{1}{2\alpha} \text{tr}[A_1, A_2]^2 = \frac{1}{\alpha} \text{tr}(A_{1122} - A_{1212})$. Under an infinitesimal homogeneous quadratic change of variables $\delta A_i = v_i^{jk} A_{jk}$ the change in the action $-\frac{1}{\alpha} \text{tr} \{([\delta A_1, A_2] + [A_1, \delta A_2])[A_1, A_2]\}$ becomes, in the large- N limit

$$\begin{aligned} -\alpha L_v S(G) &= v_i^{jk} L_{jk}^i (G_{1212} - G_{1122}) = v_i^{jk} (\delta_{1212}^{I_1 I_2} G_{I_1 j k I_2} - \delta_{1122}^{I_1 I_2} G_{I_1 j k I_2}) \\ &= G_{11212} (2v_1^{11} + v_2^{12} + v_2^{21}) - G_{11122} (2v_1^{11} + v_2^{21} + v_2^{12}) \\ &\quad + G_{12122} (2v_2^{22} + v_1^{12} + v_1^{21}) - G_{11222} (2v_2^{22} + v_1^{12} + v_1^{21}). \end{aligned} \quad (4.36)$$

Since the moments that appear are independent, we set the coefficients to zero:

$$2v_1^{11} + v_2^{12} + v_2^{21} = 0 \quad \text{and} \quad 2v_2^{22} + v_1^{12} + v_1^{21} = 0. \quad (4.37)$$

But these conditions are identical to those for a quadratic vector field to preserve the measure of a 2-matrix model (3.35). Since the above two equations are independent, we have a $2^3 - 2 = 6$ parameter family of non-anomalous homogeneous quadratic symmetries of the 2-matrix Yang-Mills model given in (3.36). It is remarkable that every measure preserving linear and quadratic vector field also preserves the action of the YM 2-matrix model and CS 3-matrix model in the large- N limit. We wonder if this continues to hold for higher rank symmetries or more matrices.

Cubic symmetry. The Lie bracket of two non-anomalous rank-2 vector fields (if $\neq 0$) is a rank-3 non-anomalous vector field (since they form a Lie algebra). This is a way of generating new symmetries. Consider two quadratic symmetries of the YM model

$$L_v = aL_{21}^2 - \frac{a}{2}L_{11}^1 \quad \text{and} \quad L_u = bL_{22}^1. \quad (4.38)$$

which correspond to the choices $v_2^{21} = a$, $v_1^{11} = -\frac{a}{2}$ while all other v_i^{jk} vanish, and $u_1^{22} = b$ and all other u_i^{jk} vanish. Their commutator is

$$[L_v, L_u] = ab \left\{ [L_{21}^2, L_{22}^1] - \frac{1}{2}[L_{11}^1, L_{22}^1] \right\} = ab \left\{ L_{212}^1 - L_{222}^2 + \frac{3}{2}L_{221}^1 + \frac{1}{2}L_{122}^1 \right\} \quad (4.39)$$

The non-vanishing components of the resulting non-anomalous cubic symmetry are

$$w_1^{212} = ab, \quad w_1^{221} = \frac{3}{2}ab, \quad w_1^{122} = \frac{1}{2}ab, \quad \text{and} \quad w_2^{222} = -ab. \quad (4.40)$$

One can also check explicitly that this defines a simultaneous symmetry of the action and the measure. For example, the change in action is

$$\delta \text{tr} S = ab \text{tr} \left\{ \left([A_{212} + \frac{3}{2}A_{221} + \frac{1}{2}A_{122}, A_2] - [A_1, A_{222}] \right) [A_1, A_2] \right\} = 0. \quad (4.41)$$

The conditions for a homogeneous cubic vector field to be volume preserving are

$$w_i^{\text{min}} + w_i^{\text{nim}} = 0 \quad \text{and} \quad w_i^{\text{imn}} + w_i^{\text{inm}} + w_i^{\text{mni}} + w_i^{\text{nmi}} = 0, \quad (4.42)$$

and w_i^{jkl} satisfy these conditions as well.

4.4.4 Quadratic symmetries of 2-matrix Gaussian + YM model

Having determined homogeneous quadratic symmetries of gaussian and YM models, we get those for gaussian+YM model $\text{tr } S(A) = \text{tr} [\frac{m^2}{2}(A_1^2 + A_2^2) - \frac{1}{2\alpha}[A_1, A_2]^2]$ by taking their intersection. For, there can be no cancelation between rank 3 & 5 tensors from the action of a homogeneous quadratic vector field on the gaussian and YM terms. Since every quadratic measure preserving vector field also preserves the YM action, the intersection is the same family $\delta A_1 = a[A_1, A_2]$, $\delta A_2 = b[A_1, A_2]$, $a, b \in \mathbf{R}$ as for the gaussian (section 4.4.1).

5. Supplementing loop equations with Ward identities

5.1 Gaussian

LE of the Gaussian are not underdetermined. We do not need the WI in this case. Nevertheless, the Gaussian does have non-anomalous symmetries (sections 4.3.1, 4.4.1), which lead to non-trivial WI. The LE along with WI are an overdetermined system in this case. Nevertheless, the WI are consistent with the LE and there is no contradiction, as we have shown in section 2. To illustrate this, consider a Λ -matrix model with unit covariance $\text{tr } S(A) = \frac{1}{2}\delta^{ij} A_i A_j$. The unique solution to the LE states that the odd rank correlations vanish and the even ones G_K are determined by a planar version of Wick's theorem involving a sum over non-crossing partitions of K into pairs of indices. For example $G_{ij} = \delta_{ij}$, $G_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}$ etc. The WI corresponding to linear non-anomalous symmetries are (v_j^k are anti-symmetric)

$$L_v G_K = v_j^k L_{k_j}^j G_K = \sum_{i=1}^n v_j^k \delta_{k_i}^j G_{k_1 \dots k_{i-1} k k_{i+1} \dots k_n} = \sum_{i=1}^n v_{k_i}^k G_{k_1 \dots k_{i-1} k k_{i+1} \dots k_n} = 0. \quad (5.1)$$

Thus, the moments must be $o(\Lambda)$ invariant tensors. We can check that these WI are consistent with the LE. For example, with $n = 1$ we get the WI $v_l^k G_k = 0$, for all antisymmetric v_l^k . But there are anti-symmetric v_k^j with non-vanishing determinant, and G_j must lie in their kernel which is trivial. So $G_j = 0$, as implied by the LE. The WI for odd n are trivially satisfied by solutions to the LE, since odd rank moments vanish. WI for $n = 2$ are

$$v_{k_1}^k G_{kk_2} + v_{k_2}^k G_{k_1 k} = 0, \quad \forall \text{ antisymmetric } v_l^k. \quad (5.2)$$

For $G_{ij} = \delta_{ij}$, the l.h.s. becomes $v_{k_1}^{k_2} + v_{k_2}^{k_1}$ which vanishes on account of anti-symmetry of v . Similarly we can check that the WI are consistent with the LE for $n = 4, 6, \dots$.

We could do the same for quadratic symmetries. Let us consider the two-matrix gaussian model. The WI $L_v G_K = 0$ following from quadratic non-anomalous symmetries (section 4.4.1) $L_v = a(L_{12}^1 - L_{21}^1) + b(L_{12}^2 - L_{21}^2)$ for arbitrary real a, b are

$$\delta_K^{L1M} G_{L12M} = 0 \quad \text{and} \quad \delta_K^{L2M} G_{L21M} = 0. \quad (5.3)$$

These WI are consistent with the LE (for $|K| \leq 4$, that we checked, these WI are consequences of cyclicity and do not contain new information). For more nontrivial use of the WI we must progress to non-gaussian multi-matrix models whose LE are underdetermined.

5.2 Gaussian plus Yang-Mills

The matrix integrals for correlations of the 2-matrix Gaussian + YM model, whose action is

$$\text{tr } S(A) = \text{tr} \left[\frac{m}{2}(A_1^2 + A_2^2) - \frac{1}{2\alpha}[A_1, A_2]^2 \right], \quad (5.4)$$

converge. Recall that the commutator of hermitian matrices is anti-hermitian, and the square of an anti-hermitian matrix is non-positive. Thus, the quartic term is non-negative. The quadratic term ensures that as any matrix element goes to $\pm\infty$, the action goes to $+\infty$. Thus, the Boltzmann weight $e^{-N \text{tr } S}$ vanishes at least exponentially fast as any matrix element goes to $\pm\infty$. Thus, all polynomial observables have finite expectation values. From this we conclude that the LE and WI are rigorously valid. In section 2.2 we obtained the LE for $|I| < 4$. They left a number of correlations undetermined. In section 4.3.4 we found that linear non-anomalous symmetries of this model form the $o(2)$ Lie algebra parameterized by v_j^i such that $v_1^1 = v_2^2 = 0$ and $v_2^1 = -v_1^2$. The corresponding WI, which we will use to supplement the LE, read $TG_K = 0$ for all words K , where $T = L_1^2 - L_2^1$. These are listed in appendix B for moments of rank up to 4. They imply that all G_i vanish. The only G_{ij} that might be non-vanishing are $G_{11} = G_{22}$. All 3-point G_{ijk} vanish. 4-point correlations vanish except possibly $G_{1111}, G_{2222}, G_{1212}, G_{1122}$ and their cyclic permutations. They must, however, satisfy the relations $G_{1111} = G_{2222}$ and $G_{1111} = 2G_{1122} + G_{1212}$. Some of these conditions could also have been got from the LE, (2.2). We need one more condition on rank-2 moments and two more conditions on rank-4 moments to determine all moments of rank ≤ 4 . The LE for $|I| = 1$ gives one new condition

$$G_{11} = 1 + \frac{1}{\alpha}(2G_{1212} - 2G_{1122}). \quad (5.5)$$

The LE for $|I| = 2$ (section 2.2) relate 3 and 5 point correlations. Using the fact that all 3-point correlations vanish, they tell us that $G_{11212} = G_{11122}$ and $G_{12122} = G_{11222}$. Supplementing these LE with the WI for 5-point correlations $TG_{ijklm} = 0 \forall ijklm$ (which we do not list explicitly), we are able to conclude that all rank-5 correlations vanish.

Thus far, we have found that the only correlations with rank ≤ 5 that could be non-vanishing are $G_{11}, G_{22}, G_{1111}, G_{2222}, G_{1212}$ and G_{1122} , up to cyclic permutations. We have found 4 relations among these 6 unknowns:

$$G_{11} = G_{22}, G_{1111} = G_{2222}, G_{1111} = 2G_{1122} + G_{1212}, G_{11} = 1 + \frac{1}{\alpha}(2G_{1212} - 2G_{1122}) \quad (5.6)$$

Thus, by use of the WI, we have reduced the underdeterminacy of the LE. We could proceed further in this manner. The LE for $|I| = 3$ relate rank-4 and rank-6 moments, while the WI for rank-6 moments give further conditions on rank-6 moments. We could also look for additional conditions using the WI from quadratic symmetries found in section 4.4.4, but we postpone that. Our purpose here was only to illustrate the general framework we have developed. In a separate paper, we hope to return to a more thorough study of the correlations of this model using the LE and WI and their comparison with other approaches [26, 27] or monte-carlo simulations [28].

5.3 Chern-Simons 3-matrix model

The CS 3-matrix model has action $\text{tr } S(A) = \frac{2i\kappa}{3} \epsilon^{ijk} \text{tr } A_{ijk} = 2i\kappa \text{tr } A_1[A_2, A_3]$. We expect its matrix integrals to diverge. To see this, go to a basis where A_2 is diagonal, then the action is independent of the diagonal elements of A_3 , due to the commutator. So integration over the diagonal elements of A_3 would diverge. Our derivation of the LE and WI holds at best formally for this model. We do not know whether the LE and WI are a consistent system for this action. Nevertheless, we consider them formally to illustrate the general framework. We find no inconsistency, at least for correlations up to rank 3. The LE of the CS 3-matrix model are under determined (section 2.2). WI corresponding to non-anomalous linear symmetries (4.12) are obtained by imposing the conditions $L_v G_K = 0$ for all K and traceless v_j^i :

$$v_i^j \delta_K^{K_1 i K_2} G_{K_1 j K_2} = 0 \quad \forall \text{ traceless } v_j^i. \tag{5.7}$$

These are the conditions that G_K be (cyclic and hermitian) invariant tensors of $SL_3(\mathbf{R})$. We will work out the WI explicitly for $|K| = 0, 1, 2, 3$. For K empty, this is a vacuous condition, so put $K = k$ to get the WI $v_k^j G_j = 0$ for all traceless v_k^j . There are traceless v_k^j with non-vanishing determinant, and G_j must lie in the kernel of such linear transformations. But this kernel is trivial, so $G_j = 0$. For $K = kl$ we get

$$v_l^j G_{kj} + v_k^j G_{jl} = 0 \quad \forall \text{ traceless } v. \tag{5.8}$$

Putting $k = l = 1$ we get $v_1^1 G_{11} + v_1^2 G_{12} + v_1^3 G_{13} = 0$. But this must hold for all real v_1^1, v_1^2, v_1^3 so that $G_{11} = G_{12} = G_{13} = 0$. Putting $k = l = 2$ we get $G_{21} = G_{22} = G_{23} = 0$. Finally putting $k = l = 3$ we get $v_3^1 G_{31} + v_3^2 G_{32} - (v_1^1 + v_2^2)$. Again $v_3^1, v_3^2, v_1^1, v_2^2$ are freely specifiable so that $G_{31} = G_{32} = G_{33} = 0$. From this we conclude that all $G_{ij} = 0$. This is of course consistent with the remaining WI gotten by putting $k = 1, l = 2$ etc since $G_{ij} = 0$ is an obvious solution of the homogeneous system $v_l^j G_{kj} + v_k^j G_{jl} = 0$.

As for WI²⁵ for rank-3 correlations,²⁶ we set $K = klm$ and get

$$v_k^j G_{jlm} + v_l^j G_{kjm} + v_m^j G_{klj} = 0 \quad \forall \text{ traceless } v. \tag{5.10}$$

$k = l = m = 1$ gives $v_1^1 G_{111} + v_1^2 G_{112} + v_1^3 G_{113} = 0$ whence $G_{111} = G_{112} = G_{113} = 0$. Similarly, putting $k = l = m = 2$ and $k = l = m = 3$ we get $G_{122} = G_{222} = G_{223} = G_{133} = G_{233} = G_{333} = 0$. The only remaining undetermined rank-3 correlations are G_{123} and G_{132} . The remaining WI are either vacuous (e.g. $k = 1, l = 2, m = 3$) on account of v being traceless or (e.g. $k = l = 1, m = 2$) give $G_{123} + G_{132} = 0$.

²⁵ L_i^i (no sum over i) is a number operator, it counts the number of i 's in a given moment. For example, consider $G_{I_i J_i K_i L}$, where none of the multi-indices I, J, K, L contain an i . Then

$$L_i^i G_{I_i J_i K_i L} = G_{I_i J_i K_i L} + G_{I_i J_i K_i L} + G_{I_i J_i K_i L} = 3G_{I_i J_i K_i L}. \tag{5.9}$$

The number operators commute $[L_i^i, L_j^j] = 0$. Thus $\sum_i L_i^i$ measures the rank of a given moment. This can be used to get most of the rank-3 moments of the CS model by employing WI involving $L_1^1 - L_2^2$ and $L_1^1 - L_3^3$.

²⁶Accounting for cyclicity and hermiticity, the space of rank-3 tensors is 11 dimensional, see appendix D.

To summarize, the WI due to linear non-anomalous symmetries imply that all correlations of rank ≤ 3 vanish except for G_{123} and G_{132} (and their cyclic permutations), and these are related by the WI $G_{123} + G_{132} = 0$. WI remedy the underdeterminacy of LE of the CS model. The only non-trivial condition from the LE was (see section 2.2.2 of ref. [20])

$$\Im(G_{123} - G_{132}) = -\frac{1}{2\kappa} \tag{5.11}$$

This, along with the WI $G_{123} + G_{132} = 0$ now allows us to determine *all* correlations up to rank 3, the only non-vanishing ones (up to cyclic symmetry) are²⁷

$$G_{123} = \frac{1}{4i\kappa} \quad \text{and} \quad G_{132} = -\frac{1}{4i\kappa} \tag{5.12}$$

We checked that this result is consistent with the WI corresponding to quadratic non-anomalous symmetries obtained in section 4.4.2. So at least up to rank-3 moments, the WI cure the underdeterminacy problem of the LE! We could proceed in this manner to higher rank correlations.

5.4 2-matrix Yang-Mills: a cautionary tale

The matrix integrals for correlations of the YM 2-matrix model $\text{tr} S(A) = -\frac{1}{2\alpha} \text{tr} [A_1, A_2]^2$ do not converge [25] due to a similar argument as given for the CS (section 5.3). Thus, our derivation of the WI and LE is not strictly valid. We cannot be certain that they form a consistent system. In fact, we find that the WI and LE for this model do not form a consistent system when considering rank-4 correlations. Despite several checks, we could find no calculational error. We do not know the deeper reason for this inconsistency, but suspect it could have something to do with the lack of convergence of matrix integrals invalidating our derivation of the WI and LE. Thus, it is probably good to be cautious in formal use of the WI and LE.

The LE of the YM 2-matrix model are underdetermined, (section 2.2). Recall that the LE do not determine any moments of rank 1, 2 or 3. Here, the WI come to the rescue. Recall (4.3.3), that the non-anomalous linear symmetries of this model form the Lie algebra $sl_2(\mathbf{R})$, spanned by $L_1^1 - L_2^2, L_2^1, L_1^2$. The WI $L_v G_K = 0$ for $|K| \leq 3$ suffice to determine all 1, 2, and 3 point correlations, and imply they are all zero. Let us also consider WI for moments of rank 4:

$$\begin{aligned} (L_1^1 - L_2^2)G_{1111} &= 4G_{1111} = 0, & (L_1^1 - L_2^2)G_{2222} &= -4G_{2222} = 0, \\ (L_1^1 - L_2^2)G_{1112} &= 2G_{1112} = 0, & L_2^1 G_{1112} &= G_{1212} + 2G_{1122} = 0, \\ L_2^1 G_{1212} &= 2G_{2221} = 0 & \Rightarrow G_{2221} &= 0. \end{aligned} \tag{5.13}$$

While the WI determine many rank-4 correlations, they give only one relation between G_{1212} and G_{1122} . To fix them we use the simplest conditions coming from the first of the two LE

$$I = 1 : \quad 2G_{1212} - 2G_{1221} = -\alpha ; \quad I = 2 : \quad 2G_{2212} - 2G_{2221} = 0. \tag{5.14}$$

²⁷Hermiticity and cyclicity mean $G_{123}^* = G_{132}$ which implies $\Re G_{123} = \Re G_{132}$ and $\Im G_{123} = -\Im G_{132}$

The second equation is vacuous, but thanks to the WI we see $G_{1222} = 0$. As for the first equation, WI provides us with the condition $G_{1212} + 2G_{1122} = 0$. So we get

$$G_{1212} = -\frac{\alpha}{3} \quad \text{and} \quad G_{1122} = \frac{\alpha}{6}. \tag{5.15}$$

However, this is not consistent with the WI for non-anomalous quadratic symmetries (section 4.4.3). In particular the WI obtained from linear symmetries together with

$$\left(L_{12}^1 - \frac{1}{2}L_{22}^2\right)G_{112} = G_{1212} + \frac{1}{2}G_{1122} = 0 \tag{5.16}$$

implies $G_{1212} = G_{1122} = 0$, which is a contradiction.

6. Some outstanding questions

A summary and discussion of the results of this paper was given in the introduction. Here, we list some questions raised by our work. (1) We have only addressed the exact determination of normalized correlations in large- N matrix models using the LE and WI. But what about the partition function or free energy? (2) It is interesting to know whether the LE and WI together determine all single-trace correlations in the large- N limit. (3) We have only discussed infinitesimal non-anomalous symmetries. Many models also possess discrete non-anomalous symmetries, which lead to useful relations among correlations. Some of these relations are actually a consequence of the LE or WI. But in general, it may be necessary to supplement the LE and WI by conditions from discrete symmetries. (4) Detailed study of LE and WI of specific multi-matrix models should clarify whether we need additional conditions to solve for all correlations. (5) It is interesting to identify matrix models with a maximal family of non-anomalous symmetries. Interestingly, we found that the 3-matrix CS model and the 2-matrix commutator-squared YM models each possesses a maximal family of linear and quadratic non-anomalous symmetries. (6) It is interesting to classify the solutions to the simplest WI. For example, the correlations that satisfy WI for linear symmetries of the Gauss+YM model must be invariant cyclic hermitian tensors of the orthogonal Lie algebra. What is the general form of such tensors? (7) We observed that for $n > 1$, Lie brackets of rank n non-anomalous vector fields are rank $n + 1$ non-anomalous vector fields, provided they are non-vanishing. It would be interesting to study this Lie algebra of non-anomalous symmetries in specific examples. Can it be infinite dimensional? If so, might the model be integrable in some sense? (8) We wonder whether the full gauge fixed Yang-Mills theory in the large- N limit has any additional non-anomalous symmetries besides Poincare invariance and BRST invariance. Our work indicates that such symmetries can be far from obvious and highly non-linear.

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A. Alternative derivation of SDE preserving hermiticity

Consider the change of variables corresponding to translation by a constant Hermitian matrix ϵ

$$(A_i)_b^a \rightarrow (A_i)_b^a + \epsilon_b^a, \quad \text{and} \quad (A_j)_b^a \rightarrow (A_j)_b^a \quad \text{for } j \neq i \quad (\text{A.1})$$

in the integral ($[dA] = \prod_k dA_k$ is the Lebesgue measure on independent matrix elements)

$$I = \int [dA] (A_I)_b^a \Phi_K e^{-N \text{tr} S^J A_J}. \quad (\text{A.2})$$

Here $\Phi_K = \frac{\text{tr}}{N} A_K$. The value of the integral should be unaltered under this change of variables²⁸

$$\begin{aligned} \delta I &= \epsilon_d^c \int [dA] \delta_I^{I_1 i I_2} (A_{I_1})_c^a (A_{I_2})_b^d \Phi_K e^{-N \text{tr} S^J A_J} \\ &+ \epsilon_d^c \int [dA] (A_I)_b^a \delta_K^{LiM} \frac{1}{N} (A_{ML})_c^d e^{-N \text{tr} S^J A_J} \\ &- \epsilon_d^c \int [dA] (A_I)_b^a \Phi_K N S^J \delta_J^{J_1 i J_2} (A_{J_2 J_1})_c^d e^{-N \text{tr} S^J A_J} = 0 \end{aligned} \quad (\text{A.3})$$

where we also used the translation invariance of the measure. Since this holds for arbitrary Hermitian ϵ we conclude²⁹

$$\delta_I^{I_1 i I_2} \langle (A_{I_1})_c^a (A_{I_2})_b^d \Phi_K \rangle + \frac{1}{N} \delta_K^{LiM} \langle (A_I)_b^a (A_{ML})_c^d \rangle = N S^{J_1 i J_2} \langle (A_I)_b^a \Phi_K (A_{J_2 J_1})_c^d \rangle. \quad (\text{A.4})$$

Contracting a with c and b with d , and dividing both sides by N^2 we get

$$\delta_I^{I_1 i I_2} \langle \Phi_{I_1} \Phi_{I_2} \Phi_K \rangle + \delta_K^{LiM} \frac{1}{N^2} \langle \Phi_{LIM} \rangle = S^{J_1 i J_2} \langle \Phi_{J_1 I J_2} \Phi_K \rangle. \quad (\text{A.5})$$

Since this must hold for every I and i , it is equivalent to the equations

$$v_i^I \delta_I^{I_1 i I_2} \langle \Phi_{I_1} \Phi_{I_2} \Phi_K \rangle + v_i^I \delta_K^{LiM} \frac{1}{N^2} \langle \Phi_{LIM} \rangle = v_i^I S^{J_1 i J_2} \langle \Phi_{J_1 I J_2} \Phi_K \rangle \quad \forall \quad v_i^I \in \mathbf{R}. \quad (\text{A.6})$$

In a completely analogous fashion, we repeat this calculation with several insertions $\Phi_{K_1} \cdots \Phi_{K_n}$ and get the Schwinger-Dyson equations obtained earlier in (2.8).

²⁸The following formula is useful: $\frac{\partial (A_I)_b^a}{\partial (A_i)_c^d} = \delta_I^{I_1 i I_2} (A_{I_1})_e^a \delta_f^d \delta_c^e (A_{I_2})_b^f = \delta_I^{I_1 i I_2} (A_{I_1})_c^a (A_{I_2})_b^d$. Contracting a with b one gets $\frac{\partial \text{tr} A_I}{\partial (A_i)_c^d} = \delta_I^{I_1 i I_2} (A_{I_1})_c^a (A_{I_2})_a^d = \delta_I^{I_1 i I_2} (A_{I_2 I_1})_c^d$.

²⁹It is shown in Adler's book [29] page. 26 that if $\text{tr}(\epsilon H) = 0$ for all Hermitian ϵ then $H = 0$

A.1 Other possible changes of variables in matrix integrals

LE are underdetermined, so do the G_I satisfy other equations? The WI *are* such equations and with the LE, may go a long way towards fixing G_I . Here we consider two other types of changes of variable in matrix integrals to see if they give new equations. However, we do not find any.

Consider an infinitesimal change $\delta A_i = v_i^I A_I$ where v_i^I is a Hermitian matrix for each I & i ; previously they were real numbers. Conditions for invariance of partition function are

$$-NS^{I_1 i I_2} \text{tr} (A_{J I_2 I_1} v_i^J) + \delta_J^{J_1 j J_2} \text{tr} (v_j^J A_{J_1}) \text{tr} A_{J_2} = 0 \quad (\text{A.7})$$

Since these must hold for arbitrary matrix elements $[v_i^I]_b^a$ we get

$$-NS^{I_1 i I_2} (A_{J I_2 I_1})_b^a + \delta_J^{J_1 j J_2} (A_{J_1})_b^a \text{tr} A_{J_2} = 0 \quad (\text{A.8})$$

But these are not equations for trace invariants. To get equations for G_I , we must take a trace, divide by N^2 and take expectation values. But this leads to the LE derived before.

Next we consider an infinitesimal change $\delta A_i = [v_i^I, A_I]$ with arbitrary hermitian matrices v_i^I . These types of change of variable do not appear in BRST transformations but do appear in gauge transformations $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda + [A_\mu(x), \Lambda(x)]$. Invariance of partition function implies

$$\delta_I^{I_1 j I_2} \text{tr} (v_i^I A_{I_1}) \text{tr} A_{I_2} - \delta_I^{I_1 j I_2} \text{tr} A_{I_1} \text{tr} (v_i^I A_{I_2}) + S^{I_1 j I_2} [A_{I_2 I_1}, A_I] = 0. \quad (\text{A.9})$$

Using the arbitrariness of v_i^I we get

$$\delta_I^{I_1 j I_2} (A_{I_1})_b^a \text{tr} A_{I_2} - \delta_I^{I_1 j I_2} \text{tr} A_{I_1} (A_{I_2})_b^a = S^{I_1 j I_2} [A_{I_2 I_1}, A_I]_b^a. \quad (\text{A.10})$$

As before, we must take a trace to get an equation for the G_I , but in fact we get a triviality. Thus, we have not found any equations for G_I in addition to the LE and WI.

B. WI for Gaussian+YM model

Below is a list of Ward identities $TG_K = 0$ for moments of rank up to $|K| = 4$ in the 2-matrix Gaussian+YM model. They correspond to the non-anomalous linear vector field $T = L_1^2 - L_2^1$.

$$\begin{aligned} TG_1 &= -G_2 = 0; & TG_2 &= G_1 = 0; & TG_{11} &= -G_{21} - G_{12} = -2G_{12} = 0 \Rightarrow G_{12} = 0; \\ TG_{12} &= G_{11} - G_{22} = 0 \Rightarrow G_{11} = G_{22}; & TG_{22} &= 2G_{12} = 0 \Rightarrow G_{12} = 0; \\ TG_{111} &= -3G_{112} = 0 \Rightarrow G_{112} = 0; & TG_{112} &= G_{111} - 2G_{122} = 0 \Rightarrow G_{111} = 2G_{122} = 0; \\ TG_{122} &= 2G_{112} - G_{222} = 0 \Rightarrow G_{222} = 2G_{211} = 0; & TG_{222} &= 3G_{122} \Rightarrow G_{122} = 0; \\ TG_{1111} &= -4G_{1112} = 0; & TG_{1112} &= G_{1111} - 2G_{1122} - G_{1212} = 0 \Rightarrow G_{1111} = 2G_{1122} + G_{1212}; \\ TG_{1122} &= -2G_{1222} + 2G_{1112} = 0 \Rightarrow G_{1112} = G_{1222}; & TG_{1212} &= 2G_{1112} - 2G_{2221} = 0; \\ TG_{1222} &= 2G_{1122} + G_{1212} - G_{2222} = 0; & \text{and } TG_{2222} &= 4G_{1222} = 0. \end{aligned} \quad (\text{B.1})$$

C. Is there a model for which Ward identities determine all G_I ?

Are there any non-trivial multi-matrix models where the WI are sufficiently numerous to determine all (or a maximal set of) correlations without using the LE? We seek models with a very large family of non-anomalous symmetries. Trying to answer the corresponding question in two dimensional quantum field theory has proven very fruitful, as evidenced by the progress in 2d conformal field theory. The latter are so symmetrical that a maximal family of correlations can be determined by milking conformal invariance. Here we make an elementary observation. The WI $L_\nu G_K = 0$ are a system of homogeneous linear equations. So they are either underdetermined (if the determinant of the system vanishes) or admit only the trivial solution $G_K = 0, \forall K$. Though the WI can give us much information on correlations, they cannot determine all of them except in the trivial case where they are all zero. For example in a 2-matrix model, if we consider an extreme (and probably unrealistic) case where all measure preserving vector fields are also action preserving,³⁰ then it follows that all correlations of rank up to 4 vanish. This leaves open the question of identifying non-trivial models with a maximal family of non-anomalous symmetries, i.e. the ones for which the WI are most useful.

D. Cyclically symmetric tensors of rank n

The real dimension of the space of cyclic hermitian tensors $G_{i_1 \dots i_n}$ on a vector space V of dimension Λ (i.e. $1 \leq i_1, \dots, i_n \leq \Lambda$) is

$$c(n, \Lambda) = \frac{1}{n} \sum_{d|n} \phi(d) \Lambda^{n/d} \tag{D.1}$$

where $\phi(d)$ is Euler's totient phi-function³¹ and the sum is over all divisors of n . $c(n, \Lambda)$ is the number of independent correlations of rank- n in a Λ matrix model. We thank the mathematician Jean Yves Thibon of Université de Marne-la-Vallée, France for sharing this formula with us. This answers a question posed in appendix A of ref. [20]. It comes from the character of $GL(V)$

$$\frac{1}{n} \sum_{d|n} \phi(d) p_d^{n/d}(\xi) \quad \text{where} \quad p_d(\xi) = \sum_{i=1}^{\Lambda} \xi_i^d \tag{D.2}$$

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³⁰There may be no non-trivial action with this property. We find the YM 2-matrix model and CS 3-matrix models are each such that every measure preserving linear and quadratic symmetry is also action preserving.

³¹ $\phi(d)$ = number of positive integers less than or equal to d and coprime to d .

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