

A Model of Interacting Partons for Hadronic Structure Functions ¹

by

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Abstract

We present a model for the structure of baryons in which the valence partons interact through a linear potential. This model can be derived from QCD in the approximation where transverse momenta are ignored. We compare the valence quark distribution function predicted by our model with that extracted from global fits to Deep Inelastic Scattering data. The only parameter we can adjust is the fraction of baryon momentum carried by valence partons. Our predictions agree well with data except for small values of the Bjorken scaling variable.

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1 Introduction

This thesis is about the structure of the proton. The proton, along with the neutron is one of the constituents of the atomic nucleus. For a long time since their discovery, it was not known whether the proton and neutron had substructure, and if so, what their constituents were like. The Deep Inelastic Scattering Experiments [1] of the early 1970s found that the proton was made of point-like constituents called quarks or partons.

These experiments were similar in spirit to the alpha particle scattering experiments of Rutherford, which established that the atom contains a point-like nucleus. He scattered alpha particles against a thin gold foil and found that there was a small probability for them to scatter through wide angles. This would not be possible if the positive charge in an atom was uniformly distributed. Moreover, the scattering cross-section he measured was that of a point-like nucleus carrying all the positive charge of the atom! Subsequent experiments showed that the nucleus was made of protons and neutrons. Charge radii measured in elastic electron-proton scattering showed that the proton was not elementary. What were its constituents like?

The Deep Inelastic Scattering experiments scattered electrons against protons. This time, the scattering was inelastic. The inclusive scattering cross-section was measured and expressed in terms of ‘structure functions’. The structure functions describe the structure of the proton. These experiments used the electromagnetic force between the electron and the constituents of the proton to study the ‘strong force’ that held the proton together. The electromagnetic force is mediated by the exchange of a photon. By making the wave-length of the photon small enough, it was possible to ‘look’ deep within the proton.

The startling discovery of the Deep Inelastic Scattering Experiments was that as long

as one looked closely enough, the structure functions did not depend on the wave-length of the photon! The constituents of the proton did not have any length-scale associated with them! This phenomenon is called scaling: the constituents of the proton appeared point-like.

The parton model was proposed by Bjorken, Feynman and others [2] as a simple explanation of scaling in Deep Inelastic Scattering. The proton was thought of as being made of point-like constituents called partons. These were identified with the quarks which were until then, hypothetical particles. Structure functions can be expressed in terms of parton distribution functions. These describe the distribution of partons within the proton.

Soon after, Quantum ChromoDynamics (QCD), the fundamental theory of strong interactions was discovered. Scaling was understood as a consequence of asymptotic freedom in QCD [3]. Small violations of scaling were predicted by perturbative QCD. These have been confirmed by experiments. However, the initial parton distributions cannot be calculated within perturbative QCD. They have as yet not been understood theoretically. However, these parton distributions are so important to high energy hadron collisions that they have been measured and extracted from experimental data in great detail [4, 5, 6].

Understanding the structure functions of quarks in a proton is not unlike understanding the orbits of planets around the sun or the wavefunctions of electrons in an atom. Understanding the latter has proven extremely fruitful in all of science. In the case of the proton, we are dealing with the strong force rather than the gravitational or electromagnetic.

In this thesis, we shall present a model of interacting partons to explain the distribution of valence quarks in the proton. The quarks are assumed to be relativistic particles interacting with each other through a linear potential in the ‘null’ coordinate. Their momenta transverse to the direction of the collision will be ignored in favour of the much larger longitudinal momentum. That collinear QCD can describe hadronic structure functions has also been proposed by others [7]. The ground state wave function of the proton will be the one that minimizes its energy. We will approximate the proton wave function with a product of single parton wave functions. This is analogous to the Hartree approximation in Atomic Physics. These approximations will allow us to determine the parton wave functions

as the solution to a non-linear integral equation. We shall study this equation in several ways (both analytic and numerical) and obtain a fairly complete quantitative picture of the valence quark distribution.

Finally we will compare our predictions with the parton distributions extracted from experimental data. Considering that we have at-most one parameter to adjust (the fraction f , of proton momentum carried by the valence quarks) the agreement is quite good, except for small values of the momentum fraction x_B . In this region, our model is not expected to be accurate. We may not ignore sea-quarks, gluons and transverse momenta.

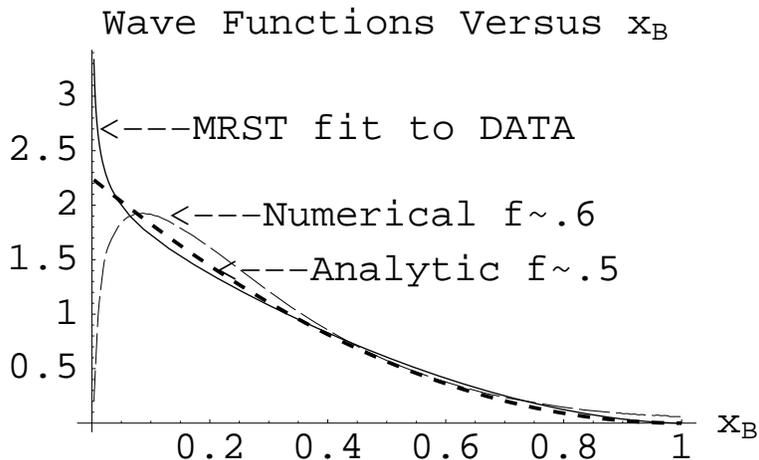


Figure 1. Comparison of the predicted valence parton wavefunction $\sqrt{\phi(x_B)}$ with the MRST [5] global fit to data. The wavefunction we predict goes to a non-zero constant at the origin. The ‘analytic’ prediction is obtained as a variational approximation. The numerical solution is not reliable in this region of small x_B . The fit to data has a mild divergence at $x_B = 0$.

These results have been described in a recent publication [8]. Furthermore, this interacting parton model can be derived from QCD [9] in the limit where transverse momenta are ignored.

The question of deriving the spectrum and structure functions of hadrons from QCD is an old and important one. The meson spectrum and wave functions of two dimensional QCD were obtained by ’t Hooft [10] in 1974 by a clever summation of planar Feynman diagrams in the large N limit. Perturbation theory works in the case of mesons since they are described by small fluctuations

from the vacuum. However, the baryons remained elusive. Later, Witten [11] suggested that the baryon can be described by a Hartree Fock approximation in the large N limit of QCD. He carried out this idea in a non-relativistic context. In the early 1980s, Skyrme's [12] idea that the baryon is a topological soliton was revived by Balachandran and others [13] and shown to be consistent with QCD. Rajeev [9] developed Quantum HadronDynamics (QHD) in two dimensions. Two dimensional QHD is an equivalent reformulation of two dimensional QCD in terms of observable particles: the hadrons, rather than the quarks, which are confined within the hadrons. Baryons are the topological solitons of QHD while mesons are small fluctuations of the vacuum. The large N limit of 2dQHD reproduces [9] the meson spectrum and wave functions of 't Hooft. But it also allows us to predict the structure of the baryon. Within a variational approximation, the structure of the baryon can be estimated by a non-linear integral equation [9]. It was found that this integral equation also had a simple derivation in terms of the parton model [8, 9]. We present this parton model point of view here.

This thesis is organised into 3 main chapters. Chapter 2 provides background, a discussion of Deep Inelastic Scattering and the parton model. In Chapter 3 we present the interacting parton model and derive the integral equation satisfied by the valence quark wave function. Chapter 4 discusses how we solve for the valence quark wave function and comparison with experimental data. We do not assume much familiarity with particle physics. We will often use terms that are specialized to this branch of physics, but most of them are defined or explained at some point.

2 Deep Inelastic Scattering

In electron-proton Deep Inelastic Scattering for instance, a high energy electron scatters against a proton. The electron does not feel the strong force and its weak interaction is dominated by the electromagnetic interaction, which is mediated by the exchange of a high energy virtual photon. Thus the kinematics is described in terms of two 4-vectors: q^μ the momentum of the photon and p^μ the momentum of the proton. The photon momentum is space-like, $q^2 < 0$ while p^μ is time-like, $p^2 = M_p^2 > 0$. M_p is the rest mass of the proton. The scattering is inelastic. The proton disintegrates producing several hadrons in the final state:

$$eP \rightarrow eX$$

If we sum over all possible final states X, we get the inclusive deep inelastic cross-section, which is expressed in terms of two dimensionless scalar ‘structure functions’ F_1 and F_2 . Being Lorentz scalars, they can depend only on p^2 , $p \cdot q$ and q^2 . $p^2 = M_p^2$ is fixed by the mass of the proton. For convenience, we may take our two independent scalars as $Q^2 = -q^2$, and the dimensionless ratio $x_B = \frac{Q^2}{2p \cdot q}$, called the Bjorken scaling variable.

$$\text{So } F_{1,2} = F_{1,2}(x_B, Q^2).$$

Q^2 , being the square of the momentum transferred by the photon, sets the energy scale of the experiment. Q is inversely proportional to the wave length of the photon. An experiment at large Q^2 is therefore looking deep inside the proton. If $Q^2 \gg \frac{1}{a^2}$, we are in the deep inelastic region. $\frac{1}{a} \sim 100 \text{ MeV}$ is the inverse charge radius of the proton in its rest frame.

In the deep inelastic region, we will show that x_B can be thought of as the fractional momentum of the quark inside the proton that scatters the photon. This will be made clear in the context of the parton model to be discussed soon.

The Deep Inelastic Scattering experiments of the early 1970s [1] showed that at sufficiently large $Q^2 \gg \Lambda_{QCD}^2$, the structure functions were approximately independent of Q^2 . They depended,

not on the Lorentz Scalars Q^2 or $2p.q$ separately, but only on their ratio x_B . This phenomenon is called Bjorken scaling.

The structure functions may be expressed in terms of parton distribution functions [14]. They directly describe the structure of the proton. The parton distribution functions $\phi_a(x_B, Q^2)$ are momentum space probability distribution functions. For instance, the up quark parton distribution function gives the probability density of finding an up quark carrying a fraction x_B of proton momentum, when the proton is probed at the energy scale Q^2 . As with the structure functions, the parton distribution functions are essentially independent of Q^2 . Being probability distributions, they are the absolute squares of the parton wave functions inside the proton. The reduction from structure functions to parton distribution functions is made possible by ignoring certain correlations and the factorization theorem of perturbative QCD [15, 14]. Roughly speaking, the amplitude for the virtual photon to scatter off the proton is expressed as a sum of products of two factors: the amplitude for it to scatter elastically off a quark of given momentum and the probability of finding a quark with that momentum in the proton. While the elastic scattering off a quark of a given momentum is calculable in perturbation theory, the probability of finding such a quark in the proton is non-perturbative and yet to be calculated from first principles.

2.1 The Quark Model and Quantum ChromoDynamics

The quark model, based on hadron spectroscopy suggested that for the purpose of quantum numbers, the proton is a colorless combination of 2 flavours of quarks: up, up and down. The quarks are fermions. In addition to quantum numbers like position, spin and charge, quarks carry flavour (up, down, strange etc.) and color quantum numbers. Color is the analogue of electric charge in the theory of strong interactions. The number of colors N is 3 in nature. Quantum ChromoDynamics (QCD) is the presently accepted theory of the color force. The strong force is mediated by the exchange of spin 1 bosons called gluons. It was developed partly in analogy with Quantum ElectroDynamics (QED) which is the theory of the electromagnetic force. QCD is a quantum field theory with a non-abelian gauge symmetry. While its basic framework is understood, it has proven very hard to solve and make predictions about non-perturbative effects.

The proton is just one of several particles which take part in strong interactions. They are collectively called hadrons. Hadrons are broadly classified according to their spin. Integer spin

bosonic hadrons are called mesons while the half integer spin fermions are called baryons. The proton and neutron are the lightest and most common baryons. The pions are the lightest mesons. Free quarks have never been observed in nature: they are confined within the hadrons, which are colorless combinations of quarks.

In the late 1960s and early 70s, the parton model for hadrons was proposed by Bjorken, Feynman and others [2] to explain the phenomenon of scaling in Deep Inelastic Scattering. According to the original parton model, in the deep inelastic region, the proton behaved as though it was made up of essentially free point-like constituents called partons. These partons were identified with the quarks (u, u and d in the case of the proton).

When probed at even higher energies (larger Q^2) and small x_B , it was found that there is a small probability of finding anti-quarks (such as \bar{u} and \bar{d}) and even quarks of other flavours (strange quarks for instance) inside the proton. These did not fit directly into the original parton model. However, these additional probability distributions were measured and are described in terms of a phenomenological extension of the original parton model. We now speak of valence, sea, anti-quark and gluon distributions in the proton. The valence quarks are the 3 quarks (up, up and down in the case of the proton) of the original parton model. They are named in a loose analogy with the valence electrons of the atom. One needs to probe deeper into the proton (higher Q^2) before ‘sea’ quarks and anti-quarks become significant.

In this paper we shall primarily be interested in developing a model that predicts the x_B dependence of valence quark wave functions. Since this is the main goal, we shall ignore some other details and small corrections, which though important and measurable, obscure the main point. For instance, the wave function depends on the isospin of the baryon (proton or neutron). We shall calculate a single valence quark wave function ignoring isospin effects, which must therefore be compared to the isoscalar combination of experimentally measured up and down quark wave functions in the baryon. Correcting for isospin effects is not hard, but will not be addressed here.

In the physics of strong interactions, our experimental knowledge and precision of measurements far exceeds our theoretical understanding in most areas. In contrast with QED, where theory and experiment agree to an unprecedented degree, in the non-perturbative regime of QCD we are barely at the qualitative or 10% level!

2.2 Deep Inelastic Scattering and the Parton Model

Let us now return to the kinematics of deep inelastic collisions. Up to now we described the scattering in Lorentz invariant language. To understand the parton model of Feynman better, it will be useful to consider the ‘infinite-momentum’ frame.

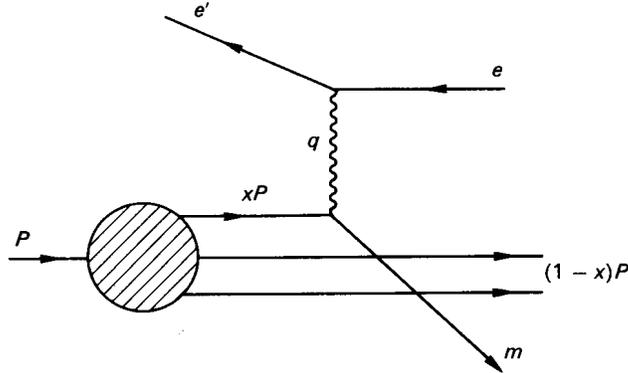


Figure 2. A parton model schematic of a Deep Inelastic Collision. Figure taken from [16].

In this inertial reference frame, the proton has a very large 3-momentum \mathbf{P} and we may ignore its rest mass in comparison $p^\mu = (|\mathbf{P}|, \mathbf{P})$. The proton is thought of as consisting of several point-like constituents, the partons. The 4-momentum of a given parton is expressed as xp^μ where $0 \leq x \leq 1$ is the fraction of proton momentum carried by the given parton. Deep Inelastic Scattering is visualized as initially, the *elastic* scattering of the photon against a parton. The partons are assumed to have negligible mass m . This is followed by a complicated process of *hadronization* in which the partons recombine to form several hadrons in the final state X .

From our point of view, that of predicting parton distribution functions, there is an important simplification that Deep Inelastic Collisions allow. The component of momentum in the direction of the collision far exceeds those transverse to it. Therefore, it is reasonable to ignore the transverse components of the parton momenta. It is this approximation that makes the problem of predicting hadronic structure functions tractable from a theoretical point of view. We are essentially dealing with a problem in 2 space-time dimensions.

The parton model also gives us a very useful physical interpretation of the Bjorken scaling variable, x_B . x_B was one of the two Lorentz scalars used to parametrize the structure functions. We will show that $x_B = \frac{Q^2}{2p \cdot q}$ is the same as the fraction x of proton momentum p carried by the

parton that scatters off the photon. Momentum conservation implies that $(xp + q)^2 = m^2$ where p is the 4-momentum of the proton, q the momentum of the photon and m the mass of the struck quark. But m is negligibly small and $x^2 p^2 = x^2 M_p^2$ where M_p is the mass of the proton. In the deep inelastic region, $Q^2 \gg M_p^2$ and therefore $x \sim \frac{Q^2}{2p \cdot q}$ which we recognize as the Bjorken scaling variable x_B . Therefore we may interpret x_B in the deep inelastic region as the fraction of proton 4-momentum carried by a parton.

This parton model interpretation of x_B as the fraction of proton momentum carried by a parton suggests that $0 \leq x_B \leq 1$ at least in the deep inelastic region. This is in fact true in general. It follows from conservation of 4-momentum in a collision and the fact that the proton is the lightest baryon. Since x_B is a Lorentz scalar, it may be evaluated in any inertial reference frame. In the lab frame, a photon of 4-momentum $q^\mu = (\nu, \mathbf{q})$ collides inelastically with a proton of mass M_p at rest. The invariant (mass)² of the final state X is W^2 .

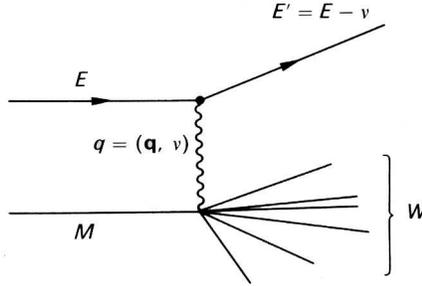


Figure 3. Kinematics of a Deep Inelastic Collision in the lab frame. Figure taken from [16].

Even in an inelastic collision, 4-momentum is conserved:

$$[(\nu, \mathbf{q}) + (M, \mathbf{0})]^2 = W^2$$

Therefore, $q^2 + 2M\nu + M^2 = W^2$, where $q^2 = q^\mu q_\mu$. Writing Q^2 for $-q^2$ we see that $Q^2 = 2M\nu - (W^2 - M^2)$. Now the proton is the lightest baryon, while the final state X includes several heavier hadrons, hence $W^2 \geq M^2$ and $x_B = \frac{Q^2}{2M\nu} \leq 1$. Moreover, since the energy transfer $\nu \geq 0$ and $Q^2 \geq 0$, we have $x_B \geq 0$. Therefore, $0 \leq x_B \leq 1$ as desired.

While we are still in the lab frame, it will be good to point out precisely what the deep inelastic limit is. We have $x_B = \frac{Q^2}{2M\nu}$. The deep inelastic region of the $x_B - Q^2$ parameter space is the limit of large Q^2 and ν keeping x_B fixed.

In the deep inelastic limit, the structure functions are essentially independent of Q^2 . They depend only on x_B . For instance, F_2 falls by about 50% as Q^2 increases from 1 to 25 GeV^2 , while the square of the elastic form factor falls by a factor of 10^6 over the same range! [16]. This invariance of the Structure Functions on the energy scale of the measurement, Q^2 is known as Bjorken Scaling and was first observed in the Deep Inelastic Scattering Experiments of the 1970s [1]. The proton when probed at high energies (small distances), appeared to consist of point-like constituents.

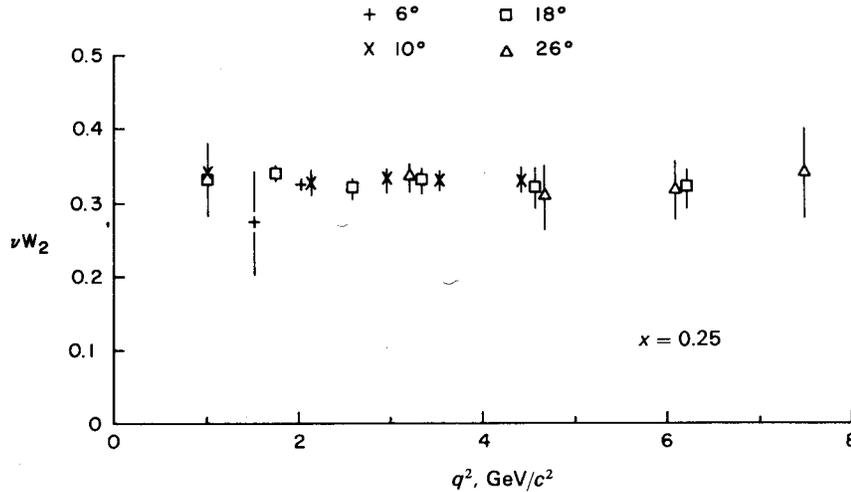


Figure 4. F_2 as a function of Q^2 at $x_B = .25$. For this choice of x_B , there is practically no Q^2 dependence of the structure function, that is, exact scaling. (After Friedman and Kendall [1] (1972).) Figure taken from [16].

Soon, however, it was found that scale invariance was only an approximate symmetry: the structure functions had a very slow (logarithmic) Q^2 dependence. The accurate prediction of the Q^2 dependence of the structure functions is one of the major successes of perturbative QCD. However, there is as yet no satisfactory theoretical understanding of the x_B dependence of the structure functions. It is essentially controlled by non-perturbative effects. The fundamental theory of strong interactions QCD has been in place for several years now, but has proven very hard to solve even approximately. We shall describe a model which allows us to calculate the x_B dependence of hadronic structure functions. This model can be derived as an approximation to the large N limit of two dimensional QCD. N here is the number of colors. It was actually obtained as an

approximation to Quantum HadronDynamics(QHD), an equivalent reformulation of 2dQCD in terms of color invariant observables [9]. However, we are able to interpret and describe this model in terms of the parton model. It is this parton model point of view that we shall describe in what follows.

Before describing the interacting parton model, let us introduce a convenient coordinate system in two space-time dimensions: the null coordinates.

2.3 Null Coordinates

It will prove extremely convenient to use a null coordinate system when describing the kinematics in the valence parton model. One useful consequence of this is that the relativistic energy-momentum dispersion relation $E = \pm\sqrt{p_1^2 + m^2}$ will no longer involve complications due to the square-roots. The sign of energy will be the same as that of momentum! In QHD, the field variable $M(p,q)$ depends on two space-time points that are separated by a null distance and thus causally connected.

As explained earlier, we will ignore the transverse momenta of the partons. We may take the longitudinal momenta of the partons to be directed along the x^1 axis. The two dimensional space-time position and momentum in cartesian coordinates are (x^0, x^1) and (p_0, p_1) . Rather than use p_1 as the kinematic variable, we will use the null component of momentum $p = p_0 - p_1$. ‘ p ’ is the momentum along a null line in 2-dimensional Minkowski space. We will use (p_0, p) as our coordinates. This is not an orthogonal coordinate sytem. However, the simplification achieved is that the mass-shell condition for a particle of mass m : $p_0^2 = p_1^2 + m^2$ is replaced by $p_0 = \frac{1}{2}(p + \frac{m^2}{p})$ where $p = p_0 - p_1$ is the null momentum.

Thus we see that for a particle with positive energy p_0 , the null momentum must be positive, unlike in cartesian coordinates. Also note that the fraction x of proton 4-momentum carried by a parton is the same as the fraction of corresponding null momenta. For brevity, we shall often refer to null momentum just as momentum. There should be no confusion since we will no longer use the cartesian coordinate p_1 .

3 The Interacting Valence Parton Model

3.1 The Quark-Quark Potential

We now describe a model for the structure of baryons in the language of the parton model. In the original quark-parton model, the baryon is made of N partons, which are essentially massless, free particles. N here is the number of colors, which is 3. We shall keep N arbitrary for now. It is true that when probed at very small distances (large momentum transfers, Q^2), the quarks inside the proton appear to be free particles. This is called asymptotic freedom [3] in QCD. However, quarks are confined within the proton. An isolated quark has never been produced. Therefore, even though the force that binds quarks together is vanishingly small at small distances, it is non-zero at intermediate distances and does not diminish at large distances. This is in stark contrast to the electromagnetic and gravitational forces which decrease with distance!

If the partons in our model were indeed free, they would not bind to form a proton. The only way the proton can have non-trivial structure functions is for the quarks to interact. Since we are describing a many-body system (N partons), it is simplest to assume an attractive two body potential between the quarks.

In Quantum Chromodynamics, the force between quarks is mediated by the exchange of massless spin 1 bosons called gluons. Gluons are the carriers of the color force just as photons are the quanta of the electromagnetic force. When QCD is dimensionally reduced to 1+1 dimensions [9], the gauge fields may be eliminated using their equations of motion. What remains is effectively a long range quark-quark force which is given by a linear two-body potential in the null coordinates.

This linear potential can also be understood in other ways. Since gluons are massless, their propagator in momentum space is $\frac{1}{q^2}$. This is the same as the photon propagator. In 4 space-time dimensions, the photon propagator corresponds to the $\frac{1}{|r-r'|}$ Coulomb potential. However, since we are in 2 space-time dimensions, we must use the appropriate Coulomb potential, which is the

Green's function of the Laplace operator (second derivative with respect to x) in 1 space dimension. This is $\frac{1}{2}|x - y|$. Notice that this potential is linear, attractive and confining.

Moreover, there is evidence that the quark-quark potential is linear as reported in [17]. This interacting parton model can be derived as an approximation to Quantum HadronDynamics (QHD) [8, 9]. In QHD, Lorentz invariance leads to the choice of a linear potential. Translation invariance implies that it can only depend on the absolute value of the separation $|x - y|$.

3.2 Wave Function of the Proton

We assume that the proton is made up of N partons (quarks). Therefore, the proton is a relativistic quantum many-body system. We will describe the proton in terms of a wave function that will depend on the quantum numbers of the partons (momenta, spin, color and flavour.)

We are interested in knowing the baryon wave function in its ground state, since that is where the proton spends most of its time. The strategy will be to write down the energy of the baryon in the state ψ and minimize this energy with respect to different choices of ψ . The configuration ψ which minimizes the energy is its ground state wave function. This is what we will compare with experimental data.

The quarks are spin $\frac{1}{2}$ fermions transforming under the fundamental representation of the color group $SU(N)$. However, the proton itself is colorless. (i.e. a color singlet) It must transform under the trivial representation of $SU(N)$. As mentioned earlier, we will work in the null coordinates. Therefore the wave function of a single parton is $\tilde{\psi}(a, \alpha, p)$. Here 'a' labels the flavour (up, down) and the spin quantum numbers of the parton. α labels color and $p = p_0 - p_1$ is the null momentum of the parton. We use $\tilde{\psi}$ to denote the momentum space wave function. The corresponding position space wave function $\psi(a, \alpha, x)$ is the inverse Fourier transform of $\tilde{\psi}(a, \alpha, p)$. Since the null momentum is always positive, $\tilde{\psi}$ must vanish for negative p . We shall often refer to this as ' $\tilde{\psi}$ has only positive support'.

We can think of $\tilde{\psi}$ as a joint probability density. It is a probability density that depends on two discrete random variables 'a' and ' α ' and one continuous random variable p , for any given parton. The proton is made up of N partons and has a wave function

$$\tilde{\psi}(a_1, \alpha_1, p_1; \dots; a_N, \alpha_N, p_N)$$

Here a_i, α_i, p_i are the spin, flavour, color and null momentum of the i^{th} parton. Since the

quarks are fermions, the proton wave function should be completely anti-symmetric under interchange of a pair of quarks, by the Pauli exclusion principle. However, since the baryon is invariant under color, the wave function must be completely anti-symmetric in color alone.

$$\tilde{\psi}(a_1, \alpha_1, p_1; \dots; a_N, \alpha_N, p_N) = \frac{\epsilon_{\alpha_1, \dots, \alpha_N}}{\sqrt{N!}} \tilde{\psi}(a_1, p_1; \dots; a_N, p_N)$$

This means the wave function is completely symmetric in spin, flavour and momentum quantum numbers. Therefore, once color has been factored out, the partons behave like a collection of bosons.

3.3 Hamiltonian of the N-Parton System

What is the energy of the proton in the state $\tilde{\psi}(a_1, p_1; \dots; a_N, p_N)$? We will express the energy as a sum of kinetic and potential energies. Consider the case $N = 1$, i.e. a proton consisting of a single parton with wave function $\tilde{\psi}(a, p)$. The ‘effective mass’ of the parton is μ_a , which could possibly depend on what its flavour is. We shall return to a discussion of the effective mass later. The relativistic kinetic energy-momentum relation expressed in terms of null-momentum is

$$p_0 = \frac{1}{2}(p + \frac{\mu_a^2}{p}).$$

The probability density that the parton indeed has null momentum p and spin-flavour ‘a’ is $|\tilde{\psi}(a, p)|^2$. Therefore, the expectation value of the kinetic energy of this single parton baryon is

$$\sum_a \int_0^\infty \frac{1}{2}(p + \frac{\mu_a^2}{p}) |\tilde{\psi}(a, p)|^2 \frac{dp}{2\pi}$$

It follows that the kinetic energy of a baryon with N partons is

$$\sum_{a_1, a_2, \dots, a_N} \int_0^\infty \sum_{i=1}^N \frac{1}{2}(p_i + \frac{\mu_{a_i}^2}{p_i}) |\tilde{\psi}(a_1, p_1; \dots; a_N, p_N)|^2 \frac{dp_1 \dots dp_N}{(2\pi)^N}$$

(We shall choose our normalization constants for Fourier transforms so that a factor of $\frac{1}{2\pi}$ consistently appears in the measure of every momentum space integral.)

The potential energy is most easily expressed in position space. As explained earlier, the partons interact through a two-body potential. i.e. the potential energy of a pair of quarks at null coordinates x and y is $g^2 v(x - y)$, where g is a coupling constant with the dimensions of mass. The potential v is linear in the separation between the quarks $v(x - y) = \frac{1}{2}|x - y|$. This yields a potential energy

$$\frac{\bar{g}^2}{2} \sum_{a_1, \dots, a_N} \int_{-\infty}^{\infty} \sum_{i \neq j} v(x_i - x_j) |\psi(a_1, x_1; \dots; a_N, x_N)|^2 dx_1 \cdots dx_N$$

for the system of N partons. The factor of $\frac{1}{2}$ is to avoid double counting. $\psi(a_1, x_1; \dots; a_N, x_N)$ is the position space wave function of the baryon. It is the inverse Fourier transform of the momentum space wave function.

$$\psi(a_1, x_1; \dots; a_N, x_N) = \int_0^{\infty} \tilde{\psi}(a_1, p_1; \dots; a_N, p_N) e^{i \sum_j p_j x_j} \frac{dp_1 \cdots dp_N}{(2\pi)^N}$$

We said that μ_a is the effective mass of ‘a’ parton of flavour a inside the proton. This must be explained. A quark is confined within the proton and cannot be isolated as a bare particle. Therefore, its bare mass is not a directly measurable quantity. All that matters is its effective mass μ_a when confined within the proton. However, one can make an indirect measurement of the ‘bare’ quark masses. At high energies, the strong force is essentially zero and the electro-weak interactions of these essentially free quarks depends on their bare masses. These have been measured and are called ‘current’ quark masses. The current quark mass of the up and down quarks, which are the valence quarks in the proton, are about 5 and 8 MeV/c². This must be contrasted with the energy scale of the strong interactions $\Lambda_{QCD} \sim 100$ MeV. We see that the bare up and down quark masses are negligible in comparison. In QCD, the limit of zero quark mass is the limit of Chiral Symmetry. It is probably best to think of the quark mass parameters m_a as measuring the extent to which Chiral Symmetry is broken in the theory. In nature, the quark masses are not exactly zero, but they are zero to a good first approximation. It is this limit that we shall mostly be interested in while comparing our results with experimental data.

Returning to the question of effective and bare quark masses, we mention that a similar situation exists in the case of an electron in a metal. It has an effective mass that is different from the mass of a free electron. The term in the energy that involves the effective mass may be re-expressed as the sum of a term involving the bare mass alone and a term involving the coupling constant. This latter term is often referred to as the self-energy term. In QED, which is an interacting theory of the electron, the ‘bare’ electron can emit and re-absorb photons. This part of the interaction can be thought of as renormalizing the bare mass of the electron, giving rise to a self-energy

However, these analogies are only to set the context. They cannot be pushed too far, in the present case, the square of the effective mass is infact negative in the limit of chiral symmetry! In

this case, the self energy term really arises in 2 dimensional QCD when a quartic operator in the potential energy is normal ordered [9]. The resulting expression for the effective mass in terms of the bare mass is $\mu_a^2 = m_a^2 - \frac{\tilde{g}^2}{\pi}$, where $\tilde{g}^2 = Ng^2$. This expression has been derived in the literature [10, 18].

Adding the potential and kinetic energies, the total energy of the proton is

$$\begin{aligned} \mathcal{E}_N(\tilde{\psi}) = & \sum_{a_1 \cdots a_N} \int_0^\infty \sum_{i=1}^N \frac{1}{2} \left[p_i + \frac{\mu_{a_i}^2}{p_i} \right] |\tilde{\psi}(a_1, p_1; \cdots a_N, p_N)|^2 \frac{dp_1 \cdots dp_N}{(2\pi)^N} \\ & + \frac{1}{2} g^2 \sum_{a_1 \cdots a_N} \int_{-\infty}^\infty \sum_{i \neq j} v(x_i - x_j) |\psi(a_1, x_1; \cdots a_N, x_N)|^2 dx_1 \cdots dx_N. \end{aligned}$$

The wave function that minimizes this energy subject to the constraint that the total probability is 1 is the ground state baryon wave function. Here the norm of $\tilde{\psi}$ is given by

$$\|\tilde{\psi}\|^2 = \sum_{a_1, \dots, a_N} \int_0^\infty |\tilde{\psi}(a_1, p_1; \cdots; a_N, p_N)|^2 \frac{dp_1 \cdots dp_N}{(2\pi)^N}$$

3.4 The Hartree Ansatz

The minimization of the above energy can be thought of as a problem in many-body theory. We don't have a general solution to such a problem and must look for approximation methods. An important simplification can be made since we are describing the ground state of a bosonic system, since we have factored out the totally anti-symmetric color part of the wave function. Mean field theory gives a good approximation to the ground state where each boson moves in the mean field created by the others. Moreover, in the ground state, each of the bosons can be assumed to occupy the same single particle state. Therefore, in the mean field approximation, we make the ansatz:

$$\tilde{\psi}(a_1, p_1; \cdots; a_N, p_N) = \delta(\sum_i p_i - P) \prod_{i=1}^N \tilde{\psi}(a_i, p_i).$$

In the language of probability distributions, we are assuming that the joint probability distribution of the N partons is well approximated by the product of N independent (identical) single parton distributions. Since the null momenta are positive we impose the condition that the null momenta of the partons add up to that of the baryon (P). Therefore, the single parton wave functions must vanish for $p < 0$ and $p > P$. (Infact, the condition is weaker, all we need is the delta function

factor on the momentum sum above. But the difference is a correlation which is suppressed in the limit of a large number of partons, N .) Thus we are ignoring quark-quark correlations. When comparing with data, we will express the wave function in terms of the dimensionless fractional momentum $x = \frac{p}{P}$. In Section 2.2 we showed that this fractional momentum is nothing but the Bjorken scaling variable x_B in the deep inelastic limit.

In practice, this is a very reasonable assumption. It underlies most of the phenomenological description of Deep Inelastic Scattering. The very fact that one can speak of an up quark momentum distribution in the proton, without mentioning the simultaneous down quark momentum at which it was measured means that it is a good approximation to ignore correlations.

Such a mean field approximation is common in many-body physics and is probably best known in the context of the Hartree approximation of atomic physics. There, we are solving the Schrödinger equation for a many electron atom. The number of electrons in a neutral atom is the same as the atomic number Z . The mean field approximation can be thought of as a large- Z approximation [9]. The fact that it works well even for Helium when $Z = 2$ encourages us to use it here for $N = 3$. For a more precise analogy, see the appendix to [9].

3.5 Momentum Sum Rule and Boundary Conditions

There is one further constraint that the single particle wave function must satisfy: the momentum sum rule. The expectation value of the total momentum of all the valence partons must equal the fraction f of the baryon momentum actually available to them

$$N \int_0^P p |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} = fP.$$

The valence quarks do not carry all the momentum of the baryon, as in the original parton model. The parton distributions extracted from Deep Inelastic Scattering data show that roughly half the momentum of the baryon is carried by the valence quarks. The rest is in the gluons, sea quarks and anti-quarks. We shall see that the fraction f is the only parameter on which the parton distributions predicted by our model depend.

The energy per parton is therefore:

$$E = \sum_a \int_0^P \frac{1}{2} \left[p + \frac{\mu_a^2}{p} \right] |\tilde{\psi}(a, p)|^2 \frac{dp}{2\pi} +$$

$$\frac{1}{2}\tilde{g}^2 \int_{-\infty}^{\infty} v(x-y) \sum_a |\psi(a,x)|^2 \sum_{\beta} |\psi(\beta,y)|^2 dx dy.$$

with the potential $v(x-y) = \frac{1}{2}|x-y|$ as explained in Section 3.1. The single parton wave function is determined by minimizing the energy per parton subject to the constraints: $||\tilde{\psi}|| = 1$ and $\tilde{\psi}(a,p) = 0$ unless $0 \leq p \leq P$. We will impose the boundary condition that the wave function vanish as $p \rightarrow P$. This is because the probability density of a single parton carrying all momentum of the baryon must be zero. The behavior of the wave function as $p \rightarrow 0$ will be worked out in detail in the next chapter (Section 4.4). It will turn out that for a positive quark mass $m > 0$, the wave function vanishes at the origin too. In the limiting case $m = 0$, the wave function goes to a non-zero value at the origin.

The condition that the momentum space wave function vanish outside $[0,P]$ looks rather different in position space. It says that $\psi(a,x)$ must be the value along the real line, of an entire function. This constraint can be hard to implement especially if we look for numerical solutions. Hence we work in momentum space.

We also make a comment on the support of the wave function. We have imposed the constraint that $\tilde{\psi}(p)$ vanish for $p > P$. i.e. the null momentum of a parton must not exceed that of the baryon. However, the momentum sum rule states that $N \int_0^P p |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} = fP$, where f is the fraction of baryon momentum carried by the N valence partons. But this is just N times the mean (first moment) of the single parton distribution! We see that the single parton wave function is primarily concentrated around the region $p = \frac{fP}{N}$. For large enough N , it should be reasonable to allow the wave functions to be non-zero beyond P , since they are vanishingly small for large p anyway. Thus we may replace the upper limit of integration P , by ∞ especially while obtaining analytic approximations. Note that we are not saying that $P \rightarrow \infty$. This is not a Lorentz invariant statement. P is the null momentum of the baryon and has a fixed value in any given inertial reference frame.

3.6 Positivity of the Energy

We may reformulate this constrained minimization problem as the solution of a certain non-linear integral equation. Before proceeding with this, let us make some slight simplifications. We will set all the parton masses equal to each other. In nature, the up and down current quark masses are

both roughly zero. Further, let us look for a single parton wave function that is non-zero for only a single spin-flavour index ‘a’ : $\tilde{\psi}(a, p) = \delta_{a,1}\tilde{\psi}(p)$.

This breaks the U(M) invariance of our model spontaneously. This symmetry can be restored later by the collective variable method as in the theory of solitons, though we shall not address these issues here.

With these simplifications, the energy is given by:

$$E = \int_0^P \frac{1}{2} \left[p + \frac{\mu^2}{p} \right] |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} + \frac{1}{2} \tilde{g}^2 \int_{-\infty}^{\infty} v(x-y) |\psi(x)|^2 |\psi(y)|^2 dx dy.$$

We take a moment to check that the energy is positive (and hence bounded below), so that its minimization is a well posed problem! This is immediate since the integrands of both the kinetic and potential energies are positive functions for $\mu^2 = m^2 - \frac{\tilde{g}^2}{\pi} \geq 0$. The physically interesting region is the limit of zero quark mass $m \rightarrow 0$. This corresponds to a negative value of μ^2 . This case is more subtle, the kinetic ($\int_0^P \frac{1}{2} \left[p + \frac{m^2}{p} \right] |\tilde{\psi}(p)|^2 \frac{dp}{2\pi}$) and potential energies ($\frac{1}{2} \tilde{g}^2 \int_{-\infty}^{\infty} v(x-y) |\psi(x)|^2 |\psi(y)|^2 dx dy$) remain positive while the self energy term ($\int_0^P \frac{1}{2} \left[\frac{-\tilde{g}^2}{\pi p} \right] |\tilde{\psi}(p)|^2 \frac{dp}{2\pi}$) is negative. However, it can be shown that this self energy term is cancelled by a portion of the potential energy, when expressed in momentum space. In practice, this will not be a problem since we will approach the physically interesting region from positive values of μ^2 and see that a sensible limit exists.

3.7 The Energy in Momentum Space

The constraint on the position-space wave function is that it be the boundary value of an entire function. The same constraint is much simpler in momentum space: it must vanish outside the interval $[0,P]$. Therefore, as mentioned in Section 3.5, it will be convenient to express the potential energy in momentum space. The energy per parton is

$$E = \int_0^P \frac{1}{2} \left[p + \frac{\mu^2}{p} \right] |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} + \frac{1}{2} \tilde{g}^2 \int_{-\infty}^{\infty} v(x-y) |\psi(x)|^2 |\psi(y)|^2 dx dy.$$

The potential energy is best understood in the language of electrostatics. It is just the electro-static potential energy of a charge density $\rho(x) = \tilde{g}|\psi(x)|^2$ in one space dimension, where the Green’s function is $\frac{1}{2}|x-y|$. \tilde{g} here is analogous to the unit of electric charge. The Poisson integral formula then gives the electrostatic potential

$$V(x) = g \int_{-\infty}^{\infty} |\psi(y)|^2 \frac{|x-y|}{2} dy$$

Rather than refer to $V(x)$ as the electrostatic potential, we must call it the mean potential due to the partons. It is not to be confused with the 2-body potential $v(x-y)$ between the quarks! $V(x)$ then satisfies Poisson's equation $V''(x) = \tilde{g}|\psi(x)|^2$. The boundary conditions are $V(0) = \tilde{g} \int_{-\infty}^{\infty} |\psi(y)|^2 \frac{|y|}{2} dy$ and as $|x| \rightarrow \infty, V(x) \rightarrow \frac{\tilde{g}|x|}{2}$, which is just the asymptotic form of the potential of a charge \tilde{g} localized around the origin.

We shall rewrite all of this in momentum space, but will have to be careful since we will find that the mean potential $\tilde{V}(p)$ has a $\frac{-1}{p^2}$ singularity at the origin in momentum space. The momentum space integrals we get will therefore be singular and we will define them as Finite Part integrals in the sense of Hadamard [19]. These 'Finite Part' prescriptions are not to be thought of as a witch-craft that allows one to assign finite values to divergent integrals, but as a way of restating the above boundary conditions in momentum space. Ultimately, it is the physics that determines what the correct definition of an apparently divergent quantity is. As we shall see, these definitions will turn out to be very natural and motivated by general principles such as analytic continuation. It is therefore likely that these are the 'correct' definitions in other contexts too.

We first rewrite Poisson's equation for the mean potential in momentum space by Fourier transforming: $-p^2\tilde{V}(p) = \tilde{W}(p)$ where $\tilde{W}(p)$ is the convolution $\int_0^P \tilde{\psi}(p+q)\tilde{\psi}(q)\frac{dq}{2\pi}$. We see that the mean potential is singular at the origin: $\tilde{V}(p) \rightarrow \frac{-1}{p^2}$ as $p \rightarrow 0$. It will be useful to note that $\tilde{W}(0) = 1, \tilde{V}(-p) = \tilde{V}^*(p)$ and $\tilde{W}'(0) = -\frac{1}{2}|\tilde{\psi}(0)|^2$. Also, by a choice of phase, the wave function $\tilde{\psi}(p)$ may be taken to be real. Therefore $\tilde{V}(p)$ is real and even.

Therefore, the energy in momentum space is:

$$E[\tilde{\psi}] = \int_0^P \frac{1}{2}(p + \frac{p^2}{p})|\tilde{\psi}(p)|^2 \frac{dp}{2\pi} + \frac{\tilde{g}}{2} \mathcal{FP} \int \frac{dpdq}{(2\pi)^2} \tilde{V}(p)\tilde{\psi}(q)\tilde{\psi}^*(q-p)$$

$$\text{where } \tilde{V}(p) = \frac{-\tilde{g}}{p^2} \int_0^P \tilde{\psi}^*(p+r)\tilde{\psi}(r)\frac{dr}{2\pi}$$

We see that the integrand in the potential energy has a $\frac{1}{p^2}$ singularity and we must define it as a 'Finite Part' (FP) integral. We will postpone a discussion of Finite Part integrals till we actually have to evaluate them in the next chapter.

3.8 Integral Equation for the Ground State Wave Function

We must minimize the above energy functional with respect to the constraint that the norm of ψ be 1. This is equivalent to minimizing $F[\tilde{\psi}] = E[\tilde{\psi}] + \lambda|\tilde{\psi}|^2$. λ here is the Lagrange multiplier enforcing the constraint. Since the functional we are minimizing is quartic in ψ , λ is *not* the same as the ground state energy. Therefore the groundstate wave function satisfies the equation $\frac{\delta F[\tilde{\psi}]}{\delta \tilde{\psi}} = 0$, which when written out more explicitly reads

$$\frac{1}{2}(p + \frac{\mu^2}{p})\tilde{\psi}(p) + \tilde{g}FP \int_0^P \tilde{V}(p-q)\tilde{\psi}(q)\frac{dq}{2\pi} = \lambda\tilde{\psi}(p)$$

Since \tilde{V} is quadratic in $\tilde{\psi}$, we see that this is a non-linear singular integral equation for $\tilde{\psi}$.

In the next chapter, we will explain how the singular integrals are defined and obtain a solution to the equation. We must mention that there is a well-developed theory of linear integral equations, even those with Cauchy-type singularities [19]. But the non-linearities and severity of the singularities in our case mean that we will have to develop special techniques to deal with our integral equation.

The strategy will be to understand the ground state wave function from several different points of view. No single technique may give us the exact solution. We will develop approximation methods which are valid in distinct special cases. By patching these together, we will obtain both a qualitative and quantitative understanding of the solution.

3.9 Parameters in the Model

Let us conclude this chapter with a discussion of the parameters in the theory. \tilde{g} is the coupling constant of the theory. It has dimensions of mass and is related to the coupling constant of two-dimensional QCD by $\tilde{g} = g^2N$. g is related to the coupling constant of the four dimensional theory through a multiplicative factor of the order of the inverse transverse size of the proton. This arises when we ‘integrate over’ transverse coordinates to arrive at this ‘effective’ two-dimensional model. Though we do not predict the value of the coupling constant, we will see that it cancels out when we calculate the valence quark distribution.

N is the number of colors, which is the same as the number of valence quarks. Most of the approximations we have made become exact when $N \rightarrow \infty$. We will use the value $N = 3$ when comparing with data.

μ is the ‘effective’ quark mass and is related to the current quark mass by $\mu^2 = m^2 - \frac{g^2}{\pi}$. We will be interested in the limit of chiral symmetry when $m \rightarrow 0$. The energy expressed above has logarithmic divergences in both the potential and self energies when m is exactly zero. However, we shall define the theory as the limit $m \rightarrow 0$. A Frobenius analysis will show that the valence quark wave function has a sensible limit as $m \rightarrow 0$. This can be thought of as an infrared regulation. But we emphasize that there is no real divergence in the problem, requiring renormalization. In any event, nature is balanced on a knife-edge at a small, positive value of m !

The limit where m is small keeping \tilde{g} fixed is the extreme relativistic limit. The binding energy of the system far exceeds the rest-mass of the partons. We will also have occasion to consider the non-relativistic limit in which $\mu \gg \tilde{g}$. This is not the physically interesting case as it corresponds to a baryon made up of heavy quarks. However, in this limit the integral equation will reduce to a non-linear ordinary differential equation and will give us another way of studying the solutions.

λ is the Lagrange multiplier enforcing the constraint on the norm of $\tilde{\psi}$. It has no direct physical meaning and is not equal to the energy of the state $\tilde{\psi}$.

P is the null momentum of the baryon, which in its restframe, is equal to its rest-mass. $\tilde{\psi}$ is non zero only for $0 \leq p \leq P$. But when we express the wave function in terms of the dimensionless Lorentz invariant variable $x_B = \frac{p}{P}$, we will see that the structure functions do not depend on which value of P we used (i.e. which inertial reference frame we picked). \tilde{g} too will cancel out. One can think of P as being measured in units of \tilde{g} . When the dimensionless structure functions are expressed in terms of the dimensionless variable x_B , they no longer depend on \tilde{g} .

The only free parameter in the theory then, is the fraction f of baryon momentum carried by the valence partons. It enters through the momentum sum rule $N \int_0^P p |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} = fP$.

4 Determination of the Valence Quark Wave Function

4.1 Estimation of the Ground State using Variational Ansatzes

The variational principle of quantum mechanics allows us to estimate the ground state energy and wave function of a quantum mechanical system. The ground state is the one that actually minimizes the energy. Therefore, by minimizing the energy within a sub-class of functions in the Hilbert space, we may obtain an upper bound for the ground state energy and also a qualitative understanding of the ground state wave function. In practice, a judicious choice of a 1 or 2 parameter family of variational ansatzes motivated by physical principles can often come spectacularly close to the true ground state energy.

The energy per parton is given by

$$E = \int_0^P \frac{1}{2} \left[p + \frac{\mu^2}{p} \right] |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} + \frac{1}{2} \tilde{g}^2 \int_{-\infty}^{\infty} v(x-y) |\psi(x)|^2 |\psi(y)|^2 dx dy.$$

As a first approximation, we may work on the interval $[0, \infty)$ with a function that decays as $p \rightarrow \infty$, as justified in Section 3.5. A simple choice is $\tilde{\psi}_a(p) = C p e^{-ap}$ with $a > 0$, $p \geq 0$. Here ‘a’ is a variational parameter controlling the rate of decay of the wave function in momentum space. C is fixed by normalization. In position space, this positive momentum function has an analytic continuation to the upper half plane and a double pole at $x = -ia$: $\psi(x) = \frac{C'}{(x+ia)^2}$.

We may calculate the energy of such a state and minimize with respect to a. It is clear on dimensional grounds that the term $\int_0^\infty \frac{1}{2} p |\tilde{\psi}(p)|^2 \frac{dp}{2\pi}$ scales like $\frac{1}{a}$ while all other terms in the energy scale like a. (‘a’ has dimensions of inverse momentum or length.) Thus there is a value of ‘a’ at which the energy is minimized among this class of functions.

$$E_{ground\ state} \leq \frac{1}{2} \sqrt{I_1(m^2 I_2 + \tilde{g}^2 I_3)},$$

where I_1 , I_2 and I_3 are positive dimensionless pure numbers obtained from the integrations.

We have also tried another class of variational ansatzes: $\tilde{\psi}(p) = Cp^\alpha(1-p)^\beta$. This time, we work on the finite interval $[0, P]$ and set $P = 1$ for convenience. Thus p is the fraction of baryon momentum carried by the quark. This is the same as the Bjorken Scaling variable x_B . The energy calculations in this case are more formidable but can be performed in terms of hypergeometric functions. Moreover, we work in momentum space and use the definitions of singular integrals, which will be presented in Section 4.3. The results, however, are encouraging and easy to state. Among the choices $\alpha, \beta = 1, 2, 3$ we find that the energy is minimized for the choice $\alpha = 1$ and $\beta = 2$, even as $m \rightarrow 0$. There is a further decrease in energy when we consider wave functions of the form $\tilde{\psi}(p) = Cp^\nu(1-p)^2$ for fractional values of α , $0 < \alpha < 1$. Our conclusion is that in the limit of chiral symmetry (zero current quark mass), the energy is minimized in the limit where $\alpha \rightarrow 0$. We thus have a qualitative understanding of the momentum space wave function as $m \rightarrow 0$. It goes to a non-zero constant as $p \rightarrow 0$ and then falls off roughly as $(1-p)^2$ as $p \rightarrow 1$.

We shall see that this picture is re-inforced and made more precise by the other approximation methods we develop to study the integral equation for $\tilde{\psi}$.

4.2 Non-relativistic Limit

The non-relativistic limit is the limit where the binding energy of the system is small compared to the rest mass of the quarks. It can be achieved by making the quark mass large compared to the coupling constant ($m \gg \tilde{g}$). In the case of the proton, this is not a very interesting limit since the up and down quark masses are essentially zero. However, we will be able to understand our system from a slightly different perspective this way. Besides, it also gave us a way of checking our numerical procedures. The expression for the energy is:

$$E = \int_0^P \frac{1}{2} \left(p + \frac{\mu^2}{p} \right) |\tilde{\psi}(p)|^2 + \frac{\tilde{g}^2}{2} \int_{-\infty}^{\infty} |\psi(x)|^2 \frac{|x-y|}{2} |\psi(y)|^2 dx dy.$$

The relativistic energy-null momentum dispersion relation is $p_0 = \frac{1}{2} \left(p + \frac{\mu^2}{p} \right)$, $p > 0$. We see that p_0 has a minimum when $p = \mu$. In the extreme case of no interaction at all ($\tilde{g} = 0$), the total energy is just the kinetic energy. In this case the energy is minimized when all the null-momentum of the parton is concentrated at the value $p = \mu$. In other words, $|\tilde{\psi}(p)|^2$ is a delta function at $p = \mu$.

The effect of having a small, but non-zero interaction $\tilde{g} > 0$ is to broaden out the ground state wave function. In any case, it is concentrated around the minimum of the dispersion curve. Therefore we may replace the dispersion relation by its quadratic approximation in the neighbourhood of its minimum, $p = \mu$. The result should not be surprising. We get the new dispersion relation

$$p_0 = \mu + k^2/2\mu,$$

where $k = p - \mu$ is the shifted momentum variable and $\mu = \sqrt{m^2 - \frac{g^2}{\pi}} \sim m$ is the mass of the heavy quark.

We immediately recognize this as the non-relativistic energy of a particle of mass μ and momentum k . Putting this into the expression for the energy per parton, we have

$$E = \mu + \int \frac{k^2}{2\mu} |\tilde{\chi}(k)|^2 \frac{dk}{2\pi} + \frac{\tilde{g}}{2} \int_{-\infty}^{\infty} V(x) |\chi(x)|^2 dx.$$

where $\tilde{\chi}(k) = \tilde{\psi}(\mu + k)$ and $\chi(x)$ is its Fourier transform. $V(x)$ is the effective potential satisfying Poisson's equation with boundary conditions as in Section 3.7:

$$V''(x) = g|\chi(x)|^2$$

To minimize this energy we vary with respect to $\chi(x)$ and impose the constraint $\|\chi\| = 1$. The result is a Schrödinger-like equation.

$$\frac{-1}{2\mu} \chi''(x) + \frac{\tilde{g}}{2} V(x) \chi(x) = \lambda \chi(x)$$

It is non-linear, since V itself depends on χ . However, it can be solved by iteration. We start with a guess for the ground state wave function $\chi_0(x)$ and calculate $V_0(x)$ by solving Poisson's equation. Using this approximate effective potential $V_0(x)$, we solve the eigenvalue problem

$$\frac{-1}{2\mu} \chi_1''(x) + \frac{\tilde{g}}{2} V_0(x) \chi_1(x) = \lambda \chi_1(x)$$

for the first iterate wave function $\chi_1(x)$. $\chi_1(x)$ is used to determine a new effective potential $V_1(x) = \tilde{g} \int_{-\infty}^{\infty} |\chi_1(x)|^2 \frac{|x-y|}{2} dx$. (This is the solution of Poisson's equation satisfying the boundary conditions given in Section 3.7). The process is iterated till it converges (or until successive iterates are the same up to numerical errors, if one is working numerically). The resulting ground state

$\chi(x)$ may be transformed back to momentum space and re-expressed in terms of $p = k + \mu$. We thus obtain an approximate ground state wave function and energy for the N parton system in the non-relativistic limit. We shall not discuss the details here.

We mention the procedure primarily because it gives us a general method of obtaining approximate solutions to certain non-linear equations. It is essentially this procedure of iteration that will be used in the numerical solution to our non-linear integral equation for the ground state parton wave function. It will ofcourse be more complicated since we have a non-linear integral equation with singularities.

4.3 Finite Part Integrals

We have mentioned the need to define singular integrals where the integrand has a $\frac{1}{q^2}$ singularity. For instance, consider the integral equation for the ground state wave function $\tilde{\psi}(p)$:

$$\left[\frac{1}{2} \left(p + \frac{\mu^2}{p} \right) - \lambda \right] \tilde{\psi}(p) + \tilde{g}^2 \mathcal{FP} \int_0^P \tilde{V}(p-q) \tilde{\psi}(q) \frac{dq}{2\pi} = 0,$$

where $\tilde{V}(p) = -\frac{1}{p^2} \int_0^P \tilde{\psi}^*(p+q) \tilde{\psi}(q) \frac{dq}{2\pi}$.

Hence the integrand in the equation for $\tilde{\psi}(p)$ is singular. We will define it as a ‘Finite Part’ integral in the sense of Hadamard [19]. In general, all our singular integrals can be reduced to those of the form

$$\mathcal{FP} \int_0^P \frac{f(q)}{q^2} dq$$

where f itself is not singular at the origin. In fact, we will assume that f is C^1 at the origin from the right. This means that f possesses a continuous right-hand derivative at the origin. We shall rewrite the above integral as

$$\mathcal{FP} \int_0^P \frac{f(q)}{q^2} dq = \int_0^P \frac{f(q) - f(0) - qf'(0)}{q^2} dq + f(0) \mathcal{FP} \int_0^P \frac{dq}{q^2} + f'(0) \mathcal{FP} \int_0^P \frac{dq}{q}$$

The first integral on the right is non-singular at the origin and is an ordinary Riemann Integral, since we have subtracted out the singular parts. However, the last two integrals on the right still need to be defined. They are of the form $\mathcal{FP} \int_0^P \frac{q^\nu}{q^2} dq$. We know that for $\nu > 1$, this is just a Riemann Integral and has the value $\frac{P^{\nu-1}}{\nu-1}$. We shall now define it even for $\nu < 1$ by analytically continuing this formula.

$$\text{i.e. } \mathcal{FP} \int_0^P \frac{q^\nu}{q^2} dq = \frac{P^{\nu-1}}{\nu-1} \text{ for } \nu \neq 1.$$

This clearly runs into trouble for $\nu = 1$. In this special case, we shall define it to be the average of the values for $\nu = 1 + \epsilon$ and $\nu = 1 - \epsilon$ as $\epsilon \rightarrow 0$.

This is easily seen to imply that $\mathcal{FP} \int_0^P \frac{dq}{q} = \text{Log}(P)$. Therefore we have the following definition:

$$\begin{aligned} \mathcal{FP} \int_0^P \frac{q^\nu}{q^2} dq &= \frac{P^{\nu-1}}{\nu-1} & \nu \neq 1. \\ \mathcal{FP} \int_0^P \frac{q^\nu}{q^2} dq &= \text{Log}(P) & \nu = 1. \end{aligned}$$

We must mention a few cautionary remarks. The definition for $\nu = 1$ violates the change of variable formula for the scale transformation $q \rightarrow \lambda q$ by the amount $\text{Log}\lambda$. We must take this into account when making a scale transformation in the case $\nu = 1$. For $\nu \leq 1$, the Finite Part integral of a positive function may be negative. For example

$$\mathcal{FP} \int_0^1 \frac{dq}{q^2} = -1. \text{ and } \mathcal{FP} \int_{-\infty}^{\infty} \frac{dq}{q^2} = 0.$$

However, we also note that these definitions reduce to the familiar Riemann integrals whenever they exist in the usual sense. Furthermore, the usual Cauchy Principal value for integrals with a simple pole that does not lie at a limit of integration continues to hold. In particular then,

$$\mathcal{FP} \int_{-\infty}^{\infty} \frac{dq}{q} = 0$$

We emphasize that these definitions are necessary since we would like to rewrite position space potential energy integrals in terms of momentum variables. These complications arise because the integral kernel of the Green's function $\frac{|x|}{2}$ expressed in momentum space takes the form $\frac{-1}{p^2}$. These definitions ensure that the same integrals evaluated in position and momentum space give the same results. For instance, the position space mean potential $V(x)$ satisfies Poisson's equation $V''(x) = |\psi(x)|^2$ along with a pair of boundary conditions. Poisson's equation in momentum space for the mean potential $\tilde{V}(p)$ is

$$-p^2 \tilde{V}(p) = \int_0^P \tilde{\psi}^*(p+r) \tilde{\psi}(r) \frac{dr}{2\pi}$$

The boundary conditions translate to rules for integrating the $\frac{1}{p^2}$ singularity appearing in $\tilde{V}(p)$.

As an illustration and check, let us show that our definitions do indeed imply the equality of the Green's functions expressed in position and momentum space. We would like to prove that

$$-\frac{|x|}{2} = \mathcal{FP} \int_{-\infty}^{\infty} \frac{1}{p^2} e^{ipx} \frac{dp}{2\pi}$$

The proof will involve using our definition for Finite Part Integrals since the integrand on the right is singular. We will use contour integration to complete the proof.

For $x = 0$, both sides are zero according to our definition. It suffices to prove the stated result for $x > 0$ since both sides are even functions of x as is easily checked.

$$\begin{aligned} \mathcal{FP} \int_{-\infty}^{\infty} \frac{1}{p^2} e^{ipx} \frac{dp}{2\pi} &= \int_{-\infty}^{\infty} \frac{e^{ipx} - 1 - ipx}{p^2} \frac{dp}{2\pi} + \mathcal{FP} \int_{-\infty}^{\infty} \frac{dp}{2\pi p^2} + ix \mathcal{FP} \int_{-\infty}^{\infty} \frac{dp}{2\pi p} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

I_3 is just the Cauchy principal value and is zero as observed before. I_2 is also zero by the Finite Part prescription. I_1 is the Riemann integral of an entire function of p along the real axis. It may be evaluated using a semicircular contour in the upper-half plane. After making some estimates, we have $I_1 = \frac{-x}{2}$ for $x > 0$. Therefore, the desired result follows.

4.4 The Small p Behavior of the Valence Quark Wave Function

We shall describe a Frobenius-type analysis that gives us the behavior of the wave function $\tilde{\psi}(p)$ for small positive p . The integral equation for the wave function is:

$$\begin{aligned} \left[\frac{1}{2} \left(p + \frac{\mu^2}{p} \right) - \lambda \right] \tilde{\psi}(p) + \tilde{g}^2 \mathcal{FP} \int_0^P \tilde{V}(p-q) \tilde{\psi}(q) \frac{dq}{2\pi} &= 0 \\ \text{where } \tilde{V}(p) &= -\frac{1}{p^2} \int_0^P \tilde{\psi}^*(p+q) \tilde{\psi}(q) \frac{dq}{2\pi}. \end{aligned}$$

Keeping only the singular terms for small p , we have:

$$\frac{1}{2} \left(m^2 - \frac{\tilde{g}^2}{\pi} \right) \frac{1}{p} \tilde{\psi}(p) + \tilde{g} \mathcal{FP} \int_{p-P}^p \tilde{V}(q) \tilde{\psi}(p-q) \frac{dq}{2\pi} = 0$$

$\tilde{V}(q) \sim \frac{-1}{q^2}$ for small q . Now we assume a power law behavior $\tilde{\psi}(p) \sim p^\nu$ for small $p > 0$ and derive an equation for ν .

$$\frac{1}{2} \left(\frac{m^2}{\tilde{g}^2} - \frac{1}{\pi} \right) = \mathcal{FP} \int_{1-\frac{p}{P}}^1 \frac{(1-y)^\nu}{y^2} \frac{dy}{2\pi}$$

where $y = \frac{q}{p}$. Since $p \ll P$, we have

$$\frac{\pi m^2}{\tilde{g}^2} - 1 = \mathcal{FP} \int_0^1 \frac{(1-y)^\nu + (1+y)^\nu}{y^2} dy + \int_1^\infty \frac{(1+y)^\nu}{y^2} dy$$

The first of these integrals is singular and we evaluate it according to the definition given earlier. The result is a transcendental equation for ν :

$$\frac{\pi m^2}{\tilde{g}^2} - 1 = \int_0^1 \frac{(1+y)^\nu + (1-y)^\nu - 2}{y^2} dy - 2 + \int_1^\infty \frac{(1+y)^\nu}{y^2} dy$$

which is of the form $\pi m^2/\tilde{g}^2 - 1 = h(\nu)$. It is easily seen that for $m = 0$, $\nu = 0$ is a solution. In the limit of zero current quark mass, the critical wave function tends to a constant at the origin. Calculating $h(\nu)$ shows that for a positive quark mass m , the wave function goes to zero at $p = 0$ and rises like a power law $\tilde{\psi}(p) \sim p^\nu$, $\nu > 0$. The following plot shows $h(\nu)$. The solution ν for a given value of $m \geq 0$ is the point at which the horizontal line $\pi \frac{m^2}{\tilde{g}^2} - 1$ intersects the curve.

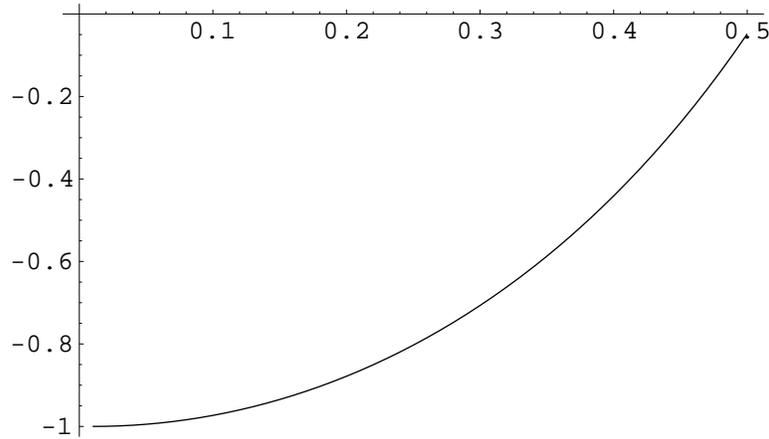


Figure 5. $h(\nu)$ as a function of ν . The solution ν for a given value of $m \geq 0$ is the point at which the horizontal line $\pi \frac{m^2}{\tilde{g}^2} - 1$ intersects the curve. Notice that for $m = 0$, $\nu = 0$ is a solution: in the limit of Chiral Symmetry, the wave function goes to a non-zero constant at the origin.

Our conclusion that the critical momentum space wave function tends to a non-zero constant as $p \rightarrow 0$ implies that it is discontinuous at the origin. Therefore, the critical position space wave function decays like $\frac{1}{x}$ at infinity. This is very reasonable from another point of view. In the soliton model [12] the baryon is thought of as being made up of an infinite number of pions. It has been shown that this picture is consistent with QCD [13]. But in that case, it should be possible to

approximate the baryon wave function with the meson wave function for large spatial x . At large distances we are essentially far away from the proton. The meson wave function has been calculated by 't Hooft [10]. In the limit $m \rightarrow 0$ the 't Hooft meson wave function too has a discontinuity in momentum space and consequently decays like $\frac{1}{x}$ as $x \rightarrow \infty$, just as we obtain here.

This Frobenius-like analysis is very useful. It gives us a clear quantitative understanding of the small p behavior of the wave function in the physically interesting region $m \rightarrow 0$. We have that as $m \rightarrow 0$, $\tilde{\psi}(p) \sim p^\nu$ with $\nu \rightarrow 0$. i.e. the wave function rises sharply near the origin. At the critical point $m = 0$, the wave function goes to a non-zero constant at $p = 0$. We had already got a scent of this behavior from our variational estimates! Together with the numerical solution, which will be valid except for small p , we will have almost a complete understanding of the valence quark wave function.

4.5 Numerical Solution

The numerical solution of this problem is *not* straightforward since the kernel of the integral equation is singular. We need a reliable method of numerical quadrature for integrals such as

$$\mathcal{FP} \int_0^P f(p, q) \rho(q) dq$$

when the weight function $\rho(q)$ has a singularity at $q = 0$ like $\frac{1}{q^2}$. We need to subdivide the interval $[0, P] = \cup_{r=1}^n [b_r, b_{r+1}]$ into subintervals. Within each subinterval we choose a set of points $q_{jr}, j = 1, \dots, \nu_r$. We approximate the integral by a sum

$$\mathcal{FP} \int_0^P f(p, q) \rho(q) dq = \sum_{jr} w_{jr} f(q_{jr}).$$

The weights w_{jr} are determined by the condition that within each subinterval $[b_r, b_{r+1}]$, the integral of a polynomial of order $\nu_r - 1$ is reproduced exactly:

$$\mathcal{FP} \int_{b_r}^{b_{r+1}} q^k \rho(q) dq = \sum_{j=1}^{\nu_r} w_{jr} q_{jr}^k.$$

This is the usual method of numerical quadrature [19, 20] except that the integral on the left hand side is singular for $b_r = 0$ and $k = 0, 1$. In these cases we can evaluate the left hand side analytically as the finite part in the sense of Hadamard. (The main difference from the usual

situation is that the moments on the left hand side of the above equation are not all positive.) The weights are then determined by solving the above system of linear equations.

Given an approximate mean potential $\tilde{V}_s(p)$ we can convert the linear integral equation

$$\frac{1}{2}[p + \frac{\mu^2}{p} - 2\lambda_s]\tilde{\psi}_{s+1}(p) + \tilde{g}^2 \mathcal{FP} \int_0^\infty \tilde{V}_s(p-q)\tilde{\psi}_{s+1}(q)\frac{dq}{2\pi} = 0$$

into a matrix eigenvalue problem by the above method of quadrature. We use the ground state eigenfunction so determined to calculate numerically the next approximation \tilde{V}_{s+1} for the mean potential. This process is iterated until the solution converges. Having determined the wavefunction, we must impose the momentum sum rule to determine \tilde{g} . We used Mathematica to implement this numerical procedure. An approximate analytic solution was used as a starting point for the iteration.

4.6 Comparison with Experimental Data

Now we turn to the question of the comparison of our model with data from Deep Inelastic Scattering. It is customary to describe the parton distributions as functions of the Bjorken scaling variable $0 \leq x_B \leq 1$ which is the fraction of the null component of momentum carried by each parton. This means we must rescale momenta to the dimensionless variable $x_B = \frac{p}{P}$. The probability density of a parton carrying a fraction x_B of the momentum is then

$$\phi(x_B) = \frac{P}{2\pi} |\tilde{\psi}(x_B P)|^2.$$

(The factor of $\frac{1}{2\pi}$ is needed because $\phi(x_B)$ is traditionally normalized to one with the measure dx_B rather than $\frac{dx_B}{2\pi}$.)

It is important to note that the only dimensional parameter in our theory, \tilde{g} , cancels out of the formula for $\phi(x_B)$: it only serves to set the scale of momentum and when the wavefunction is expressed in terms of the dimensionless variable x_B , it cancels out. The wave function only depends on the ratio $\frac{m^2}{\tilde{g}^2}$ and is independent of \tilde{g} in the limit of zero current quark mass. We have set μ^2 to the critical value (within numerical errors), which is the value corresponding to chiral symmetry; i.e., zero current quark mass. The number of colors we fix at $N = 3$. Thus the only free parameter in our theory is the fraction f of the baryon momentum carried by the valence partons. The parameters N and f appear in the combination $N_{\text{eff}} = \frac{N}{f}$.

We have ignored the isospin of the quarks in the above discussion. We should therefore compare our structure functions with the isoscalar combination of the valence quark distributions of a baryon, $\phi(x_B)$. It is not difficult to take into account the effects of isospin.

The parton distributions have been extracted from scattering data by several groups of physicists [4, 5, 6]. In the figure below we plot our wavefunctions and compare them to that extracted from data by the MRST collaboration, at $Q^2 = 1 \text{ GeV}^2$. We agree remarkably well with data except for small values of x_B . The agreement is best when the fraction of the baryon momentum carried by the valence partons is about $f \sim 0.5$.

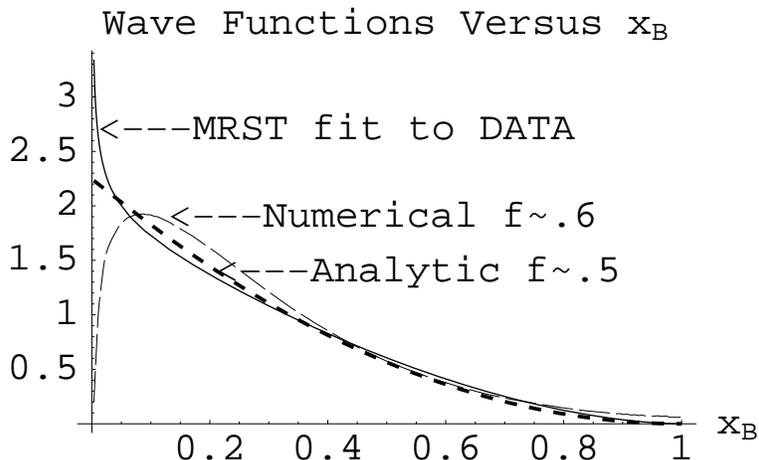


Figure 1. Comparison of the predicted valence parton wavefunction $\sqrt{\phi(x_B)}$ with the MRST [5] global fit to data. The wavefunction we predict goes to a non-zero constant at the origin. The numerical solution is not reliable in this region. The ‘analytic’ wave function is the variational estimate that minimizes the energy. The fit to data has a mild divergence at the origin.

Our model does not predict the observed behavior of the parton distributions for small x_B : our probability distribution tends to a constant for small x_B (Sections 4.1, 4.4) although due to numerical errors this is not evident in the numerical solution. The observed distributions have an integrable singularity there: roughly speaking, $\phi(x_B) \sim x_B^{-0.5}$ for small x_B . The approximations we made clearly break down in the small x_B region: transverse momenta and sea quarks can no longer be ignored, indeed even gluons need to be considered. We will study these effects in future publications.

5 Conclusion

This thesis has been about the structure of the proton. The Deep Inelastic Scattering experiments showed that the proton is made of point-like constituents, called quarks or partons. The structure of the proton is described in terms of parton distribution functions. These are momentum-space probability distributions of quarks inside the proton. Parton distribution functions are the analogues of electron wave functions in an atom, and are of great importance in particle physics. They are measured in Deep Inelastic Scattering experiments. However, we do not yet have a theoretical understanding of the x_B (momentum fraction or Bjorken Scaling variable) dependence of the parton distribution functions.

In this thesis we solve the problem of predicting valence quark distributions in the proton. We have presented a model of interacting partons for the structure of baryons. The valence quarks interact through a linear potential in the null coordinate. This model can be derived from QCD in the approximation where transverse momenta are ignored. We obtain the valence quark wave function as the solution to a non-linear integral equation. We understand the behavior of the solution both analytically and numerically. Our prediction is compared with the baryon structure function extracted from global fits to Deep Inelastic Scattering data. The only parameter we can adjust is the fraction of baryon momentum carried by the valence partons. Our predictions agree well with data except for small values of the Bjorken Scaling variable, x_B , where our model is not expected to be accurate. We cannot ignore gluons, sea quarks and transverse momenta. We shall study these effects in future publications.

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