# Large$-N$ Limit as a Classical Limit: Baryon in Two-Dimensional QCD and Multi-Matrix Models

by

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# Curriculum Vitae

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# Abstract

In this thesis, I study the limit of a large number of colors ($N$) in a non-abelian gauge theory. It corresponds to a classical limit where fluctuations in gauge-invariant observables vanish. The large-dimension limit for rotation-invariant variables in atomic physics is given as an example of a classical limit for vector models.

The baryon is studied in Rajeev's reformulation of two-dimensional QCD in the large-$N$ limit: a bilocal classical field theory for color-singlet quark bilinears, whose phase space is an infinite grassmannian. In this approach, 't Hooft's integral equation for mesons describes small oscillations around the vacuum. Baryons are topological solitons on a disconnected phase space, labelled by baryon number. The form factor of the ground-state baryon is determined variationally on a succession of increasing-rank submanifolds of the phase space. These reduced dynamical systems are rewritten as interacting parton models, allowing us to reconcile the soliton and parton pictures. The rank-one ansatz leads to a Hartree-type approximation for colorless valence quasi-particles, which provides a relativistic two-dimensional realization of Witten's ideas on baryon structure in the $1/N$ expansion. The antiquark content of the baryon is small and vanishes in the chiral limit. The valence-quark distribution is used to model parton distribution functions measured in deep inelastic scattering. A geometric adaptation of steepest descent to the grassmannian phase space is also given.

Euclidean large-$N$ multi-matrix models are reformulated as classical systems for $U(N)$ invariants. The configuration space of gluon correlations is a space of non-commutative probability distributions. Classical equations of motion (factorized loop equations) contain an anomaly that leads to a cohomological obstruction to finding an action principle. This is circumvented by expressing the configuration space as a coset space of the automorphism group of the tensor algebra. The action principle is interpreted as the partial Legendre transform of the entropy of operator-valued random variables. The free energy and correlations in the $N \to \infty$ limit are determined variationally. The simplest variational ansatz is an analogue of mean-field theory. The latter compares well with exact solutions and Monte-Carlo simulations of one and two-matrix models away from phase transitions.

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# Chapter 1

# Introduction

## 1.1 QCD and the Large-$N$ Classical Limit

In this section, we motivate and provide the historical context for the issues studied in this thesis. We use analogies with celestial mechanics and atomic physics, since they are familiar and more mature physical theories. However, the analogies are not to be taken too literally. A summary of the thesis and a short survey of the literature is also included in this introduction.

### The Standard Model: Strong and Electroweak Interactions

At the turn of the twenty-first century, the standard model of particle physics for electroweak and strong interactions is at an intermediate stage of development, one that is similar to a stage in mechanics after Newton's force laws had been discovered, but hamiltonian and lagrangian mechanics were still being developed. The latter are not merely elegant repackagings of the force laws in terms of single functions, but provide a way to go beyond the two body problem in a systematic manner, and passage to continuum mechanics. Moreover, it is difficult to imagine doing statistical mechanics without the idea of a hamiltonian. What is more, these developments played a key role in the discovery of the "next theory", quantum mechanics. Drawing an analogy with atomic physics, we are at a stage after the discovery of quantum mechanics, but before its reformulation in terms of path integrals. Path integrals are indispensable to the

theoretical development that followed quantum mechanics, quantum field theory.

Spectacular experimental discoveries, such as patterns found in the hadronic spectrum [40, 83], electroweak gauge bosons [89], and scaling in deep inelastic scattering [38], parallel the discovery of the detailed orbits of planets and moons, or of discrete atomic spectra. On the other hand, equally deep theoretical discoveries such as the gauge principle [106], electroweak unification [102], asymptotic freedom in non-abelian gauge theories [57], and perturbative renormalizability of unbroken and spontaneously-broken gauge theories [56] remind us of the discovery of the inverse-square force law between the sun and planets, or of the uncertainty principle and the Schrodinger equation for electrons in atoms.

Based on experiment and guiding principles such as gauge invariance and renormalizability, the lagrangian of the standard model has now been almost completely defined. The standard model is based on non-abelian gauge theories (Yang-Mills theories) for the force carriers: gluons for the strong force, and the $W$ and $Z$ electroweak gauge bosons, which are generalizations of the photon of electromagnetism. These bosons couple to fermionic matter particles, the quarks and leptons (which include the electron). Finally, an untested part of the standard model consists of scalar fields that are expected to provide spontaneous breaking of the electroweak gauge symmetry.

Quantum Chromodynamics (QCD) is the unbroken non-abelian gauge theory describing the interactions of quarks and gluons. In a gauge theory, the basic gluon and quark fields have some redundant degrees of freedom that are not observable. The true dynamical degrees of freedom (the "hadrons") are determined by the principle of gauge invariance. In QCD, electric charge is replaced by color charge of quarks and gluons. Unlike photons, which are electrically neutral, gluons carry color charge. QCD also has the property of asymptotic freedom which, at high momentum transfers, suggests that quarks and gluons behave as free particles (up to logarithmic corrections). This is in contrast to the electrostatic force between charges or the gravitational force between planets and the sun, which grow stronger at short distances. On the other hand, the strong force does not fall off at large distances, but leads instead to the phenomenon of confinement. In fact, the "fundamental particles", quarks and gluons, have never been

isolated. Unlike the electron, which can be removed from an atom by ionization, quarks and gluons seem to be confined inside (colorless, gauge-invariant) bound states of the strong interactions. These are called mesons (such as the pion) and baryons (such as the proton and neutron). In addition, there are glueballs, which are gauge-invariant bound-states of gluons alone. However, there is no quantum number that distinguishes glueballs from mesons. These are collectively referred to as hadrons.

### The success of perturbation theory around the $\hbar \to 0$ classical limit

Non-abelian gauge theories, being quantum field theories, can be illuminated through perturbation around a classical limit. A classical limit is one in which the fluctuations in *some* observables vanish. In the limit of vanishing Planck's constant, the fluctuations in *all* observables of a gauge theory, not just those that are gauge-invariant, vanish. So it is not surprising that the loop expansion around the free-field classical limit of $\hbar \to 0$ ("perturbation theory") has been tremendously successful in making quantitative predictions about electroweak interactions. The latter are not just weak at currently accessible energies, but also involve a spontaneously-broken gauge symmetry. Thus, *all* states, not just the gauge-invariant ones, and in particular, the gauge bosons $W^{\pm}, Z^0$, are observable.

As for the strong interactions, because of asymptotic freedom, perturbation theory makes accurate predictions about some aspects of their high-energy behavior. In particular, logarithmic scaling violations, and other phenomena where quarks and gluons can be treated as almost free, are well described by perturbation theory around the $\hbar \to 0$ limit. Indeed, QCD is the only known renormalizable four-dimensional theory with the correct high-energy behavior. This is primarily why it is believed to be the correct theoretical model for strong interactions. (The quantum theory can also be studied around non-trivial classical solutions in the $\hbar \to 0$ limit, such as solitons and instantons [85]).

### Why study QCD? What is left to do?

The discovery of the basic lagrangian, and the verification of its predictions for short distances, is only the first step in understanding dynamics. The finite-time and asymptotic behavior of solutions of the differential equations of classical mechanics hold a lot

of information and surprises not evident over short times. In celestial mechanics, elliptical orbits of planets could be derived easily from Newton's equations of motion. But the effects of Jupiter on the motion of other planets, or the estimation of the perihelion shift of Mercury, required reformulations of the theory and the development of new approximation methods, such as classical perturbation theory. These eventually led to the discovery of general relativistic corrections to newtonian gravity. Another example is the study of the stability of the solar system, which in turn led to the development of chaotic dynamics and Kolmogorov-Arnold-Moser theory [9].

We have a similar situation in QCD. Perturbation theory around the trivial vacuum in the $\hbar \to 0$ limit allows us to examine the strong interaction at only short distances, where quarks and gluons are almost free. This tells us very little about the long-distance behavior or about bound states of quarks and gluons. In fact, the problem is quite severe, since *all* observed particles are hadronic bound states. A mechanism that explains both qualitatively and quantitatively the confinement of quarks and gluons within hadrons has not been found. The empirically deduced long-range linear potential between quarks has also not been derived from QCD. How quarks and gluons bind to form hadrons, in other words the structure functions of these bound states (such as the proton), cannot be understood by merely perturbing around the vacuum. Though gluons are massless, the lightest observed particle of the theory is massive. Understanding this mass gap in the hadronic spectrum is an outstanding challenge. There are other peculiarities in the mass spectrum of hadrons. For example, the linearly rising Regge trajectories, where the angular momenta of hadronic resonances (short-lived particles) are linearly related to the squares of their masses, suggests that quarks are held together by a string with constant tension [44]. Chiral symmetry-breaking is yet another phenomenon one would like to establish in QCD. Perhaps its most dramatic manifestation is that practically massless quarks (up and down, $5 - 7$ MeV) bind to form the proton, which is over a hundred times as heavy (938 MeV). The situation is such that, even though QCD is believed to be the correct theory, it has not been possible, aside from large-scale numerical simulations and certain non-relativistic situations involving mesons with two heavy quarks, to calculate the mass or wave function (structure function) of even a

single observed particle in the theory! However, there are many simplified models such as dimensional reductions, strong coupling expansions, low energy effective theories or supersymmetric generalizations that exhibit some of these features. One hopes that further progress can be made by combining new physical and mathematical ideas. Of course, there is a wealth of experimental data [83] and increasingly accurate numerical simulations (see for example [25, 98]) with which to compare. Finally, we can hope that a deeper understanding of QCD will lead us to physical principles and mathematical structures, that are needed for any future physical theory that observation may require us to invent. For example, a deep appreciation for the hamiltonian and Poisson bracket formulation of classical mechanics led Dirac to invent the canonical formalism, which is more convenient for quantum field theory than the original Schrodinger equation of single particle quantum mechanics.

### Alternative Classical Limits

Quantum theories often have more than one classical limit in which some observables do not fluctuate. Usually, each is characterized by assuming a limiting value for some parameter. Different classical limits are useful for formulating different phenomena characterizing the same underlying quantum theory. For instance, in atomic physics, the $\hbar \to 0$ classical limit gives no indication of why the atom is stable. However, there are other classical limits in atomic physics. The limit of a large number of dimensions is a classical limit where rotationally invariant variables do not fluctuate. This limit provides a "classical" explanation for why the electron does not fall to the minimum of the Coulomb potential (see §1.3). Atomic Hartree-Fock theory is yet another classical limit that provides a way of studying many-electron atoms [87].

QCD has a different classical limit from $\hbar \to 0$. This is the limit of a large number of colors $N$, where the structure group of the gauge theory is $SU(N)$. There appear to be three colors in nature. Fluctuations in gauge-invariant observables *alone* vanish in the large-$N$ limit. Thus, we should expect the large-$N$ classical limit to be a better approximation than the $\hbar \to 0$ limit for unbroken gauge theories with strongly-coupled gauge fields, where it is only the gauge invariant states that are observed. This is the regime of interest for the strong-interaction phenomena mentioned previously. The large-

$N$ limit, as an approximation for non-abelian gauge theories, was originally proposed by 't Hooft in a perturbative context [55]. In this limit, planar Feynman diagrams were found to dominate. 't Hooft calculated the spectrum of mesons in the large-$N$ limit of two-dimensional QCD by summing such planar diagrams [53], and found an infinite tower of bound states, and the analogs of linearly-rising Regge trajectories. The planar diagrams also suggested a connection to a string model of hadrons which was expected from the linearly rising potential and Regge trajectories. We will see later that summing such planar diagrams only gives the linear approximation around the vacuum of the large-$N$ limit. Nevertheless, several other empirical facts about the strong interaction also seemed likely to be accommodated in the large-$N$ limit (see Witten [104]). For example, Zweig's rule, an empirical deduction concerning, for instance, the mixing of mesons with glue or exotic states, is exact in the linear approximation to the large-$N$ limit where mesons are pure $q\bar{q}$ states [104]. Thus, there are theoretical and phenomenological reasons to expect that $N = \infty$ is a good starting point for an approximation to QCD, though only $N = 3$ corresponds to nature.

Despite the success of the diagrammatic point of view, it obscures the classical nature of the large-$N$ limit. Though it is believed that the large-$N$ limit of QCD is a classical limit, its formulation as a classical dynamical system is not well understood. In particular, we would like to identify the classical configuration space, the equations of motion and action or hamiltonian, and also develop the approximation methods required to understand the theory. In this thesis, we present some investigations of the large-$N$ limit as a classical limit for two simplified models of QCD: the quark structure of baryons in two-dimensional QCD, and matrix models as a simple model for gluon structure.

**More on why QCD is difficult: Divergences and Gauge Invariance**

Three of the many aspects that make QCD hard to understand are the large number of degrees of freedom, gauge invariance, and ultraviolet divergences. As currently formulated, QCD is a divergent quantum field theory. The only known way to make finite predictions in four space-time dimensions, aside from the numerical lattice-QCD approach, is by the procedure of perturbative renormalization [56]. While very successful, this procedure is inherently tied to the perturbative solution of the theory. Rather

than address the important question of finding an alternative finite formulation of the theory, we will work with regularized versions of the theory. One such regularization is a matrix model, which reduces the number of degrees of freedom. We will also work with a reduction to two dimensions, where the theory is ultraviolet finite without any need for regularization. This will allow us to focus attention on the difficulties arising from gauge invariance.

In a gauge theory, we have the unusual situation where a physical theory is formulated in terms of unobservable particles, the quarks and gluons. To accommodate the properties of the observed hadrons, it is then necessary to rewrite the theory in terms of gauge-invariant observables. The principle of gauge invariance is no longer relevant when we work with a gauge-invariant formulation of the theory. We must look for other geometric, probabilistic, or algebraic principles that play as important a role. This is still a formidable task in the full $3+1$ dimensional theory. So we address this question in the simpler contexts mentioned above. Quarks transform as $N$-component vectors under color, the $SU(N)$ structure group, while gluons transform in the adjoint representation as $N \times N$ hermitian matrices. The components of these vectors and the matrix elements are the so-called "gauge degrees of freedom" (quarks and gluons) that carry the color quantum number, and are not directly observable. Only color-invariant combinations are observable. Furthermore, gauge-invariant observables (the Wilson loop and meson observables) are non-local, which means that we need to make the passage from local gauge fields to non-local loop or string-like variables.

### The Baryon in the Large-$N$ Limit of 2d QCD

Vector models have fewer degrees of freedom, making them easier to deal with than matrix models. We therefore devote the first half of this thesis to the vector model of quarks in two-dimensional QCD, interacting via a linear potential due to longitudinal gluons. 't Hooft's work on two-dimensional QCD left a puzzle as to how to handle baryons. An early proposal of Skyrme was that baryons must arise as solitons in a theory of mesons [95]. However, 't Hooft's equation for mesons was linear, and did not support soliton solutions. Rajeev discovered a bilocal (along a null line) reformulation of two-dimensional QCD in terms of color-invariant quark bilinears. In the large-$N$ classical

limit, it is a hamiltonian dynamical system whose phase space is an infinite dimensional grassmannian manifold. The phase space is a curved manifold because the quark density matrix is a projection operator, making this theory strongly interacting even in the large-$N$ limit. This is an indication of the type of geometric ideas that may play a role in a gauge-invariant reformulation of QCD. 't Hooft's meson spectrum is recovered as a linear approximation to the equations of motion on the grassmannian around the naive vacuum. Moreover, the phase space is disconnected, with components labelled by an integer-valued baryon number. The baryon arises as a topological soliton, and we study this in this thesis. In particular, we determine the "shape", or more precisely, the form factor, which tells us, roughly, how quarks are distributed inside the proton, though in a color invariant manner, without explicit reference to color carrying quarks.

We also examine another puzzle regarding the structure of the proton: the relation between the soliton and parton pictures. The parton model is a complementary view to the soliton picture of the baryon [36, 17]. Deep inelastic scattering experiments [38] indicate that the constituents of the proton, called partons, are point-like. This is because the proton structure function is roughly scale invariant, i.e., independent of the length scale at which it is observed. This was accommodated in the parton model by postulating that the proton is made of point-like partons. The latter had to be non-interacting in order to match the observed high-energy behavior. Partons were identified with quarks and gluons after the discovery of QCD. Their non-interacting behavior at high energies was regarded as a consequence of their asymptotic freedom. The QCD-improved parton model (perturbative QCD) made accurate predictions for how scale invariance is broken, i.e., the logarithmic scale dependence of structure functions. However, the dependence of structure functions on the momentum fraction carried by a parton, which is akin to an atomic wave function, is non-perturbative and remained inaccessible. In Part I of this thesis, we derive an interacting parton model from the solitonic point of view, thereby reconciling the two disparate points of view in two space-time dimensions. We also determine the non-perturbative momentum-fraction dependence of the quark structure function within two-dimensional QCD. This two-dimensional theory provides a good approximation to Deep Inelastic Scattering, and we use it to model the quark

distributions measured in experiment (§3.5).

### Large-$N$ Matrix Models

Our analysis of quark structure still leaves the gluon distribution undetermined. Gluons are the force carriers between quarks, just as photons transmit the force between electric charges in QED. However, QED is an abelian gauge theory while QCD is a non-abelian gauge theory, leading to strong self interactions between gluons. One manifestation of this is in the ability of gluons (unlike photons) to form hadronic bound states, referred to as glueballs. The non-abelian nature of gluons also leads to another phenomenon that makes the strong interactions very different from the quantum mechanics of an atom. The electron wave function determines the shape of an atom, and, within the non-relativistic theory, the photon does not carry any momentum of an atom. But for a proton, when the momentum transferred by the probe is about 1 GeV, it is an experimental fact that quarks and gluons carry roughly equal portions of its total momentum [21]. The contribution from gluons only grows with the probe's momentum transfer [21].

Of the four possible states of polarization of the gluon in four dimensions, only two are dynamical i.e. its transverse states. The longitudinal and time components can be eliminated by gauge fixing and solving the resulting constraint equation. Within two-dimensional QCD, there are no transverse gluons, and they were ignored previously. Now we turn to dynamical gluons, namely, the propagating gluon degrees of freedom which cannot be eliminated by gauge fixing. In Part II of this thesis, we study euclidean multi-matrix models in the large-$N$ classical limit. They are a regularized version of gluon dynamics. Understanding gluon dynamics is much harder than quark dynamics. The reason is that there are many more gauge-invariant gluon observables than pure quark observables. The dot products of pairs are the only invariants of a set of vectors, while the traces of arbitrary products of a collection of matrices (the gluon correlation tensors) are all unitary invariants.

It is believed that the large-$N$ limit of matrix models comprises a classical theory. In this thesis, we identify this classical theory in a manifestly unitary invariant manner. We show that the configuration space is a space of non-commutative probability distri-

butions. The gluon field is an $N \times N$ matrix-valued random variable. These matrices at different points of space-time do not commute, and as a consequence we are dealing with a non-commutative version of probability theory (Chapter 4). The coordinates on the configuration space are gluon correlations. Their fluctuations vanish in the large-$N$ limit, and satisfy the factorized Schwinger-Dyson (or loop) equations. We identify these as the classical equations of motion (§5.1). However, a classical action is very elusive. There is an anomaly in the equations arising from the transformation from matrix elements to invariants, manifested as a cohomological obstruction to finding an action on the configuration space (§5.2.1). We circumvent this obstruction by expressing the configuration space as a coset space of a non-commutative analogue of the diffeomorphism group by an isotropy subgroup. We then find a classical action on the group that is invariant under the action of the isotropy subgroup. So, the action really lives on the quotient, i.e., the configuration space (§5.2.2). Our search for principles that determine a gauge-invariant formulation of large-$N$ matrix models leads us to the automorphism group of the free algebra and its first cohomology!

What is more, we find that the entropy of non-commutative probability theory, developed as a branch of operator algebra by Voiculescu and collaborators [99], plays a central role. Whenever we restrict the allowed observables of a physical system, we should expect an entropy due to our lack of knowledge of those observables. For example, entropy in statistical mechanics arises because we do not measure the velocities of individual gas molecules but only macroscopic variables such as pressure and density. Similarly, confinement of the color degrees of freedom should lead to an entropy in the strong interactions. In a matrix model, expressing the action in terms of unitary invariant gluon correlations rather than unobservable matrix elements, leads to an entropy. We find (§5.3) a variational principle for the solution of the classical equations of motion for gluon correlations by maximizing entropy subject to certain constraints.

### Approximation Methods

Progress in newtonian mechanics came about through at least two ways: (1) general approximation methods, or alternative formulations of the theory, and (2) exact solutions of some special systems. Examples of (1) are hamiltonian and lagrangian mechanics,

Hamilton-Jacobi theory, perturbation theory and variational principles. Examples of (2) are Jacobi's solution of the rigid body, and the theory of integrable systems.

A common theme throughout this thesis is the use of approximation methods to solve the classical theories we get in the large-$N$ limit. A classical limit is itself an approximation. However, both the classical theories we get, dynamics on an infinite grassmannian for quarks in two-dimensional QCD, and the maximization of entropy in multi-matrix models, are highly non-linear and non-local classical theories. They require the development of new non-perturbative approximation methods. Here, we take inspiration from the approximation methods of classical mechanics, atomic physics, and many-body theory. We look for analogs of variational principles, mean-field theory, Hartree-Fock theory, steepest descent, and the loop expansion.

Finding the ground-state of the baryon involves minimizing its energy on an infinite dimensional curved phase space. The first method we develop is a geometric adaptation of steepest descent for a curved phase space (§3.1). Then we find a variational approximation method that replaces the full phase space with finite dimensional submanifolds, and we study the dynamical system on reduced phase spaces (§3.2, §3.4). In the simplest case, we get an analogue of mean-field theory (Hartree-Fock theory) for a system of interacting colorless quasi-particles. This is how we are able to derive the interacting valence parton picture from the exact soliton description of the baryon (§3.2.2, §3.2.3). We also show how to go beyond this, and include anti-quarks (§3.4.2). Though the emphasis is on approximate solutions, along the way we also find the exact form factor of the baryon in the large-$N$ limit of two-dimensional QCD for massless current quarks (see (3.68)). We compare our approximate solution for the quark distribution function with numerical calculations [58] and also measurements from Deep Inelastic Scattering, and find good agreement (see §3.5).

Since the 1980s significant progress was made in finding exact solutions for partition functions and special classes of correlations of carefully chosen matrix models, such as two matrix models with specific interactions, matrix chains associated with Dynkin diagrams of simply-laced Lie algebras etc. The methods often originated in the theory of integrable systems or conformal field theory (see for example [78, 77, 97, 66, 34]). However, there

is a lack of approximation methods to handle generic matrix models in the large-$N$ limit. Even the analogs of simple methods such as mean-field theory and variational principles were previously not known. We find a variational principle that allows us to determine the gluon correlations from a finite-parameter family that best approximate the correlations of a given matrix model (§5.2). We use this variational principle to find an analog of mean-field theory for large-$N$ multi-matrix models (§5.4), and also indicate how one can go beyond mean-field theory (§B.3). These approximation methods compare favorably with exact solutions, and with a Monte-Carlo simulation away from divergences in the free energy (§5.4.1, §5.4.2). For other approaches to solving matrix models, see for instance the work of J. Alfaro et. al. [8].

**Literature on Matrix Models in High Energy Physics**

Since the 1970s, there has been a great deal of work done on large-$N$ matrix models. Some of the earlier papers are reproduced in the collection of Ref. [19]. We mention a few of the many developments. Brezin, Itzykson, Parisi and Zuber studied the euclidean one-matrix model using the saddle point method in the large-$N$ limit. They found an important relation between the quantum mechanics of a single matrix in the large-$N$ limit and a system of free fermions [18]. Migdal and Makeenko found that the Wilson loops of a large-$N$ gauge theory satisfy a closed set of "factorized loop" equations [74]. Yaffe's coherent states approach [105], and Sakita's and Jevicki's [90, 62] work on the collective field formalism of large-$N$ field theories was an important step in understanding an anomaly in the hamiltonian of the large-$N$ limit. The anomaly is one of the main differences relative to the $\hbar \to 0$ classical hamiltonian. Our recent work shows that this anomaly is in fact the non-commutative analogue of Fisher information of probability theory [1]. The papers of Cvitanovic and collaborators [26, 27] on planar analogues of some of the familiar methods of field theory, but with non-commutative sources was helpful in our algebraic formulation of the problem. Eguchi's and Kawai's [32] proposal on reducing a matrix field theory to a matrix model with a finite number of degrees of freedom, but in the large-$N$ limit, has been a recurring theme ever since.

Important breakthroughs in the study of random surfaces, two-dimensional string theory, and two-dimensional quantum gravity coupled to matter, were made after the

mid 1980s (see Ref. [42] for a review). The planar Feynman-graph expansion of large-$N$ matrix models was used as a way of discretizing a two-dimensional surface. Models with one or a finite number of matrices, and the $c = 1$ quantum mechanics of a single matrix, were of importance in these developments. The double scaling limit was developed to study surfaces obtained in the continuum limit. In the double scaling limit, the coupling constants are tuned to critical values as $N \to \infty$. This limit is not a classical limit, unlike the 't Hooft large-$N$ limit. Fluctuations in observables remain large in the double scaling limit. However, the double scaling limit allows one to include contributions from all genera in the topological expansion of Feynman diagrams.

The work of Seiberg and Witten in the early 1990s on electric-magnetic duality allowed the elucidation of vacuum structure of a large class of supersymmetric gauge theories along with the mechanisms of chiral symmetry-breaking and confinement in these cases [92].

In the mid 1990s, supersymmetric matrix models were proposed as non-perturbative definitions of M-theory and superstring theory [13, 59]. From the late 1990s onwards, there has been a great deal of work on the large-$N$ limit of supersymmetric gauge theories, catalyzed by the AdS/CFT correspondence of Maldacena [75]. Large-$N$ matrix models are also used to study the effective super potentials for $\mathcal{N} = 1$ supersymmetric gauge theories with adjoint chiral super fields, following the work of Dijkgraaf and Vafa [28, 22]. Matrix models also find applications to the problem of determining the anomalous dimensions of operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [80, 3].

Random-matrix theory has also been applied to the spectrum of the QCD Dirac operator (see especially the work of Verbaarshot et. al. [101]).

### Random matrices in other areas of Physics and Mathematics

Remarkably, random matrices have found applications in many areas of physics and mathematics outside of particle and high-energy physics. We list a few of them.

Random matrix theory originally arose from the suggestions of Wigner and Dyson in the 1950s, that the statistical properties of the spectra of complicated nuclei could be modelled by a random hamiltonian [77].

Spin systems on random two-dimensional lattices have been studied using the large-

$N$ limit of matrix models. For example, Kazakov studied the Ising model on a random two-dimensional lattice with fixed coordination number [65].

Random matrices also have deep connections to statistical properties of zeros of the Riemann zeta function [77]. Montgomery and Dyson discovered that the pair correlation of scaled zeros of the Riemann zeta function is asymptotic to that of eigenvalues of a large random unitary matrix [81]. More recently, the universal part of the moments of the zeta function on the half line have also been found to be related to those of the characteristic polynomial of a large random unitary matrix [67].

Some chaotic quantum systems have been modelled by universal properties of large-$N$ matrix models, as have the universal correlations in some mesoscopic and disordered systems [47, 20].

The work of mathematicians, including Voiculescu and collaborators, on von Neumann algebras led to the development of the field of non-commutative probability theory [99]. The correlations of large-$N$ matrix models give natural examples of non-commutative probability distributions.

## 1.2 QCD, Wilson Loop, Gluon Correlations

### Classical Chromodynamics

Classical Chromodynamics (the $\hbar \to 0$ limit of QCD) in $3+1$ space-time dimensions is a non-abelian gauge theory with structure group $SU(N)$, where the number of colors is $N = 3$ in nature. The matter fields $q^a_\alpha(x)$ are quarks, spin $\frac{1}{2}$ fermions transforming under $N_f$ copies of the fundamental representation of $SU(N)$. '$\alpha$' is a flavor index and '$a, b$' are color indices. Quarks come in $N_f$ flavors, where $N_f = 6$ (up, down, strange, charm, bottom and top in order of increasing "current" quark mass; though quarks have not been isolated and "weighed", one can define their mass using their interactions with electroweak currents). At low energies, only up and down are important for the proton and neutron. For the most part, we will ignore flavor dependence in this thesis. The bosonic gauge (gluon) fields $[A_\mu(x)]^a_b$, $\mu = 0, 1, 2, 3$ are four $N \times N$ hermitian matrix-valued fields. They are the components of a connection one-form on Minkowski

space-time $\mathrm{R}^{3,1}$, valued in the Lie algebra of $SU(N)$. Let $\mathcal{A} = \{A_\mu(x)\}$ denote the space of connections. The theory is defined by the action

$$S_0 = -\frac{N}{4g^2}\int d^4x \ \mathrm{tr}\ F^{\mu\nu}F_{\mu\nu} + \sum_{\alpha=1}^{N_f}\int \bar{q}^a_\alpha[-i\gamma^\mu[D_\mu]^b_a - m_\alpha\delta^b_a]q_{\alpha b}d^4x \tag{1.1}$$

$\gamma^\mu$ are the Dirac gamma matrices. Here the Yang-Mills field strength is the anti-hermitian matrix field $F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The covariant derivative is $D_\mu = (\mathbf{1}\partial_\mu - iA_\mu)$ where $\mathbf{1}$ is the $N \times N$ unit matrix. The coupling constant $g$ is dimensionless in $3+1$ dimensions. The classical solutions are field configurations $(A_\mu(x),\ q(x), \bar{q}(x))$ that extremize the action, the solutions of the partial differential equations $\frac{N}{g^2}D_\mu F^{\mu\nu} = \frac{N}{g^2}(\partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}]) = j^\nu$. Here $[j^\nu]^a_b = \bar{q}^a\gamma^\nu q_b$ is the quark current.

Notice that the action $S_0$ is Lorentz invariant. However, not all observables need to be Lorentz invariant. Lorentz transformations (which are isometries of Minkowski space preserving the metric $ds^2 = (dx^0)^2 - (dx^1)^2 + (dx^2)^2 + (dx^3)^2$) merely relate observables in different reference frames.

Now, the action $S_0$ is also invariant under the group of local gauge transformations $\mathcal{G} = \{U(x)\}$

$$A_\mu(x) \mapsto UA_\mu U^{-1} + U\partial_\mu U^{-1}; \qquad q(x) \mapsto U(x)q(x) \tag{1.2}$$

where $U(x)$ is a map from space-time to the structure group $SU(N)$ that tends to the identity at infinity. The principle of gauge invariance states that not just the action, but every observable of the theory must be invariant under gauge transformations! The most famous gauge-invariant observable is the parallel transport along a closed curve $\gamma : [0,1] \to \mathrm{R}^{3,1};\ \gamma(0) = \gamma(1)$, the so-called Wilson loop observable

$$W(\gamma) = \frac{\mathrm{tr}}{N}\mathcal{P}\exp\Big\{ -i\oint_0^1 A_\mu(\gamma(s))\ \dot{\gamma}^\mu(s)\ ds\Big\} \tag{1.3}$$

where $\mathcal{P}$exp stands for the path ordered exponential. One can also consider an open string or meson observable with quarks at the end points (here $\gamma$ is not a closed curve)

$$M(\gamma) = \frac{1}{N}\bar{q}(\gamma(0))^a\Big[\mathcal{P}\exp\Big\{ -i\int_0^1 A_\mu(\gamma(s))\ \dot{\gamma}^\mu(s)\ ds\Big\}\Big]^b_a q(\gamma(1))_b. \tag{1.4}$$

We shall give the physical interpretation of the Wilson loop when we discuss its expectation value in the quantum theory.

### Quantum Chromodynamics

So far, we have been discussing Classical Chromodynamics. In the path integral approach to quantization, $(A_\mu(x),\ q(x),\ \bar{q}(x))$ become random variables. Quantum Chromodynamics (QCD) is the assignment of expectation values to gauge-invariant functions of these random variables. Naively, they are obtained by averaging over the quark and gluon fields with a weight given by $e^{iS_0/\hbar}$. However, on account of the gauge invariance of the action, this functional integral is ill-defined. Rather than average over the entire space of connections $\mathcal{A}$, we should be averaging over the space of connections modulo gauge transformations $\mathcal{A}/\mathcal{G}$ with the measure induced by the Lebesgue measure on $\mathcal{A}$. $\mathcal{A}/\mathcal{G}$ parameterizes the true degrees of freedom according to the gauge principle. The idea is to choose a representative for each orbit of $\mathcal{G}$ in $\mathcal{A}$ (gauge fixing) and integrate over the coset representatives. In effect, the gauge field has only two independent components (for example, the transverse polarization states) after taking into account the relations imposed by gauge transformations. However, in order to maintain manifest Lorentz covariance, it is sometimes more convenient to retain all four components of the gauge field, and introduce Fadeev-Popov ghost fields (a pair of grassmann-valued hermitian matrix fields, $c_b^a(x), \bar{c}_b^a(x)$) that act as negative degrees of freedom. The standard implementation of this idea [61] in the so-called covariant gauges leads to the gauge fixed action

$$S(A,q,\bar{q},c,\bar{c}) = S_0 + \int d^4x \Big[ -\frac{1}{2\xi}\ \mathrm{tr}\ (\partial^\mu A_\mu(x))^2 +\ \mathrm{tr}\ \partial_\mu \bar{c}(x) D^\mu c(x) \Big] \tag{1.5}$$

The two additional terms in the action come respectively from the gauge fixing and the Fadeev-Popov determinant, which is the jacobian determinant for the induced measure. The expectation values of gauge-invariant observables $\mathcal{O}(A,q,\bar{q})$ are independent of the gauge parameter $\xi$ and are given by

$$\langle \mathcal{O}(A,q,\bar{q}) \rangle = \frac{\int d(A,q,\bar{q},c,\bar{c}) e^{iS/\hbar}\ \mathcal{O}(A,q,\bar{q})}{\int d(A,q,\bar{q},c,\bar{c}) e^{iS/\hbar}} \tag{1.6}$$

For example, if $\mathcal{O}$ is the Wilson loop observable, then

$$\langle W(\gamma) \rangle = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} G_{\mu_1 \cdots \mu_n}(\gamma(s_1), \cdots, \gamma(s_n))\ \theta(0 \le s_1 \le s_2 \le \cdots \le s_n \le 1) \tag{1.7}$$

where the gluon correlation tensors are

$$G_{\mu_1\cdots\mu_n}(x_1,\cdots,x_n) = \langle \frac{\mathrm{tr}}{N} A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\rangle \tag{1.8}$$

Though the gluon correlation tensors are not gauge-invariant, once they have been determined in any specific gauge, they may be used to compute the expectation values of other gauge-invariant quantities such as the Wilson loop. Since the Wilson loop $W(\gamma)$ and meson observables $M(\gamma)$ are gauge invariant, their expectation values can be calculated in any convenient gauge.

The gauge fixed action (1.5), is no longer invariant under local gauge transformations, but rather under global unitary transformations $q(x) \mapsto Uq(x)$, $\bar{q}(x) \mapsto \bar{q}(x)U^\dagger$, $A_\mu(x) \mapsto UA_\mu(x)U^\dagger$, $c(x) \mapsto Uc(x)U^\dagger$, $\bar{c}(x) \mapsto U\bar{c}(x)U^\dagger$. In other words, we have a matrix field theory (of both bosonic ($A_\mu$) and fermionic ($c, \bar{c}$) adjoint fields) coupled to a vector model. As it stands, this definition leads to ultra-violet divergent expectation values in $3+1$ dimensions, and we have to supplement this with rules for regularization and renormalization. In particular, the coupling "constant" is replaced by a "running coupling constant" $g^2(Q^2)$, which depends on the momentum scale $Q$. For large $Q^2$, we have the perturbative result that the coupling vanishes logarithmically: $g^2 \sim 1/\log(Q^2/\Lambda^2_{QCD})$. Renormalization introduces the dimensional parameter $\Lambda_{QCD}$ which sets the scale for the strong interactions and is to be determined experimentally ($\Lambda_{QCD} \sim 200$ MeV).

In this thesis, we study a pair of finite truncations of QCD. In Part I of this thesis we will study the vector model alone, though in two dimensions and in the null gauge where there are no ghosts, gluons can be eliminated and there are no ultra-violet divergences. In Part II we will study hermitian bosonic matrix models, which are matrix field theories where space-time has been regularized to have only a finite number of points.

**Wilson's Area Law Conjecture:** We conclude this section with Wilson's area law conjecture on the asymptotic expectation value of Wilson loop observables:

$$\langle W(\gamma)\rangle \sim e^{-\alpha' \mathrm{Ar}(\gamma)} \text{ as } \mathrm{Ar}(\gamma) \to \infty \tag{1.9}$$

where $\mathrm{Ar}(\gamma)$ is the minimal area of a surface whose boundary is $\gamma$. This is a statement about the vacuum of the pure gauge theory with no dynamical quarks. However, we

can heuristically interpret the conjecture as an asymptotically linear potential between quark sources. To see this, consider a current $j^\mu(x)$ which is concentrated on the curve $\gamma^\mu(s)$. It is regarded as the current density of a quark-antiquark pair that are produced and annihilated and whose combined trajectory is the closed curve $\gamma$. Then

$$\oint A_\mu(\gamma)\ \dot{\gamma}^\mu(s)\ ds = \int A_\mu(x) j^\mu(x) d^4x \tag{1.10}$$

and $\langle W(\gamma)\rangle$ is the probability amplitude for this process. In particular, consider a rectangular loop in the $x^1 - x^0$ plane with length $L$ and time $T$. Suppose further, that the potential energy between quarks is asymptotically linear, $E \sim \alpha' L$. $\alpha'$ is called the string tension since the linear potential corresponds to quarks being held together by a string with constant tension. Then the amplitude for this process is $e^{-ET} \sim e^{-\alpha' LT} = e^{-\alpha' Ar(\gamma)}$. Thus, the Wilson area law conjecture is a criterion for confinement of non-dynamical quarks by an asymptotically linear potential.

## 1.3 $d \to \infty$ as a Classical Limit in Atomic Physics

To motivate the idea of the large-$N$ limit as a classical limit of matrix models and gauge theories, let us first describe a much simpler but analogous idea in the more familiar area of atomic physics: the problem of determining the ground-state energy and wave function of electrons in an atom.

The $\hbar \to 0$ classical limit is *not* a good approximation, since the atom is not stable in this limit. The Coulomb potential is not bounded below and the electrons would fall into the nucleus. Of course, in the quantum theory, this is prevented by the uncertainty principle: momentum would grow without bound if we tried to concentrate the electron wave function at the minimum of the Coulomb potential. However, there is another classical limit, where the number of spatial dimensions $d \to \infty$, in which the atom has a stable ground-state! This classical limit can even be used as the starting point for an approximation method for $d = 3$.

We illustrate this idea for a hydrogenic atom of atomic number $Z$. The analogue of unitary invariance in matrix models is rotational invariance for the atom. So we will

consider the problem of minimizing the energy

$$E = \int \left[ \frac{\hbar^2}{2m} \frac{\partial \psi^*}{\partial x_i} \frac{\partial \psi}{\partial x_i} - \frac{Ze^2}{r} |\psi(\mathbf{x})|^2 \right] d^d x \tag{1.11}$$

subject to the unit norm constraint $\int |\psi(\mathbf{x})|^2 d^d x = 1$, in the zero angular momentum sector $\psi(\mathbf{x}) = \psi(r)$ where $r = (\Sigma_{i=1}^d x_i x_i)^{\frac{1}{2}}$. Note that we work in $d$ spatial dimensions but retain the three-dimensional Coulomb potential. While fluctuations in all observables vanish in the $\hbar \to 0$ classical limit, only the fluctuations in rotationally invariant observables vanish in the $d \to \infty$ classical limit that we study below.

We first transform from $x_i \mapsto r$, which introduces a jacobian $d^d x = \Omega_d r^{d-1} dr$. $\Omega_d$ is the surface area of the unit sphere $S^{d-1}$. Thus

$$E = \Omega_d \int \left[ \frac{\hbar^2}{2m} \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_i} |\psi'(r)|^2 - \frac{Ze^2}{r} |\psi(r)|^2 \right] r^{d-1} dr \tag{1.12}$$

and $< \psi | \psi >= \int \psi^*(r) \psi(r) \Omega_d r^{d-1} dr$. We absorb the jacobian by defining a *radial* wave function $\Psi(r) = \sqrt{\Omega_d} r^{(d-1)/2} \psi(r)$ so that it is normalized in a simple way $< \Psi | \Psi >= \int \Psi^*(r) \Psi(r) dr$. In terms of the radial wave function:

$$\begin{aligned} E &= \int \left[ \frac{\hbar^2}{2m} \left\{ |\Psi'(r)|^2 + \frac{(1-d)^2}{4r^2} |\Psi(r)|^2 + \frac{(1-d)}{2} \frac{1}{r} \frac{d}{dr} |\Psi(r)|^2 \right\} - \frac{Ze^2}{r} |\Psi(r)|^2 \right] dr \\ &= \int \left[ \frac{\hbar^2}{2m} |\Psi'(r)|^2 + \left\{ \frac{\hbar^2 (d-1)(d-3)}{8mr^2} - \frac{Ze^2}{r} \right\} |\Psi(r)|^2 \right] dr \end{aligned} \tag{1.13}$$

where we have integrated by parts and ignored the surface term which vanishes for square integrable wave functions. We see that a portion of the kinetic energy coming from derivatives of the jacobian and integration by parts now manifests itself as a correction to the three-dimensional Coulomb potential. The hamiltonian is

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{8m} \frac{(d-1)(d-3)}{r^2} - \frac{Ze^2}{r} \tag{1.14}$$

To take the $d \to \infty$ limit, we must re-scale to variables that have a finite limit: $\rho = \frac{r}{\sqrt{d}}$; $\pi = -\frac{i}{d} \frac{d}{d\rho}$; $\tilde{H} = \frac{H}{d}$; $\alpha = \frac{e^2}{d^{3/2}}$. The hamiltonian and commutation relation are

$$\tilde{H} = \frac{\hbar^2}{2m} \pi^2 + \frac{\hbar^2}{8m\rho^2} \frac{(d-1)(d-3)}{d^2} - \frac{Z\alpha}{\rho}; \quad [\pi, \rho] = -\frac{i}{d} \tag{1.15}$$

We see that the large $d$ limit holding $\hbar$ finite is a classical limit, $\pi$ and $\rho$ commute and fluctuations in the rotationally invariant variable $\rho$ are small. In this limit, we

get a classical mechanical system whose phase space is the half plane $\{\rho > 0, \pi\}$ with hamiltonian and Poisson bracket

$$\tilde{H} = \frac{\hbar^2}{2m}\pi^2 + \frac{\hbar^2}{8m\rho^2} - \frac{Z\alpha}{\rho}; \quad \{\rho, \pi\}_{P.B.} = 1. \tag{1.16}$$

The main difference between this classical limit and the usual $\hbar \to 0$ limit is the appearance of a centrifugal barrier to the Coulomb potential, even for zero angular momentum! Thus we have a "classical" explanation for the stability of the atom in the $d \to \infty$ limit.

The ground-state is given by the static solution $\pi = 0$, $\rho = \rho_0 = \frac{\hbar^2}{4mZ\alpha}$, which is the minimum of $V_{eff}(\rho) = \frac{\hbar^2}{8m\rho^2} - \frac{Z\alpha}{\rho}$. In other words, in the $d \to \infty$ classical limit, the electron wave function is concentrated at a radial distance of $\rho_0$ from the nucleus. To recover fluctuations in the electron position we must quantize this classical theory. The ground-state energy is $\tilde{E} = V_{eff}(\rho_0) = -\frac{2mZ^2\alpha^2}{\hbar^2}$. To quantitatively compare with the known answer in 3 dimensions we revert to the old variables $E = d\tilde{E}$ and $e^2 = \alpha\, d^{\frac{3}{2}}$ and set $d = 3$

$$\begin{aligned} E_{exact} &= -\frac{1}{2}\frac{mZ^2e^4}{\hbar^2} \\ E_{approx} &= -\frac{2}{9}\frac{mZ^2e^4}{\hbar^2} \end{aligned} \tag{1.17}$$

Alternatively, we could have compared the energy per dimension. Thus we have a crude first approximation to the ground-state energy of hydrogenic atoms. Similarly, $\rho_0$ provides a first approximation to the mean distance of the electron from the nucleus.

This idea has been extended to many electron atoms as well. One way of implementing the Pauli exclusion principle for many electron atoms is to let the number of spin states of the electron tend to infinity along with the number of dimensions. Then the wave function is taken to be totally anti-symmetric in spin quantum numbers. For example, D. Herschbach and collaborators [50] have calculated atomic energy levels in a $\frac{1}{d}$ expansion. The leading term is less accurate than other methods such as Hartree-Fock theory. However, its advantage is that the complexity does not grow as fast with the number of electrons, since the problem can be reduced to the algebraic minimization of an effective potential. Even more impressive is the spectacular accuracy they achieved by including the corrections in an asymptotic series in inverse powers of $d$. By resum-

ming this series, they obtained an accuracy of more than 9 significant figures for the ground-state of the Helium atom [51].

Thus we see that the $d \to \infty$ classical limit allows us to understand certain features of the theory that the $\hbar \to 0$ classical limit misses. It also serves as a starting point for an approximation method. We expect the $N \to \infty$ classical limit to play a similar role in matrix models and gauge theories.

The system we have just studied can also be considered as a non-relativistic $O(d)$-vector model in one dimension (time), where the position $x_i(t), i = 1, \cdots, d$ is a $d$-component vector. The restriction to zero angular momentum corresponds to requiring all observables to be $O(d)$ invariant. More generally, the large-$N$ limit of $O(N)$-vector models are of much interest especially in statistical physics. $N$ denotes the number of spin projections for example in a Heisenberg ferromagnet. The large-$N$ limit was first studied in the context of the spherical model by Stanley [96]. The $O(N)$ non-linear sigma model in three spatial dimensions describes phase transitions in three dimensions [60].

## 1.4 The Large-$N$ limit: Planar Diagrams and Factorization

The large-$N$ limit was introduced into the study of gauge theories and matrix models by 't Hooft, who showed that in the large-$N$ limit, with appropriately scaled coupling constants, the planar Feynman diagrams (or those that can be drawn on a sphere) dominate [55]. Indeed, the usual perturbative sum over Feynman diagrams can be reorganized according to the genus of the Riemann surface on which the diagram can be drawn.

Let us indicate how this works for a matrix field theory. The dynamical variable is an $N \times N$ hermitian matrix-valued scalar field $A^a_b(x)$, where $a, b$ are "color" indices. The partition function is

$$Z = \int dA exp\Big[ -\frac{N}{\hbar} \int d^d x \Big\{ \frac{1}{2} \text{ tr } \partial_\mu A(x) \partial^\mu A(x) + \frac{1}{2} \text{ tr } A^2(x) + \sum_{k \geq 3} g_k \text{ tr } A^k(x) \Big\} \Big] \quad (1.18)$$

We have kept an over all factor of $N$ multiplying the action. The limit $\hbar \to 0$ holding $N$ fixed is the usual classical limit. Letting $N \to \infty$ holding $\hbar$ fixed is a different classical limit. They are not the same because the matrix $A(x)$ depends on $N$, though not on

$\hbar$. Let us concentrate on the factors of $N$ that appear in a Feynman diagram due to the matrix-valued nature of the field. We will suppress factors of $\hbar$ and all space-time dependence, symmetry factors etc. It is convenient to use a double line notation, where propagators of $A_b^a$ are denoted by oppositely directed double lines, each carrying one of the two color indices. If we also had vectors (like quarks $q^a$), we would denote them by single lines carrying the single color index. Consider a connected Feynman graph in the perturbative expansion of the logarithm of the partition function (connected vacuum diagrams, no external legs). Suppose it has $L$ loops and $E$ propagators (edges). Vertices are due to the cubic and higher interactions. Suppose there are $V_k$ vertices of coordination number $k \geq 3$, then the total number of vertices is $V = \sum_{k\geq 3} V_k$.

Now let us think of each loop as the boundary of a face, its orientation is given by the direction in which the color index flows on the loop which is its boundary. The propagators that make up the boundaries are edges. Since we are considering connected diagrams with no external legs, we get an oriented polyhedral surface. In other words, a regularization of a Riemann surface. The number of edges, faces and loops are related to the number of handles (= genus $G$) by the formula for the Euler characteristic

$$V - E + L = \chi = 2 - 2G \tag{1.19}$$

Each loop involves a sum over colors and contributes a factor of $N$. (In the double line notation, a loop involves an inner line which forms a closed curve with colors summed and an outer line which is not a closed curve) Each propagator (inverse of the quadratic term in the action) gives a factor of $1/N$. The $V_k$ $k$-valent vertices contribute a factor of $g_k N$ each. Thus the factors of $g_k$ and $N$ associated with any Feynman diagram is

$$N^L (1/N)^E \prod_{k\geq 3} (g_k N)^{V_k} = N^{L-E} N^{\sum_{k\geq 3} V_k} \prod_{k\geq 3} g_k^{V_k} = N^{2-2G} \prod_{k\geq 3} g_k^{V_k} \tag{1.20}$$

Thus diagrams with a common power of $N$ have a fixed genus. Moreover, in the limit $N \to \infty$ holding $g_k$ fixed, the leading diagrams are those that can be drawn on a sphere (genus $G = 0$, known as planar diagrams). Suppressed by $1/N^2$, are diagrams that can be drawn on a torus ($G = 1$, sphere with one handle) and so forth. Moreover, we see that the logarithm of the partition function is proportional to $N^2$, so that we should define the large-$N$ limit of the free energy as $F = -\lim_{N\to\infty} \frac{1}{N^2} \log Z$.

Suppose we had vectors (quarks) in addition to the matrices. It turns out that diagrams with $P$ internal quark loops are suppressed by $(1/N)^P$ compared to the planar diagrams involving the matrix fields (gluons) alone. This is because a quark loop, being a single line loop, does not involve a sum over colors. Diagrams with $P$ internal quark loops correspond to Riemann surfaces with $P$ punctures.

**Large-$N$ Factorization:** These selection rules regarding the dominant diagrams as $N \to \infty$ continue to hold even when we consider diagrams with external legs, (i.e. expectation values of $U(N)$ invariants). However, there is a further simplification beyond planarity, when we consider expectation values of products of $U(N)$ invariants: they factorize. Consider a hermitian multi-matrix model. The matrices are $A_i,\ i = 1, \cdots M$ where the $i$'s can be thought of as labelling points in space time. The action is a polynomial $S(A) = S^{i_1 \cdots i_n} A_{i_1} \cdots A_{i_n}$ and correlations are given by

$$Z = \int dA e^{-N \text{ tr } S(A)}; \quad \langle f(A) \rangle = \frac{1}{Z} \int dA e^{-N \text{ tr } S(A)}\ f(A) \tag{1.21}$$

Then the expectation values of $U(N)$ invariants factorize. For example, let $\Phi_{i_1 \cdots i_n} = \frac{\text{tr}}{N} A_{i_1} \cdots A_{i_n}$. Then

$$\langle \Phi_{i_1 \cdots i_n} \Phi_{j_1 \cdots j_m} \rangle = \langle \Phi_{i_1 \cdots i_n} \rangle \langle \Phi_{j_1 \cdots j_m} \rangle + \mathcal{O}(\frac{1}{N^2}) \tag{1.22}$$

Factorization can be proven perturbatively, (i.e. in powers of the cubic and higher order coupling constants $S^{i_1 i_2 \cdots i_n}$ for $n \geq 3$). Let us give an example of factorization in the very simplest case of the gaussian $S(A) = \frac{1}{2} \sum_i A_i^2$. Then the basic two-point correlation is

$$\langle [A_i]^a_b\ [A_j]^c_d \rangle = \frac{1}{N} \delta_{ij}\ \delta^a_d\ \delta^c_b \tag{1.23}$$

Wick's theorem says that any correlation is a sum over all pairwise contractions. For example ($\delta^a_a = N$),

$$\begin{aligned} \langle \Phi_{ij}\ \Phi_{kl} \rangle &= \langle \frac{1}{N} [A_i]^a_b [A_j]^b_a\ \frac{1}{N} [A_k]^c_d [A_l]^d_c \rangle \\ &= \frac{1}{N}\frac{1}{N} \delta_{ij} \delta^a_a \delta^b_b\ \frac{1}{N}\frac{1}{N} \delta_{kl} \delta^c_c \delta^d_d + \frac{1}{N}\frac{1}{N} \delta_{ik} \delta^a_d \delta^b_b\ \frac{1}{N}\frac{1}{N} \delta_{jl} \delta^d_a \delta^b_c \\ &\quad + \frac{1}{N}\frac{1}{N} \delta_{il} \delta^a_c \delta^d_b\ \frac{1}{N}\frac{1}{N} \delta_{jk} \delta^c_a \delta^b_d \\ &= \delta_{ij}\delta_{kl} + \frac{1}{N^2} \delta_{ik}\delta_{jl} + \frac{1}{N^2} \delta_{il}\delta_{jk} = \langle \Phi_{ij} \rangle \langle \Phi_{kl} \rangle + \mathcal{O}(\frac{1}{N^2}) \end{aligned} \tag{1.24}$$

Of the three terms on the right, the first corresponds to a planar diagram, the second is non-planar (and so suppressed by $\frac{1}{N^2}$) and the third is planar, but suppressed by $\frac{1}{N^2}$ because it involves a contraction between matrices in two different traces.

The factorization of $U(N)$ invariant observables in the large-$N$ limit implies that they do not have any fluctuations. Thus the large-$N$ limit is a classical limit for these variables. Factorization also holds for the invariant observables of a vector model and also for the meson observables (1.4) of a model with both vector and matrix-valued fields.

# Part I

# Baryon in the Large-$N$ Limit of 2d QCD

## Baryon in the Large-$N$ Limit of 2d QCD

In two dimensions the gluon field has two polarization states. They can be taken as the null and time components. The null component can be set equal to zero by a choice of gauge. The time component is not a propagating degree of freedom. It is eliminated by solving its equation of motion. This leads to a linear potential between quarks. Thus, 2d QCD with $N$ colors is a relativistic vector model of interacting fermions. By summing the planar diagrams of the large-$N$ limit, 't Hooft obtained a linear integral equation for meson masses and wave functions [53]. However, it was not clear how baryons arose. Witten proposed that baryons be described by a Hartree-type of approximation in the large-$N$ limit, though in a non-relativistic context [104].

In the null gauge, when quarks are null separated, the parallel transport operator in the meson variable (1.4) is the identity. Thus, two-dimensional QCD can be formulated as a bilocal theory of quark bilinears. Rajeev [87] found a bosonization in terms of bilocal meson variables that satisfy a quadratic constraint. The latter is the projection operator constraint on the quark density matrix. We review the derivation of this theory from 2d QCD and its classical large-$N$ limit in Chapter 2. The phase space of this theory is an infinite grassmannian, a disconnected manifold with connected components labelled by an integer, the baryon number. In Chapter 3, we study the baryon, which is a topological soliton, the minimum of energy on the component of the phase space with baryon number equal to one. The ground-state form factor of the baryon is determined variationally by restricting the dynamics to finite-rank submanifolds of the phase space. This leads to a derivation of an interacting parton model. In the simplest case, this interacting parton model corresponds to a Hartree-type approximation to an $N$-boson system in the large-$N$ limit. The $N$ bosons are the quasi-particles corresponding to valence-quarks whose wave function is already totally antisymmetric in color. In the large-$N$ and chiral limits, the exact ground-state of the baryon occurs on a rank-one submanifold of the phase space, corresponding to a configuration containing only valence-quarks. We use the valence-quark distribution to model the proton structure function measured in Deep Inelastic Scattering at low momentum transfers, where transverse momenta can be ignored as a first approximation.

# Chapter 2

# From QCD to Hadrondynamics in Two Dimensions

In this chapter we review Rajeev's reformulation [87, 86] of $1+1$ dimensional QCD in the null gauge in terms of color-singlet quark bilinears: two-dimensional Quantum Hadrondynamics (QHD). The large-$N$ limit (where $N$ is the number of colors) of the color-singlet sector of two-dimensional QCD is a classical limit of QHD and is a nonlinear field theory of a constrained bilocal meson variable valid at all energies. The classical phase space is a curved manifold, the infinite dimensional grassmannian whose connected components are labelled by baryon number. The Poisson algebra of observables is a central extension of the infinite dimensional unitary Lie algebra. Hamilton's equations of motion are nonlinear integral equations for the meson field. Free mesons are small fluctuations around the vacuum. 't Hooft's linear integral equation for meson masses arises as the linear approximation. Even in the classical large-$N$ limit, Hadrondynamics contains the interactions of mesons. Moreover, baryons arise as topological solitons.

## 2.1 Large-$N$ Limit of 2d QCD

Two-dimensional QCD is a non-abelian gauge theory of quarks coupled to gluons. The quarks are two-dimensional Dirac spinors transforming as vectors in the fundamental

representation of the structure group $SU(N)$. The gluons are vector bosons, one-forms valued in the Lie algebra of $SU(N)$: traceless $N \times N$ hermitian matrices. The number of colors $N$ is 3 in nature. The action for a single flavor of quarks is

$$S = -\frac{N}{4g^2}\int \text{ tr } F^{\mu\nu}F_{\mu\nu}d^2x + \int \left[\bar{q}^a\gamma^\mu(-i\delta^b_a\partial_\mu - A^b_{\mu a})q_b - m\bar{q}^a q_a\right]d^2x \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Here $a, b$ are color indices, $g$ is a coupling constant with the dimensions of mass and $m$ is the *current* quark mass. 2d QCD is a finite quantum field theory, there are no ultra-violet divergences. Apparent infra-red divergences occur when $m$ is set to zero, but these can be avoided by considering the $m = 0$ case as a limiting case of massive current quarks.

It is convenient to work in null coordinates $t = x^0 - x^1,\ x = x^1$ in terms of which the metric is $ds^2 = dt(dt+2dx)$. $\partial_t$ is time-like and $\partial_x$ is a null vector and the initial values of fields are given on the null line $t = 0$. $x$ is regarded as space and $t$ is regarded as time. The components of momentum $Edt+pdx$ are energy $E = p_0$ and null momentum $p = p_0+p_1$. The mass shell condition $m^2 = p_0^2 - p_1^2$ becomes $m^2 = p(2E - p)$. So the energy of a free particle is $E = \frac{1}{2}(p + \frac{m^2}{p})$. We see an important advantage of null coordinates, energy and null momentum have the same sign. So quarks have positive null momentum while anti-quarks have negative momentum. Under a Lorentz transformation of rapidity $\theta$ ( $\tanh\theta = v/c$ where $v$ is the boost velocity), $t \to e^\theta t,\ \ x \to -(\sinh\theta)t + e^{-\theta}x$ and $p \to e^\theta p,\ \ E \to e^{-\theta}E + (\sinh\theta)p$. Thus, a scaling of null momentum is just a Lorentz boost.

The time and null components of the gauge field $A = A_t dt + A_x dx$ are $A_t = A_0$ and $A_x = A_0 + A_1$. We work in the null gauge $A_x = 0$. The reason to use the null gauge is that the parallel transport along a null line is the identity. So meson observables (1.4) simplify. The two-dimensional quark spinor is $q = \frac{1}{\sqrt{2}}\begin{pmatrix}\eta\\ \chi\end{pmatrix}$. The Dirac matrices in null coordinates satisfy $(\gamma^t)^2 = 0$, $(\gamma^x)^2 = -1$, $\{\gamma^t, \gamma^x\}_+ = 2$ and a convenient representation is

$$\gamma^t = \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix},\ \ \gamma^x = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\ \ \bar{q} = q^\dagger\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{2.2}$$

where $\bar{q} = q^\dagger C$ where the charge conjugation matrix satisfies $C\gamma^\mu C^{-1} = (\gamma^\mu)^T$. The action becomes

$$S = \int dtdx \Big[ \frac{N}{2g^2} \text{ tr } (\partial_x A_t)^2 + \chi^\dagger (\hat{E} - A_t)\chi + \frac{1}{2}(\eta^\dagger \hat{p}\eta - \chi^\dagger \hat{p}\chi) - \frac{m}{2}(\eta^\dagger \chi + \chi^\dagger \eta) \Big] \quad (2.3)$$

where $\hat{E} = -i\partial_t,\ \hat{p} = -i\partial_x$. $\chi$ and $\chi^\dagger$ are therefore canonically conjugate and satisfy canonical anticommutation relations

$$[\chi^{\dagger a}(x), \chi_b(y)]_+ = \delta(x-y)\delta^a_b\ , \quad [\chi_a(x), \chi_b(y)]_+ = 0 = [\chi^{\dagger a}(x), \chi^{\dagger b}(y)]_+ \quad (2.4)$$

Neither $\eta$ nor $A_t$ has time derivatives, so they are not dynamical and may be eliminated by solving their equations of motion,

$$\begin{aligned} \hat{p}\ \eta = m\chi, \qquad & -\partial_x^2 A^a_{tb}(x) = \frac{g^2}{N} : \chi^{a\dagger}(x)\chi_b(x) : \\ \eta(x) = \frac{m}{\hat{p}}\chi(x), \qquad & A^a_{tb}(x) = -\frac{g^2}{N}\int dy : \chi^{\dagger a}(y)\chi_b(y) : \frac{1}{2}|x-y| \end{aligned} \quad (2.5)$$

since $\partial_x^2 \frac{1}{2}|x-y| = \delta(x-y)$. So eliminating the longitudinal gluons leads to a linear potential between quarks. Normal ordering is with respect to the Dirac vacuum

$$\tilde{\chi}^{\dagger a}(p)|0>= 0 \text{ for } p<0, \quad \tilde{\chi}_b(p)|0>= 0 \text{ for } p>0 \quad (2.6)$$

where $\tilde{\chi}(p) = \int \chi(x)e^{-ipx}dx$. The resulting hamiltonian is

$$H = \int dx \chi^{\dagger a}\frac{1}{2}(\hat{p} + \frac{m^2}{\hat{p}})\chi_a - \frac{g^2}{2N}\int : \chi^{\dagger a}(x)\chi_b(x) :: \chi^{\dagger b}(y)\chi_a(y) : \frac{1}{2}|x-y|dxdy \quad (2.7)$$

Introduce the color-singlet meson operator (In (1.4) the parallel transport between $x$ and $y$ is the identity in the null gauge)

$$\hat{M}(x,y) = -\frac{2}{N} : \chi^{\dagger a}(x)\chi_a(y) : \quad (2.8)$$

Since the meson observable is gauge-invariant, we may calculate it in the null-gauge which is convenient for us.

The hamiltonian and momentum can be written in terms of $\hat{M}(x,y)$ after rearranging the operators in the normal ordered products (Note: $[dp] = \frac{dp}{2\pi}$)

$$\frac{H}{N} = -\frac{1}{2}\int \frac{1}{2}(p + \frac{\mu^2}{p})\tilde{\hat{M}}(p,p)[dp] \quad + \quad \frac{g^2 N}{8}\int \hat{M}(x,y)\hat{M}(y,x)\frac{|x-y|}{2}dxdy$$

$$\frac{P}{N} = -\frac{1}{2}\int p\tilde{\hat{M}}(p,p)[dp] \tag{2.9}$$

where $\tilde{\hat{M}}(p,q) = \int dxdy e^{-i(px-qy)}\hat{M}(x,y)$. $\mu^2 = m^2 - \frac{g^2N}{\pi}$ is a finite renormalization of current quark mass, a kind of self energy coming from the rearrangement of currents in the potential energy term. The Lorentz invariant squared mass is $P(2H-P)$. The commutation relations can be expressed as

$$[\hat{M}(x,y),\hat{M}(z,u)] = \frac{1}{N}\Big[\delta(y-z)(\epsilon(x,u)+\hat{M}(x,u))-\delta(x-u)(\epsilon(z,y)+\hat{M}(z,y))\Big] \tag{2.10}$$

where $\epsilon(x,y)$ is the kernel of the Hilbert transform

$$\epsilon(x,y) \equiv \epsilon(x-y) = \mathcal{P}\int \text{ sgn }(p)e^{ip(x-y)}[dp]. \tag{2.11}$$

$\frac{1}{N}$ plays the role of $\hbar$ in a conventional quantum system and the large-$N$ limit holding $\tilde{g}^2 = g^2N$ fixed is a classical theory of the color-invariant dynamical variable $M(x,y)$ with Poisson brackets $-iN[\ ,\ ] \mapsto \{\ ,\ \}$

$$\{M(x,y),M(z,u)\} = -i\Big[\delta(y-z)(\epsilon(x,u)+M(x,y))-\delta(x-u)(\epsilon(z,y)+M(z,y))\Big] \tag{2.12}$$

From these, one can derive Hamilton's classical equations of motion. What are the allowed values of $M(x,y)$, i.e. the classical phase space? From the definition in terms of quark fields, the classical variable $M$ is a hermitian operator with integral kernel $M(x,y)^* = M(y,x)$. The fact that quarks are fermions survives the bosonization and is encoded in a quadratic constraint on $M(x,y)$. The large-$N$ limit of the quark density matrix $\rho(x,y) = \frac{1}{N}\chi^{\dagger i}(x)\chi_i(y)$ must be a projection when restricted to color-singlet states: $\rho^2 = \rho$. After normal ordering, $M(x,y) = \delta(x-y)-\epsilon(x,y)-2\rho(x,y)$ satisfies the quadratic constraint $(\epsilon+M)^2 = 1$. Moreover, baryon number $B = -\frac{1}{2}$ tr $M$ which must be an integer is a topologically conserved quantum number. Thus the phase space is a disconnected manifold with connected components labelled by baryon number. It is the infinite dimensional (restricted) grassmannian [84]. In the next section we will quickly review this classical theory on the grassmannian, 2d Classical Hadrondynamics, which is equivalent to the color-singlet sector of the large-$N$ limit of two-dimensional QCD. The quantization of this classical theory, 2d Quantum Hadrondynamics has been shown by

Rajeev [87] to be equivalent to the color-singlet sector of 2d QCD for all energies and values of $N$ and $\hbar$. We will see that this large-$N$ classical theory is very different from the $\hbar \to 0$ limit, Classical Chromodynamics.

## 2.2 Classical Hadron Theory on the Grassmannian

The color-singlet sector of two-dimensional QCD is an interacting theory of Dirac fermions (quarks) which come in $N$ colors, with the constraint that only states that are $SU(N)$ invariant are permitted. In the null coordinates we are using, the sign of kinetic energy and null momentum are the same. So consider the direct sum decomposition of the one particle complex Hilbert space $\mathcal{H} = L^2(R) = \mathcal{H}_- \oplus \mathcal{H}_+$ into positive and negative momentum functions $\mathcal{H}_+ = \{\psi | \tilde{\psi}(p) = 0 \text{ for } p < 0\}$ and $\mathcal{H}_- = \{\psi | \tilde{\psi}(p) = 0 \text{ for } p > 0\}$. Here $\psi(x) = \int_{-\infty}^{\infty} \tilde{\psi}(p) e^{ipx} [dp]$. The operator $\epsilon$ introduced earlier is the sign of momentum, $\epsilon\tilde{\psi}(p) = \pm\tilde{\psi}(p)$ for $\psi \in \mathcal{H}_\pm$. Thus $\epsilon^\dagger = \epsilon$ and $\epsilon^2 = 1$. In momentum space we can regard $\epsilon$ as a large diagonal matrix with eigenvalues $-1$ for the negative momentum states and eigenvalues $+1$ for positive momentum states. Thus, $\epsilon$ represents the Dirac vacuum where negative momentum states are completely occupied and positive ones empty. The novelty is that, since quarks come in $N$ colors, each completely occupied state is occupied by a color-invariant combination of $N \to \infty$ quarks.

More generally, for any orthogonal decomposition $\mathcal{H} = L_- \oplus L_+$, there is a hermitian operator $\Phi$ which is $-1$ on elements of $L_-$ and $+1$ on those of $L_+$. As before, $\Phi^2 = 1$. $\Phi$ represents a more general state than the Dirac vacuum, it has eigenvalue $\pm 1$ on states that are completely empty or filled.

Loosely, the phase space of large-$N$ QCD, the restricted grassmannian of $\mathcal{H}$, is the set of all its subspaces that do not differ too much from $\mathcal{H}_-$. By the above correspondence, it is also the set of hermitian idempotent operators $\Phi$ which do not deviate too much from the vacuum $\epsilon$.

More precisely, since observables are measured by their deviation from their values in the Dirac vacuum, we work with the hermitian operator $M = \Phi - \epsilon$. Then the restricted

grassmannian is the infinite dimensional manifold

$$Gr(\mathcal{H},\epsilon) = \{M^\dagger = M, (\epsilon + M)^2 = 1,\ \text{tr}\ M^\dagger M < \infty\} \tag{2.13}$$

This is the phase space of Classical Hadrondynamics. We recognize that $M$ is just the large-$N$ limit of the meson operator $\hat{M}(x,y) = -\frac{2}{N} : \chi^{\dagger a}(x)\chi_a(y) :$ of the previous section. The Grassmannian describes only those states of the multi-particle Fock space of the quantum theory $\Sigma_{r,s=0}^{\infty}\Lambda^s\mathcal{H}'_- \otimes \Lambda^r\mathcal{H}_+$ that are wedge products of single particle states (up to a scalar multiple). These are the coherent states, ones with a good classical limit. The more general states of the Fock space which are linear combinations of such wedge products are not included. Thus the grassmannian is embedded in the projective space of the fermionic Fock space with a large co-kernel: the Plucker embedding [24, 79]. We will return to this when deriving the interacting quark model from a variational approximation to the classical theory on the grassmannian (see §3.2.4, §3.4.3).

As we saw earlier, the grassmannian is a disconnected manifold, each connected component being labelled by the topologically invariant integer $B = -\frac{1}{2}\ \text{tr}\ M$ which is baryon number. Though neither $\Phi$ nor $\epsilon$ has a finite trace, the trace of the difference is well defined.

The restricted grassmannian carries an action of the restricted unitary group $U(\mathcal{H},\epsilon)$. $\Phi \mapsto U\Phi U^\dagger$ preserves the conditions $\Phi^2 = 1, \Phi^\dagger = \Phi,\ \text{tr}\ (\Phi - \epsilon)^2 < \infty$ if

$$g\ \in\ U(\mathcal{H},\epsilon) = \{U | U^\dagger U = UU^\dagger = 1,\ \text{tr}\ [\epsilon, U]^\dagger[\epsilon, U] < \infty\} \tag{2.14}$$

The action on $M$ is

$$M \mapsto UMU^\dagger + U\epsilon U^\dagger - \epsilon. \tag{2.15}$$

Moreover, any $\Phi$ may be diagonalized by a unitary $U$ and brought to a standard form with eigenvalues $\pm 1$. However, points on disconnected components of the phase space cannot be connected by a unitary transformation since the latter preserve $B$, which is integer-valued. Thus, the action is transitive on each connected component. Indeed, the grassmannian is the homogeneous space $U(\mathcal{H},\epsilon)/U(H_+) \times U(H_-)$. We will use this transitive action of the unitary group to show how to include anti-quarks and 'sea' quarks in a simple way, starting from valence-quarks alone (see §3.4.4).

The Lie algebra of the restricted Unitary group is easily obtained by taking $U = e^u \simeq 1 + u$, we get

$$\underline{U}(\mathcal{H}, \epsilon) = \{u | u = -u^\dagger,\ \mathrm{tr}\ [\epsilon, u]^\dagger [\epsilon, u] < \infty\} \tag{2.16}$$

The infinitesimal action on $M$ is $M \mapsto [u, \epsilon + M]$. This brings us to the Poisson structure on the phase space. The fermion bilinears $M$ form a representation of a central extension of this restricted unitary Lie algebra, as we see from the Poisson brackets (2.12). This is, in fact, the natural Poisson bracket on the grassmannian: it is invariant under the action of the unitary group. This requirement can be used to obtain the Poisson brackets of non-linear functions of $M$ as well:

$$\{f(M), M\} = -i[f'(M), \epsilon + M] \tag{2.17}$$

The dynamics is specified by the hamiltonian we obtained from the large-$N$ limit of 2d QCD, which may be regarded as a real-valued quadratic function on the classical phase space

$$\begin{aligned} E(M) = \lim_{N\to\infty} \frac{H}{N} &= -\frac{1}{2}\int \frac{1}{2}(p + \frac{\mu^2}{p})\tilde{M}(p,p)[dp] \\ &\quad + \frac{\tilde{g}^2}{8}\int M(x,y)M(y,x)\frac{|x-y|}{2}dxdy \end{aligned} \tag{2.18}$$

Total momentum is

$$\bar{P} = P/N = -\frac{1}{2}\int p\tilde{M}(p,p)[dp] \tag{2.19}$$

Time evolution is given by Hamilton's equations

$$\frac{dM}{dt} = \{E(M), M\} = -i[E'(M), \epsilon + M] \tag{2.20}$$

where the integral kernel of $E'(M)$ is (Note: $\mathcal{FP}$ is a finite part integral, a rule for integrating the $\frac{1}{r^2}$ singularity, see Appendix A)

$$E'(M)(p,q) = -\frac{1}{4}2\pi\delta(p-q)(p + \frac{\mu^2}{p}) - \frac{g^2}{4}\mathcal{FP}\int \frac{1}{r^2}\tilde{M}(p+r, q+r)[dr] \tag{2.21}$$

Thus, the equations of motion are quadratically non-linear integral equations for $\tilde{M}(p,q)$. A further complication comes from the quadratic constraint $(\epsilon + M)^2 = 1$. Thus the large-$N$ limit of 2d QCD is a highly non-linear interacting theory of mesons.

Let us first consider static solutions, given by $[E'(M), \epsilon + M] = 0$. The simplest static solutions are given by the extrema of the energy $E'(M) = 0$. The minimum of energy on each connected component of the phase space describes the ground-state of Classical Hadrondynamics for a given baryon number.

On the baryon number zero component of the phase space, the obvious static solution is $M = 0$, the vacuum. Small oscillations around the vacuum in the $B = 0$ sector describe an infinite number of non-interacting mesons. To see this, we linearize the equations of motion around $M = 0$:

$$\begin{aligned} \frac{i}{2}\frac{d\tilde{M}(p,q)}{dt} &= -\frac{1}{2}(K(p) - K(q))\tilde{M}(p,q) - \frac{\tilde{g}^2}{4}\mathcal{FP}\int [ds]\frac{1}{s^2}\Big[ \\ &\quad \tilde{M}(p+s, q+s)\ \mathrm{sgn}\ (q) - \ \mathrm{sgn}\ (p)\tilde{M}(p+s, q+s)\Big] \end{aligned} \tag{2.22}$$

where $\tilde{K}(p) = \frac{1}{2}(p + \frac{\mu^2}{p})$ is the kinetic energy. Moreover, the constraint $(\epsilon + M)^2 = 1$ when linearized becomes $\Big[\ \mathrm{sgn}\ (p) + \ \mathrm{sgn}\ (q)\Big]\tilde{M}(p,q) = 0$ so that one may take $p \geq 0, q \leq 0$. The equation of motion after eliminating the constraint is

$$\frac{i}{2}\frac{d\tilde{M}(p,q)}{dt} = -\frac{1}{2}[\tilde{K}(p) - \tilde{K}(q)]\tilde{M}(p,q) + \frac{1}{2}\tilde{g}^2\mathcal{FP}\int \frac{\tilde{M}(p+s, q+s)}{s^2}[ds] \tag{2.23}$$

Since the vacuum $M = 0$ is translation invariant, the total momentum $P = p - q$ is a conserved quantity. The remaining independent variable can be taken as momentum fraction $\xi = p/P,\ 0 \leq \xi \leq 1$, and we look for small oscillations $\tilde{M}(p, q, t) = e^{i\omega t}\chi(\xi)$. The resulting eigenvalue problem for $\omega$ is

$$\mathcal{M}^2\chi(\xi) = [\frac{\mu^2}{\xi} + \frac{\mu^2}{(1-\xi)}]\chi(\xi) - \frac{\tilde{g}^2}{\pi}\mathcal{FP}\int_0^1 \frac{\chi(\eta)}{(\xi - \eta)^2}d\eta \tag{2.24}$$

where $\mathcal{M}^2 = 2\omega P - P^2$ is the square of the meson mass. This is the linear integral equation for the meson spectrum of 2d QCD in the large-$N$ limit. It was originally obtained by 't Hooft by summing planar Feynman diagrams [53].

Thus, the 't Hooft linear integral equation for meson masses corresponds to the linear approximation around the vacuum, to our non-linear and non-local formulation of large-$N$ 2d QCD. We do not know of any simple way of deriving this non-linear theory by summing planar diagrams. It is only in this linearized approximation that mesons are non-interacting in the large-$N$ limit.

Heuristically, we may regard free mesons as an infinitesimal departure from the vacuum since they involve only the promotion of $\mathcal{O}(1)$ quarks from the Dirac sea (where each fully occupied state contains $\mathcal{O}(N)$ quarks) to a positive momentum state. Thus the large-$N$ limit is crucial.

What then is a finite disturbance to the vacuum? It must involve the addition, removal or rearrangement of $\mathcal{O}(N)$ quarks. The simplest example of such a state is a baryon. It is an old idea of Witten that baryons may be described in a Hartree Fock approximation in the large-$N$ limit. He outlined this in a non-relativistic context [104]. In what follows we will develop the fully relativistic theory of the baryon in two dimensions and see that a variational approximation to this theory is indeed a kind of Hartree-Fock approximation to a many body theory. We turn to the study of the baryon now.

# Chapter 3

# Ground-state of the Baryon

The ground-state of the baryon in the large-$N$ limit is the static solution corresponding to a minimum of energy on the baryon number $B = 1$ component of the phase space. Thus we have a realization of Skyrme's idea [93, 94, 95, 11] that baryons must arise as solitons of a theory of mesons. The essential novelty is that our classical bilocal theory of the meson variable $M(x, y)$ is not a low energy effective theory, but is equivalent to the large-$N$ limit of 2d QCD for all energies. In particular, this theory has an infinite number of mesons, unlike many low energy effective theories which contain only a finite number of mesonic excitations. This chapter is based on our papers [70, 63, 64]. Other useful references are [86, 14, 87].

We first describe a numerical method for determining the minimum energy configuration, based on a geometric adaptation of steepest descent to the curved grassmannian phase space.

Then we will present a variational approach. The primary hurdle in the minimization is the quadratic constraint satisfied by $M(x, y)$. An ascending family of sub-manifolds of the grassmannian is found, corresponding to operators of increasing rank. They allow us to replace the constraint with simpler ones. The minimum on these sub-manifolds provides a variational approximation to the true minimum. The dynamics on these reduced phase spaces is shown to correspond to interacting models of quarks and anti-quarks. Thus we reconcile two seemingly disparate pictures of the baryon: the soliton

and parton pictures.

The minimum energy configuration $\tilde{M}(p,q)$ is the form factor of the baryon. The diagonal elements $\tilde{M}(p,p)$ are the quark and anti-quark probability densities in the baryon in the large-$N$ limit. We determine these by a variational approximation. As an application, we use these to model the quark distributions measured in deep inelastic scattering (§3.5).

Small oscillations around the ground-state of the baryon would correspond to excited states of the baryon, such as $\Delta$ and $N^*$. We do not discuss these here. One could also study the minimum of energy on the sectors of higher baryon number; they correspond to two-dimensional nuclei.

## 3.1 Steepest Descent on the Grassmannian

To determine the ground-state of the baryon, we must minimize the energy $H$ (2.18) on the baryon number one component of the phase space holding the momentum $P$ (2.19) fixed. A Lorentz invariant formulation is to minimize the squared mass $\mathcal{M}^2(M) = P(2H - P)$ subject to the Pauli exclusion principle quadratic constraint $(\epsilon + M)^2 = 1$ and baryon number $B = -\frac{1}{2}\,\mathrm{tr}\,M = 1$ constraint. The quadratic constraint complicates matters significantly since we are minimizing a non-linear function of $M(x,y)$ on a non-linear manifold, the grassmannian.

The first method we tried to solve this problem with, is steepest descent. The latter is a way of minimizing a real-valued function, usually on a linear space. It is an iterative technique: beginning with a conveniently chosen initial configuration, we move a small distance in a direction opposite to the gradient of the function. The procedure is repeated at the new point. After a sufficiently large number of iterations, we reach a minimum of the function.

Now, the grassmannian is not a linear space, due to the quadratic constraint. It is good to keep in mind the sphere embedded in three dimensional euclidean space when thinking of the grassmannian. Indeed, the sphere is the simplest finite dimensional grassmannian. A tangent vector $V$ to the grassmannian at the point $\Phi = \epsilon + M$ must

satisfy $[\Phi, V]_+ = 0$. This comes from linearizing the constraint $\Phi^2 = 1$ at the point $\Phi$. We can think of the anticommutator as the generalization of the dot product: the condition that a vector be tangential to the sphere is implemented by requiring that its dot product with the radius vector at that point vanish. Now, the gradient vector $\mathcal{M}'(M)$ is in general not tangential to the grassmannian. So if we move in a direction opposite to it, we will fly off the grassmannian! Actually, there is more bad news: if we move in a straight line in the space of *all* hermitian operators $M$, along the tangential projection $T$ of $\mathcal{M}'(M)$, we will still leave the grassmannian. This is analogous to moving along a tangent vector to the sphere. Now $T = \frac{1}{2}[\Phi, Y]$ where $Y = \frac{1}{2}[\Phi, \mathcal{M}'(\Phi)]$. To understand the formula for the tangential projection, think of the commutator as the analogue of the cross product. Taking the cross product of an arbitrary vector $\mathcal{M}'$ with the radius vector $\Phi$ produces a vector $Y$ tangential to the sphere, but rotated by a right angle. We need to take a second cross product with the radius vector $\Phi$ to get the tangential projection $T$.

A way to stay on the grassmannian is to move along a curve $\gamma(\tau) = \Phi(\tau) - \epsilon$ which starts out tangential to the tangential projection $T$ of the gradient vector $\mathcal{M}'$. The natural choice for a curve is the great circle, i.e. the geodesic. The great circle through a point $\Phi$ on the sphere tangential to a vector $T$ is obtained by rotating the radius vector $\Phi$ about an axis orthogonal to $T$. In our situation, $Y$ is the necessary vector orthogonal to $T$. Moreover, the analogue of rotation is the action of the unitary group on the grassmannian (see §2.2). So a rotation about the axis $Y$ is implemented by the unitary operator $e^{\tau Y}$. In other words, the geodesic through $\Phi$ tangential to $T$ and parameterized by $\tau$ is

$$\Phi(\tau) = e^{\tau Y} \Phi e^{-\tau Y} \tag{3.1}$$

We are finally ready to formulate the steepest descent algorithm for the grassmannian by taking small steps along the geodesic that goes in the direction of maximal reduction of squared mass.

1. Start with an initial configuration $M_1$ and a small $\tau$.

2. At the $k^{\text{th}}$ configuration $M_k$ evaluate the gradient $\mathcal{M}'(M_k)$ and the rotation gen-

erator $Y_k = \frac{1}{2}[\epsilon + M_k, \mathcal{M}'(M_k)]$.

3. Let $M_{k+1} = e^{\tau Y_k}[\epsilon + M_k]e^{-\tau Y_k} - \epsilon$ ($M$ transforms inhomogeneously unlike $\Phi$ (2.15))

4. Repeat till there is no further reduction in the mass of the baryon.

We implemented this steepest descent method numerically using Mathematica. The value of the time step $\tau$ was arrived at empirically to ensure quick convergence. As for the initial configuration, we used the solution of the separable ansatz, a more analytic variational approximation method which we turn to next. We do not present the numerical results here, some earlier numerical results are given in the paper [70].

## 3.2 Separable Ansatz: Formulation

### 3.2.1 Rank-One Configurations

$\tilde{M}(p,q)$ depends on a pair of variables. In Feynman's valence parton model [17, 36], the valence-quark distribution is a function of a single momentum variable. Since we know that this gives a reasonable description of the proton, it must be a good approximation to assume that $\tilde{M}(p,q)$ is built out of a single function of one variable, i.e. it is a separable or rank-one operator:

$$M_1 = -2\ \psi \otimes \psi^\dagger \quad \text{i.e.} \quad \tilde{M}_1(p,q) = -2\ \tilde{\psi}(p)\tilde{\psi}^*(q) \tag{3.2}$$

The quadratic constraint $(\epsilon + M)^2 = 1$ becomes

$$(\epsilon - 2\psi \otimes \psi^\dagger)^2 = \epsilon^2 - 2\epsilon\psi \otimes \psi^\dagger - 2\psi \otimes \psi^\dagger\epsilon + 4\psi \otimes \psi^\dagger\psi \otimes \psi^\dagger \tag{3.3}$$

The constraint is satisfied if $\tilde{\psi}(p)$ vanishes for $p < 0$ i.e. $\epsilon\psi = \psi$ and is of unit norm $\psi^\dagger\psi = 1$. The former means this configuration contains no anti-quarks. The latter ensures $M_1$ is a configuration of baryon number one: $B = -\frac{1}{2}\ \mathrm{tr}\ M_1 = \int_0^\infty \tilde{\psi}(p)\tilde{\psi}^*(p)[dp] = 1$.

Since $M_1$ is unaltered by a change in the phase of $\psi$, the phase space of the rank-one ansatz is the projective space on positive momentum wave functions $\mathcal{P}(\mathcal{H}_+)$. The Poisson brackets of $\tilde{M}(p,q)$ are satisfied if

$$\{\tilde{\psi}(p), \tilde{\psi}(p')\} \ = \ 0 \ = \ \{\tilde{\psi}^*(p), \tilde{\psi}^*(p')\},$$

$$\{\tilde{\psi}(p), \tilde{\psi}^*(p')\} \quad = \quad -i\ 2\pi\delta(p-q) \tag{3.4}$$

The hamiltonian when written in terms of $\psi$ is

$$E_1(\psi) = \int_0^\infty \frac{1}{2}[p + \frac{\mu^2}{p}]|\tilde{\psi}(p)|^2[dp] + \frac{1}{2}\tilde{g}^2 \int |\psi(x)|^2|\psi(y)|^2 \frac{|x-y|}{2} dxdy. \tag{3.5}$$

Thus, we have a self contained hamiltonian dynamical system on the reduced phase space of rank-one configurations.

### 3.2.2 Quantizing the Separable Ansatz: Interacting Valence Quarks

In order to make precise the relation of the separable ansatz to an interacting valence-quark model, we quantize the classical theory of the previous section. We denote the parameter that measures quantum corrections by $\frac{1}{N}$. The constraint on norm is implemented by restricting attention to states $|V>$ satisfying

$$< V| \int_0^\infty \hat{\tilde{\psi}}^*(p)\hat{\tilde{\psi}}(p)[dp]|V >= 1. \tag{3.6}$$

The Poisson brackets go over to commutation relations

$$[\hat{\tilde{\psi}}(p), \hat{\tilde{\psi}}(p')] = 0 = [\hat{\tilde{\psi}}^\dagger(p), \hat{\tilde{\psi}}^\dagger(p')],\ [\hat{\tilde{\psi}}(p), \hat{\tilde{\psi}}^\dagger(p')] = \frac{1}{N} 2\pi\delta(p-p'). \tag{3.7}$$

A representation for these commutation relations is given by the canonical commutation relations of bosonic creation-annihilation operators:

$$[\hat{\tilde{b}}(p), \hat{\tilde{b}}(p')] = 0 = [\hat{\tilde{b}}^\dagger(p), \hat{\tilde{b}}^\dagger(p')], \quad [\hat{\tilde{b}}(p), \hat{\tilde{b}}^\dagger(p')] = 2\pi\delta(p-q). \tag{3.8}$$

where $\hat{\psi} = \frac{1}{\sqrt{N}}\hat{b}$, $\hat{\psi}^\dagger = \frac{1}{\sqrt{N}}\hat{b}^\dagger$. In terms of these the constraint becomes

$$< V| \int_0^\infty \hat{b}^\dagger(p)\hat{b}(p)[dp]|V >= N \tag{3.9}$$

Thus we have a system of $N$ bosons, so $N$ must be an integer. The hamiltonian operator becomes

$$N\hat{E}_1 = \int_0^\infty \frac{1}{2}[p + \frac{\mu^2}{p}]\tilde{b}^\dagger(p)\tilde{b}(p)\ [dp] + \frac{\tilde{g}^2}{2N} \int b^\dagger(x)b^\dagger(y)\frac{|x-y|}{2} b(y)b(x)\ dxdy. \tag{3.10}$$

The baryon must be in a simultaneous eigenstate of the hamiltonian and momentum operators, whose eigenvalues $E_N$ and $P$ are the energy and total momentum of the baryon. The momentum operator is

$$\hat{P} = \int_0^\infty p\ b^\dagger(p)b(p)\ [dp]. \tag{3.11}$$

If we work in the momentum basis, an $N$ particle momentum eigenstate is

$$|p_1, p_2, \cdots, p_N >= b^\dagger(p_1)b^\dagger(p_2)\cdots b^\dagger(p_N)|0> . \tag{3.12}$$

A general state $|\psi>$ containing $N$ particles is given by the wave function $\tilde{\psi}(p_1, \cdots, p_N)$, where

$$|\psi >= \int_0^\infty [dp_1]\cdots[dp_N]\tilde{\psi}(p_1, \cdots, p_N)|p_1, \cdots, p_N > . \tag{3.13}$$

The expectation value of the hamiltonian in such a state is

$$\begin{aligned} < \psi|N\hat{E}|\psi > \ &= \ \int_0^\infty \sum_{i=1}^N \frac{1}{2}[p_i + \frac{\mu^2}{p_i}]|\tilde{\psi}(p_1, \cdots, p_N)|^2[dp_1]\cdots[dp_N] \\ &\quad + \frac{\tilde{g}^2}{2N}\int_0^\infty \sum_{i\neq j}\frac{|x_i - x_j|}{2}|\psi(x_1, \cdots, x_N)|^2 dx_1 \cdots dx_N. \end{aligned} \tag{3.14}$$

It is now clear that we have a system of $N$ bosons interacting through a linear two-body potential in the null coordinates. What are these bosons? They are just the valence-quarks of the parton model, whose dependence on color has been factored out. Though quarks are fermions, the total baryon wave function must be totally antisymmetric in the color indices so that it is invariant under $SU(N)$. Thus, the baryon wave function (suppressing spin and flavor dependence) $\tilde{\Psi}$ factorizes as

$$\tilde{\Psi}(a_1, p_1; a_2, p_2; \cdots; a_N, p_N) = \epsilon_{a_1, a_2, \cdots a_N}\tilde{\psi}(p_1, p_2, \cdots, p_N). \tag{3.15}$$

Here $a_k$ denote color indices. Thus, if we ignore color, valence-quarks behave like bosons.

In the parton model, the momentum fraction carried by a quark satisfies $0 \leq \xi = \frac{p}{P} \leq 1$. How do we recover this in our picture? Null momentum is positive for a quark. Thus, the momentum of the particles created by $b^\dagger(p)$ is always positive whence the total momentum $\hat{P}$ is a positive operator. On an $N$ particle momentum basis state, $\hat{P}$ is just the sum of individual momenta, each of which is positive:

$$\hat{P}|p_1, p_2 \cdots, p_N >= [p_1 + p_2 + \cdots p_N]|p_1, p_2, \cdots, p_N > . \tag{3.16}$$

An eigenstate of $\hat{P}$ with eigenvalue $P$ must satisfy

$$[p_1 + \cdots p_N]\,\tilde{\phi}(a_1, p_1; \cdots a_N, p_N) = P\tilde{\phi}(a_1, p_1; \cdots a_N, p_N). \tag{3.17}$$

Since each of the momenta $p_i$ are positive, it follows that $0 \leq \frac{p_i}{P} \leq 1$. Thus, quantizing the rank-one approximation to our classical hadron theory gives us an interacting valence-quark model [70, 63].

### 3.2.3 Hartree Approximation and the Large-$N$ limit

It is an old idea of Witten [104] that for the baryon, the large-$N$ limit should correspond to a sort of Hartree approximation to a many body theory. We have a relativistic implementation of this idea in two dimensions.

We would like to find approximate simultaneous eigenstates of $\hat{E}_1$ and $\hat{P}$. The ground-state of a many boson system can often be described by mean-field theory: each boson moves in the field created by all the others. Moreover, all the bosons can be assumed to occupy the same single particle state in this ground-state. The naive choice is a wave function $\psi$ which is a product of single particle wave functions. But such a choice would not be a momentum eigenstate, so we enforce it by requiring the momenta to add up to $P$.

$$\tilde{\psi}(p_1, p_2, \cdots, p_N) = 2\pi\delta(\sum_i p_i - P)\tilde{\psi}(p_1)\tilde{\psi}(p_2)\cdots\tilde{\psi}(p_N). \tag{3.18}$$

In particular, the fraction of momentum carried by each parton is less than one $\tilde{\psi}(p) = 0$ unless $0 \leq p/P \leq 1$. In terms of the single parton wave function, the energy (3.14) of the baryon per quark becomes ($N(N-1) \sim N^2$)

$$E_1(\psi) = \int_0^P \frac{1}{2}[p + \frac{\mu^2}{p}]|\tilde{\psi}(p)|^2[dp] + \frac{\tilde{g}^2}{2}\int |\psi(x)|^2|\psi(y)|^2\frac{|x-y|}{2}dxdy. \tag{3.19}$$

Here, we are retaining the condition $0 \leq p \leq P$. The remaining correlations imposed by the factor $\delta(\sum_i p_i - P)$, are suppressed for large-$N$. As a compromise to the momentum eigenstate condition we will require the expectation value of $\hat{P}$ to equal $P$. This is analogous to using the canonical ensemble as an approximation to the micro-canonical ensemble. This gives rise to the momentum sum rule:

$$N\int_0^P p|\tilde{\psi}(p)|^2[dp] \;=\; P. \tag{3.20}$$

The wave function also satisfies the normalization condition $\int_0^P |\tilde{\psi}(p)|^2\ [dp]\ =\ 1$.

We see that the Hartree approximation to the interacting valence-quark model has reproduced our classical hadron theory restricted to the separable submanifold of the phase space, except for the requirement that the wave function vanish beyond $p = P$. This subtle difference is a semi-classical (finite $N$) effect. To recover the true classical limit, we must let $N$ become truly infinite. Since the total momentum is an extensive variable, $P \to \infty$ as $N \to \infty$. So for $N = \infty$, the valence-quark wave function is not required to vanish for any finite value of $p$.

### 3.2.4 Plücker Embedding and Density Matrix

In order to find the Fock space state corresponding to the classical rank-one ansatz, we use an infinite dimensional version of the Plücker embedding [24, 79, 87] of the grassmannian in the Fock space $\mathcal{F} = \sum_{r,s=0}^{\infty} \Lambda^s \mathcal{H}'_- \otimes \Lambda^r \mathcal{H}_+$. Here $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$ is the splitting into positive and negative momentum subspaces.

Given a point $M$, the state in $\mathcal{F}$ to which it is mapped by the Plucker embedding, is the wedge product of the occupied states. These are the eigenstates of the density matrix $\rho = \frac{1}{2}(1 - M - \epsilon)$ with eigenvalue 1 or equivalently, the $-1$ eigenstates of $\Phi = \epsilon + M$. In the case of the rank-one ansatz $M = -2\psi \otimes \psi^\dagger$, we look for states $\chi$ with

$$\Phi_1 \chi = (\epsilon - 2\psi \otimes \psi^\dagger)\chi = -\chi \tag{3.21}$$

Every negative momentum state (i.e. those with $\epsilon\chi = -\chi$) is a solution. The only other solution is $\chi = \psi$ since, $\epsilon\psi = \psi$ and $\psi^\dagger\psi = 1$. These are the filled states. Thus the Fock space state (denoted $|M_1>$) corresponding to the rank-one ansatz $M_1$ is a semi-infinite wedge product:

$$\cdots f^{2'} \wedge f^{1'} \wedge \psi \tag{3.22}$$

Here $f^{j'}$ is a dual basis in $\mathcal{H}_-$. The only interesting information in this state is its departure from the vacuum. If we denote by $|\tilde{0}>$ the wedge product $\cdots f^{2'} \wedge f^{1'}$ we see that $|\tilde{0}>$ is the filled negative energy sea. Then $|M_1>$ is obtained by populating $\psi$, starting from the vacuum $|\tilde{0}>$.

$$|M_1> = b_\psi^\dagger |\tilde{0}> \tag{3.23}$$

Here, the quasi-particle created by $b^\dagger_\psi$ is not a quark, but the colorless quasi-particle of the rank-one ansatz. This point of view is easier for computations and has the advantage of avoiding the unobservable color quantum number altogether.

The quantized rank-one baryon can also be described in terms of the creation annihilation operators of colored quarks. Let $|V>$ denote the second quantized state of the baryon. In the Hartree approximation, it is determined by the condition that the expectation value of the density operator in $|V>$ must equal the classical density matrix of the rank-one ansatz:

$$< V|\tilde{\hat{\rho}}(p,q)|V >= \tilde{\rho}_1(p,q) \tag{3.24}$$

For the rank-one baryon $\tilde{\rho}_1(p,q) = \tilde{\psi}(p)\tilde{\psi}^*(q) + \frac{1}{2}(2\pi\delta(p-q))(1 - \text{ sgn } p)$. A short calculation shows that

$$|V >= q^{1\dagger}_{\tilde{\psi}} \cdots q^{N\dagger}_{\tilde{\psi}}|0> . \tag{3.25}$$

satisfies the above requirement. Here, $q^{a\dagger}_{\tilde{\psi}}$ creates a quark with color $a$ in the state $\tilde{\psi}$. These operators satisfy canonical anti-commutation relations (CAR): $\{q_{iu}, q^{j\dagger}_v\} = \delta^j_i < u, v >$ with respect to the Dirac vacuum $|0>$: $q^{a\dagger}_{\tilde{\psi}_-}|0>= 0$ and $q_{b\tilde{\psi}_+}|0>= 0$ where $\tilde{\psi}_-(p)$ vanishes for $p \geq 0$ and $\tilde{\psi}_+(p)$ for $p \leq 0$.

This formalism will be useful when we generalize the theory to include sea and anti-quarks (section 3.4.2). Before doing so, we discuss the actual determination of the ground-state of the baryon within the rank-one approximation.

## 3.3 Separable Ansatz: Solution

The energy of the baryon in the separable ansatz is

$$E/N = \int_0^P \frac{1}{2}(p + \frac{\mu^2}{p})|\tilde{\psi}(p)|^2[dp] + \frac{\tilde{g}^2}{2}\int_{-\infty}^{\infty} |\psi(x)|^2|\psi(y)|^2\frac{|x-y|}{2}dxdy \tag{3.26}$$

where $\mu^2 = m^2 - \frac{\tilde{g}^2}{\pi}$. Its momentum is

$$\bar{P} = P/N = \int_0^P p|\tilde{\psi}(p)|^2[dp] \tag{3.27}$$

A Lorentz invariant formulation is to minimize the squared mass, which in null coordinates is $\mathcal{M}^2 = 2EP - P^2$ (§2.1):

$$\frac{\mathcal{M}^2}{N^2\tilde{g}^2} = \Big[\int_0^P p|\tilde{\psi}(p)|^2[dp]\Big] \quad * \quad \Big[\Big(\frac{m^2}{\tilde{g}^2} - \frac{1}{\pi}\Big)\int_0^P \frac{|\tilde{\psi}(p)|^2}{p}[dp] + \int_{-\infty}^{\infty} |\psi(x)|^2|\psi(y)|^2\frac{|x-y|}{2}dxdy\Big]. \tag{3.28}$$

while holding $\int_0^P |\tilde{\psi}(p)|^2[dp] = 1$. The only parameter in the theory is the dimensionless ratio $\nu = \frac{m^2}{\tilde{g}^2}$. The total baryon momentum $P$ can be scaled out by expressing momenta in terms of the ratio $\xi = \frac{p}{P}$. It is convenient to introduce notation for the kinetic, self and potential energies:

$$KE = \int_0^P \frac{1}{p}|\tilde{\psi}(p)|^2[dp], \qquad SE = -\frac{1}{\pi}\int_0^P \frac{1}{p}|\tilde{\psi}(p)|^2[dp]$$
$$PE = \int_{-\infty}^{\infty} dxdy|\psi(x)|^2|\psi(y)|^2\frac{1}{2}|x-y|. \tag{3.29}$$

Then,

$$\frac{\mathcal{M}^2}{\tilde{g}^2N^2} = \bar{P} * (\nu KE + SE + PE). \tag{3.30}$$

### 3.3.1 Potential Energy in Momentum space

While the kinetic and self energies are simple in momentum space, the potential energy is more easily expressed in position space. However, the integral equation for minimization of energy is simplest in momentum space. Moreover, the condition that $\tilde{\psi}(p)$ vanish for negative momenta is harder to implement in position space. It implies that $\psi(x)$ is the boundary value on the real line of a function analytic in the upper half plane. So it is useful to rewrite the potential energy in momentum space. However, the kernel of the integral operator is singular, and we take some care to define it correctly. We write

$$PE = \int_{-\infty}^{\infty} |\psi(x)|^2V(x)dx, \text{ where } V(x) = \int_{-\infty}^{\infty} |\psi(y)|^2\frac{|x-y|}{2}dy. \tag{3.31}$$

The self-consistent potential $V(x)$ obeys Poisson's equation $V''(x) = |\psi(x)|^2$ along with a pair of boundary conditions.

$$V(0) = \int_{-\infty}^{\infty} |\psi(y)|^2\frac{|y|}{2}dy, \quad V'(0) = \frac{1}{2}\int_{-\infty}^{\infty} |\psi(-y)|^2 \text{ sgn } (y)dy \tag{3.32}$$

If we define the momentum space wave function and potential by

$$\psi(x) = \int_0^P \tilde{\psi}(p) e^{ipx} [dp] \text{ and } \tilde{V}(r) = \int_{-\infty}^{\infty} e^{-irx} V(x) dx \tag{3.33}$$

then the potential energy becomes

$$PE = \mathcal{FP} \int [dp] \int [dr] \tilde{\psi}(p) \tilde{\psi}^*(p+r) \tilde{V}(r). \tag{3.34}$$

When the limits of integration are not indicated, they are from the lower to upper limit of the interval over which the integrand is supported. Poisson's equation in momentum space becomes

$$\tilde{V}(p) = \mathcal{FP} \frac{-1}{p^2} \int [dq] \tilde{\psi}(q) \tilde{\psi}^*(q-p) = \mathcal{FP} \frac{-1}{p^2} \tilde{W}(p) \tag{3.35}$$

We see that $\tilde{V}(p) \sim \frac{-1}{p^2}$ is singular for small $p$ and use $\mathcal{FP}$ to denote an appropriate "finite part". $\tilde{W}(-p) = W^*(p)$ is supported from $-P$ to $P$. The rules for integrating this singularity are given by the boundary conditions to Poisson's equation:

$$\mathcal{FP} \int \frac{-1}{p^2} \tilde{W}(p) [dp] = \int |\psi(y)|^2 \frac{|y|}{2} dy \text{ and} \tag{3.36}$$

$$-i\mathcal{FP} \int \frac{1}{p} \tilde{W}(p) [dp] = \frac{1}{2} \int_0^{\infty} \{-|\psi(y)|^2 + |\psi(-y)|^2\} dy \tag{3.37}$$

We will primarily be interested in the ground-state of the baryon. Assuming that the ground-state wave function is real, $\tilde{W}(p)$ is even. In Appendix A we show (for real wave functions $\tilde{\psi}(p)$ which vanish at the origin like $p^a, p > 0, a > 0$) that $\tilde{W}'(0) = 0$ and

$$\mathcal{FP} \int_{-P}^{P} \frac{\tilde{W}(p)}{p^2} [dp] = \int_{-P}^{P} \frac{\tilde{W}(p) - \tilde{W}(0)}{p^2} [dp] - \frac{\tilde{W}(0)}{\pi P} \tag{3.38}$$

Thus we can rewrite the potential energy as

$$PE = -\mathcal{FP} \int_{-P}^{P} \frac{|\tilde{W}(r)|^2}{r^2} [dr] = \frac{1}{\pi P} + 2 \int_0^P \frac{|\tilde{W}(0)|^2 - |\tilde{W}(r)|^2}{r^2} [dr] \tag{3.39}$$

Here $\tilde{W}(0) = 1$ and $|\tilde{W}(r)|^2 = |\tilde{W}(-r)|^2$ even if $\tilde{\psi}(p)$ is not real. Notice that in the large-$N$ limit, where $P = \infty$, the term $\frac{1}{\pi P}$ may be omitted. We can now derive the integral equation for the minimization of the baryon mass.

### 3.3.2 Integral Equation for Minimization of Baryon Mass

The condition for the minimization of $\mathcal{M}^2$ is equivalent to a non-linear integral equation for the wave function $\tilde{\psi}(p)$. Recall that

$$\frac{\mathcal{M}^2}{\tilde{g}^2 N^2} = \bar{P} * (\nu KE + SE + PE). \tag{3.40}$$

We vary this with respect to $\tilde{\psi}^*(p)$ and impose the constraint on the norm through a Lagrange multiplier $\lambda$. The condition for an extremum is:

$$\bar{P}(\nu\ \delta KE + \delta SE + \delta PE) + \delta\bar{P}(\nu KE + SE + PE) = \lambda\tilde{\psi}(p). \tag{3.41}$$

Here

$$\delta PE = -\mathcal{FP}\ \delta \int_{-P}^{P} \frac{|W(r)|^2}{r^2}[dr] \quad \text{and} \quad \frac{\delta W(r)}{\delta\tilde{\psi}^*(p)} = \tilde{\psi}(p+r). \tag{3.42}$$

Thus,

$$\begin{aligned} \delta PE &= -\mathcal{FP}\int_{-p}^{-p+P}[dr]\frac{\tilde{\psi}(p+r)\tilde{W}^*(r)}{r^2} - \mathcal{FP}\int_{p-P}^{p}[dr]\frac{\tilde{\psi}(p-r)\tilde{W}^(r)}{r^2} \\ &= 2\mathcal{FP}\int_{-P+p}^{p}[dr]\tilde{V}(r)\tilde{\psi}(p-r) \end{aligned} \tag{3.43}$$

This leads to the non-linear integral equation

$$\begin{aligned} \bar{P}\Big[\frac{\nu}{p}\tilde{\psi}(p) &- \frac{1}{\pi p}\tilde{\psi}(p) + 2\mathcal{FP}\int_{-P+p}^{p}\tilde{V}(r)\tilde{\psi}(p-r)[dr]\Big] \\ &+ p\tilde{\psi}(p)(KE + SE + PE) = \lambda\tilde{\psi}(p). \end{aligned} \tag{3.44}$$

We can write $\delta PE$ in a more convenient form, so that the self energy term $-\frac{1}{\pi p}\tilde{\psi}(p)$ is cancelled. Let us work in the limit $N = \infty$ so that $P = \infty$. Since $W(-r) = W^*(r)$, we have

$$\delta PE(p) = -2\mathcal{FP}\int_0^p \frac{[dr]}{r^2}[\tilde{W}(r)\tilde{\psi}(p-r) + \tilde{W}(-r)\tilde{\psi}(p+r)] + 2\int_p^\infty V(-r)\tilde{\psi}(p+r)[dr] \tag{3.45}$$

The second integral is finite, but with hindsight we add and subtract $\frac{\tilde{\psi}(p)}{\pi p}$ from it. We may rewrite the first integral using the definition of the "finite part" (Appendix A). Note that $\tilde{W}(0) = 1$.

$$\delta PE = \frac{\tilde{\psi}(p)}{\pi p} - 2\int_0^p \frac{[ds]}{s^2}[\tilde{\psi}(p-s)\tilde{W}(s) + \tilde{\psi}(p+s)\tilde{W}(-s) - 2\tilde{\psi}(p)]$$

$$-2\int_p^\infty \frac{[ds]}{s^2}[\tilde{\psi}(p+s)\tilde{W}(-s)-\tilde{\psi}(p)]. \tag{3.46}$$

Thus $\delta SE$ is cancelled by the first term in $\delta PE$. The integral equation becomes:

$$\begin{aligned}&\frac{\nu}{p}\tilde{\psi}(p)-2\int_0^p \frac{[ds]}{s^2}[\tilde{\psi}(p-s)\tilde{W}(s)+\tilde{\psi}(p+s)\tilde{W}(-s)-2\tilde{\psi}(p)]\\ -\quad &2\int_p^\infty \frac{[ds]}{s^2}[\tilde{\psi}(p+s)\tilde{W}(-s)-\tilde{\psi}(p)]+\frac{(\nu KE+SE+PE)}{\bar{P}}p\tilde{\psi}(p)=\frac{\lambda}{\bar{P}}\tilde{\psi}(p)\end{aligned} \tag{3.47}$$

The quantities $\bar{P}, KE, SE, PE$ (defined in §3.3) depend on the wave function and are to be determined self-consistently. We now study this integral equation, to find the ground-state of the baryon. Let us begin with the behavior of the wave function for small arguments.

### 3.3.3 Small Momentum Behavior of Wave function

To understand the behavior of the valence-quark wave function for small momenta, we do a Frobenius-type analysis of the integral equation for the minimization of the squared mass of the baryon:

$$\begin{aligned}\lambda\tilde{\psi}(p) \quad = \quad &\bar{P}\Big[\frac{\nu}{p}\tilde{\psi}(p)-\frac{1}{\pi p}\tilde{\psi}(p)+2\mathcal{FP}\int_{-P+p}^{p}\tilde{V}(r)\tilde{\psi}(p-r)[dr]\Big]\\ &+p\ \tilde{\psi}(p)(\nu KE+SE+PE)\end{aligned} \tag{3.48}$$

For small $p$, we may ignore the last term on the RHS and the one on the LHS, since the other terms are more singular in the $p\to 0$ limit. Dividing by $\bar{P}>0$ we get the integral equation

$$(\nu-\frac{1}{\pi})\frac{1}{p}\tilde{\psi}(p)+2\mathcal{FP}\int_{p-P}^{p}\tilde{V}(q)\tilde{\psi}(p-q)[dq]=0 \tag{3.49}$$

where the self-consistent potential $\tilde{V}(q)\sim\frac{-1}{q^2}$ for small q. Now we assume a power law behavior $\tilde{\psi}(p)\sim p^a$ for small $p>0$ and derive an equation for $a$.

$$(\nu-\frac{1}{\pi})=2\mathcal{FP}\int_{1-\frac{P}{p}}^{1}\frac{(1-y)^a}{y^2}\frac{dy}{2\pi} \tag{3.50}$$

where $y=\frac{q}{p}$. Since $p<<P$, we have

$$\pi\nu-1=\mathcal{FP}\int_0^1\frac{(1-y)^a+(1+y)^a}{y^2}dy+\int_1^\infty\frac{(1+y)^a}{y^2}dy \tag{3.51}$$

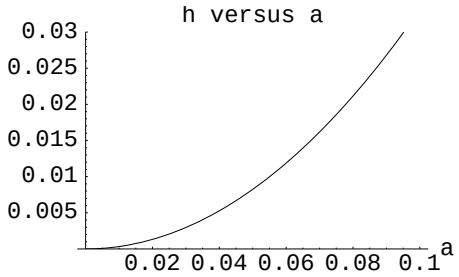


Figure 3.1: $h(a)$. The solution for the exponent $p^{a(m)}$ is the point at which the horizontal line $\frac{\pi m^2}{\tilde{g}^2}$ intersects the curve $h(a)$. For $m = 0$, $a = 0$ is a solution: in the limit of chiral symmetry, the wave function goes to a non-zero constant at the origin.

The first of these integrals is singular and we evaluate it according to the definition of the finite part (Appendix A). The result is a transcendental equation for $a$:

$$\pi\nu = h(a) \equiv \int_0^1 \frac{(1+y)^a + (1-y)^a - 2}{y^2} dy - 1 + \frac{{}_2\mathrm{F}_1(1-a, -a, 2-a, -1)}{1-a} \qquad (3.52)$$

Here ${}_2F_1$ is the Hypergeometric function and $\nu = \frac{m^2}{\tilde{g}^2}$. It is easily seen that for $m = 0$, $a = 0$ is a solution. In the limit of chiral symmetry, $\tilde{\psi}(p) \sim p^0$ as $p \to 0$. Calculating $h(a)$ shows that for a positive current quark mass $m$, the wave function rises like a power $\tilde{\psi}(p) \sim p^a$, $a > 0$. The plot shows $h(a)$. The solution for the exponent $a(m)$ for $m \geq 0$ is the point at which the horizontal line $\frac{\pi m^2}{\tilde{g}^2}$ intersects the curve. For sufficiently small $\frac{m^2}{\tilde{g}^2}$, we may solve the transcendental equation for $a$, $\pi\frac{m^2}{\tilde{g}^2} = h(a)$ analytically. Expanding $h(a)$ in a power series we get for small $a$, $h(a) \to \frac{\pi^2}{3}a^2$. Therefore, $a \to \sqrt{\frac{3}{\pi}}\frac{m}{\tilde{g}}$.

Our conclusion that in the chiral limit the momentum space wave function tends to a non-zero constant as $p \to 0$ implies that it is discontinuous at $p = 0$ since $\tilde{\psi}(p) = 0$ for $p < 0$. Therefore, the corresponding position space wave function decays like $\frac{1}{x}$ as $|x| \to \infty$. This may be expected from another point of view. The baryon we are studying, is a soliton of the meson field $\hat{M}(x, y)$. It should be possible to approximate the baryon wave function with the meson wave function for large spatial x. At large distances we are essentially far away from the baryon. The meson wave function has been calculated by 't Hooft [53] in the large-$N$ limit of 2d QCD. In the chiral limit, the 't Hooft meson wave function has a discontinuity at $p = 0$ and consequently decays like $\frac{1}{x}$ as $|x| \to \infty$. This is a non-trivial consistency check.

### 3.3.4 Exact Solution for $N = \infty, m = 0$: Exponential Ansatz

There is an exact solution to the integral equation for minimization of baryon squared mass in the chiral and large-$N$ limits: $\tilde{\psi}(p) \sim e^{-p}, p > 0$ where $b$ is fixed by the Lorentz reference frame. Moreover, we find that the mass of the baryon vanishes for this particular rank-one configuration. Therefore, in the chiral and large-$N$ limits, the exact minimum of $(mass)^2$ occurs for a configuration with no anti-quarks. While this is to be expected in the non-relativistic limit $\frac{m^2}{\tilde{g}^2} \to \infty$ [68], it is rather surprising in this ultra-relativistic chiral limit, since conventional wisdom would suggest that as $\frac{m^2}{\tilde{g}^2} \to 0$, the phase space for production of virtual $q\bar{q}$ pairs is enhanced. We find this is not true in the two-dimensional theory.

To see this, let us derive the analytic solution. We will use a variational approach that also allows us to estimate corrections due to finite $m$ and $1/N$. From the Frobenius analysis (section 3.3.3) we know that the wave function must vanish like a power $p^a$ as $p \to 0^+$ with $a \to 0^+$ in the chiral limit. For $N = \infty$ the valence-quark wave function is not required to vanish for any finite value of $p$. But for finite $N$, it must vanish beyond $p = P$. If the former is to be a good approximation, it must fall off rapidly for large momenta.

As the simplest ansatz that satisfies these boundary conditions and also facilitates explicit calculation, we take the variational wave function $\tilde{\psi}(p) = p^a e^{-bp}$. The Lorentz invariant parameter $a$ is determined by minimizing the $(mass)^2$. In the null coordinates, a rescaling $p \to \lambda p$ is actually a Lorentz boost in the longitudinal direction. Therefore, $b$ is determined by the choice of the reference frame. For the purposes of determining $a$, let us work in a reference frame in which $b = 1$. We will fix $b$ subsequently, based on the momentum sum rule, which serves to pick a reference frame.

In the notation we introduced earlier, $\frac{\mathcal{M}^2}{\tilde{g}^2 N^2} = \bar{P} * (\nu KE + SE + PE)$. The normalized wave function for the exponential ansatz is $\tilde{\psi}_a(p) = \sqrt{\frac{\pi}{2^{-2-2a}\Gamma(1+2a)}} p^a e^{-p}$. For non-zero current quark mass, $KE$ diverges for negative $a$. Therefore, it is physically reasonable to consider only positive values of $a$. In position space,

$$\psi_a(x) = \frac{\sqrt{2^{-2a}\Gamma(1+2a)}}{\Gamma(a+\frac{1}{2})}(1 \ - \ ix)^{-a-1}. \tag{3.53}$$

Then

$$\bar{P}(a) = \frac{1+2a}{2}, \quad KE(a) = \frac{1}{a}, \quad SE(a) = -\frac{1}{\pi a}. \tag{3.54}$$

The potential energy $PE(a) = \int_{-\infty}^{\infty} |\psi_a(x)|^2 V_a(x) dx$ is determined by solving $V_a^{''}(x) = |\psi_a(x)|^2$ subject to boundary conditions $V_a(0) = \int_0^{\infty} x|\psi(x)|dx = \frac{\Gamma(a)}{2\sqrt{(\pi)}\Gamma(\frac{1}{2}+a)}$ and $V_a^{'}(0) = 0$. Integrating once,

$$V_a^{'}(x) = \frac{x\Gamma(1+a)\ {}_2\mathrm{F}_1(\frac{1}{2}, 1+a, \frac{3}{2}, -x^2)}{\sqrt{\pi}\Gamma(\frac{1}{2}+a)}. \tag{3.55}$$

And the second integration gives

$$V_a(x) = \frac{\Gamma(a)[(1+x^2)^{-a} + 2ax^2\ {}_2\mathrm{F}_1(\frac{1}{2}, 1+a, \frac{3}{2}, -x^2)]}{2\sqrt{(\pi)}\Gamma(\frac{1}{2}+a)}. \tag{3.56}$$

We arrive at

$$PE(a) = \frac{2^{1-2a}\pi^2}{\Gamma(\frac{1}{2}+2a)\Gamma(1-a)(\Gamma(\frac{1}{2}+a))^3(-\sin(\pi a) + \sin(3\pi a))}. \tag{3.57}$$

The Frobenius analysis suggests that the $(mass)^2$ is minimized if $a = 0$ in the chiral limit. Therefore, we expand $PE(a)$ in a Laurent series around $a = 0$:

$$PE(a) = \frac{1}{\pi a} + \frac{\pi a}{3} - \frac{12\zeta(3)a^2}{\pi} + O(a^3). \tag{3.58}$$

The pole at $a = 0$ in the potential energy is cancelled by a similar pole in the self energy.

Alternatively, we can calculate $PE(a)$ in momentum space using the definition of the "finite part" integrals (see §3.3.1). We use this as a check on our definition of $\mathcal{FP}$. Let

$$\tilde{\psi}_a(p) = p^a e^{-p} \frac{\sqrt{\pi}}{\sqrt{(2^{-2-2a}\Gamma(1+2a))}}. \tag{3.59}$$

Then

$$PE(a) = -2\int_0^{\infty} \frac{|\tilde{W}(s)|^2 - 1}{s^2}[ds], \tag{3.60}$$

where

$$\tilde{W}_a(s) = \int_0^{\infty} \tilde{\psi}_a(s+t)\tilde{\psi}_a^*(t)[dt] \text{ for } s \geq 0,\ \tilde{W}(-s) = \tilde{W}^*(s). \tag{3.61}$$

We get

$$\tilde{W}_a(s) = \frac{(2s)^{\frac{1}{2}+a}\sqrt{\pi}K_{\frac{1}{2}+a}(s)\csc\pi a}{\Gamma(-a)\Gamma(1+2a)} \tag{3.62}$$

where $K_n(z)$ is the modified Bessel function of the second kind. This results in the same expression for $PE(a)$ as before. Collecting the results, the baryon $(mass)^2$ is:

$$\frac{\mathcal{M}^2}{\tilde{g}^2 N^2} = (\frac{1+2a}{2})[\frac{\nu}{a} - \frac{1}{\pi a} + \frac{2^{1-2a}\pi^2}{\Gamma(\frac{1}{2}+2a)\Gamma(1-a)(\Gamma(\frac{1}{2}+a))^3(-\sin(\pi a)+\sin(3\pi a))}] \quad (3.63)$$

For small '$a'$, we may write

$$\frac{\mathcal{M}^2(a)}{\tilde{g}^2 N^2} = (\frac{1+2a}{2})(\frac{\nu}{a} + \frac{\pi a}{3} - \frac{12\zeta(3)a^2}{\pi} + O(a^3)) \quad (3.64)$$

**The Chiral Limit**

In the chiral limit $m = 0 \Rightarrow \nu = 0$,

$$\frac{\mathcal{M}^2(a, m=0)}{\tilde{g}^2 N^2} = (\frac{1+2a}{2})(\frac{\pi a}{2} - \frac{12\zeta(3)a^2}{\pi} + O(a^3)) \quad (3.65)$$

is the product of two factors, both of which are monotonically increasing functions for small $a$. Therefore, the baryon mass is minimized for $a = 0$, with a minimum value of 0. Since the $(mass)^2$ is a Lorentz scalar, it is unchanged if we Lorentz boost this wave function. Thus, the $(mass)^2$ must vanish for $\tilde{\psi}(p) = Ae^{(-bp)}$ for any $b > 0$.

Assuming the positivity of $(mass)^2$ on physical grounds, we conclude that in the large-$N$ and chiral limits, the absolute minimum of $(mass)^2$ occurs for a configuration containing only valence-quarks. We emphasize that the exponential solution is not just the absolute minimum of the baryon mass on the space of separable configurations but on the *entire phase space* of Hadrondynamics, in the chiral and large-$N$ limit.

The masslessness of this two-dimensional baryon may be puzzling at first. But there is a precedent for this. 't Hooft found that in the chiral limit, the lowest lying meson was massless in (the large-$N$ limit of) 2d QCD [53]. Moreover, numerical calculations of Hornbostel et. al. [58] also show that the baryons and mesons of 2-dimensional QCD are massless in the chiral limit.

**Momentum Sum Rule and Determination of '$b'$**

The valence-quark wave function in the chiral and large-$N$ limits is $\tilde{\psi}(p) = Ae^{-bp},\ p \geq 0$. Normalization gives $A = \sqrt{4\pi b}$. A Lorentz boost is just a rescaling of momenta, so the

momentum sum rule

$$\int_0^\infty p|\tilde{\psi}(p)|^2[dp] = P/N = \bar{P} \tag{3.66}$$

fixes the reference frame and determines $b = 1/2\bar{P}$. Therefore,

$$\tilde{\psi}(p) = \sqrt{\frac{2\pi}{\bar{P}}}\exp\left(\frac{-p}{2\bar{P}}\right). \tag{3.67}$$

This is the exact valence-quark wave function for $N = \infty, m = 0$. Thus the exact form factor of the baryon in the large-$N$ and chiral limits is

$$\tilde{M}(p,q) = -\left(\frac{4\pi}{\bar{P}}\right)e^{-(p+q)/2\bar{P}} \tag{3.68}$$

For finite $N$, we work in a reference frame in which the total baryon momentum is $P = 1$. In the case $N = 3$, this corresponds to $\bar{P} = P/N = \frac{1}{3}$. So to compare this wave function with that for finite $N$, $\bar{P} = 1/3$ is the appropriate choice.

### 3.3.5 Effect of Finite Current Quark Mass

We estimate the corrections due to non-zero $\nu = \frac{m^2}{\tilde{g}^2}$ variationally using the above exponential ansatz. From the Frobenius analysis, we already expect that $a \to 0^+$ as $\frac{m^2}{\tilde{g}^2} \to 0^+$. To investigate whether there is a minimum in $a$, we expand $\frac{\mathcal{M}^2(a)}{\tilde{g}^2N^2}$ in a Laurent series:

$$\frac{\mathcal{M}^2(a)}{\tilde{g}^2N^2} = \frac{\nu}{2a} + \nu + \frac{a\pi}{6} + a^2\left(\frac{\pi}{3} - \frac{6\zeta(3)}{\pi} + O(a^3)\right) \tag{3.69}$$

We see that for small $a$, the $O(\frac{1}{a})$ and $O(a)$ terms dominate and have opposing tendencies. Thus there is a positive value of $a$ for which $\mathcal{M}^2$ is minimized as fig. 3.2 shows. For any given $\frac{m^2}{\tilde{g}^2}$, we can solve the transcendental equation for the minimization of $\mathcal{M}^2$ numerically and find the optimal $a$. We show this in fig. 3.3. We see that $a \to 0^+$ as $\frac{m^2}{\tilde{g}^2} \to 0^+$, recovering the exact exponential solution in the chiral limit. However, for small enough current quark mass, we can get the asymptotic power law $a(\frac{m^2}{\tilde{g}^2})$ analytically. Minimizing in $a$, we find that $a \to \sqrt{\frac{3}{\pi}}\frac{m}{\tilde{g}}$ as we found from the Frobenius analysis (section 3.3.3). Moreover, $\frac{\mathcal{M}^2}{\tilde{g}^2N^2} \to \sqrt{\frac{\pi}{3}\frac{m^2}{\tilde{g}^2}} + \frac{m^2}{\tilde{g}^2}$ as $\frac{m^2}{\tilde{g}^2} \to 0$.

In other words, for sufficiently small current quark masses, our variational estimate for the valence-quark wave function is $\tilde{\psi}(p) = Ap^ae^{-bp}$, with $a = \frac{m}{\tilde{g}}\sqrt{\frac{3}{\pi}}$. The variational

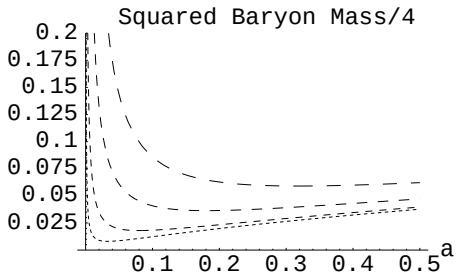


Figure 3.2: Estimate of $\frac{1}{4}$ baryon mass$^2$ in units of $\tilde{g}^2N^2$ as a function of $a$. The curves top to bottom are for $\frac{m^2}{\tilde{g}^2} = .05, .02, .005, .001$. As $\frac{m^2}{\tilde{g}^2} \to 0^+$, both the minimum value of $(mass)^2$ and the value of $a$ that minimizes it tend to $0^+$, recovering the exact massless exponential solution in the chiral limit.

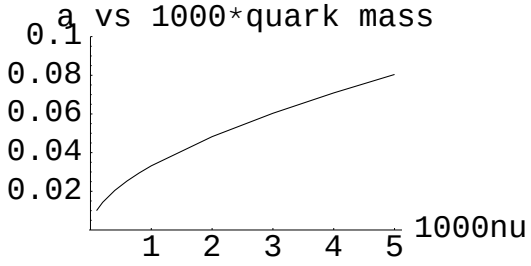


Figure 3.3: The power $a$ in the wave function $\tilde{\psi}(p) = Ap^a e^{-bp}$ plotted as a function of $1000\frac{m^2}{\tilde{g}^2}$.

upper bound for the baryon mass is $\mathcal{M} = \tilde{g}N(\frac{\pi}{3})^{\frac{1}{4}}(\frac{m}{\tilde{g}})^{\frac{1}{2}}$. Though the baryon mass is of order $N$, in the chiral limit, the proportionality constant vanishes in two dimensions!

### 3.3.6 Variational ground-state for Finite $N$

We have seen (section 3.2.3) that the leading effect of finite $N$ is to restrict the range of quark momenta to $0 < p < P$. Thus $\tilde{\psi}(p)$ vanishes for $p > P$. For finite, but large-$N$ we expect the ground-state wave function to be well approximated by the exact exponential solution (3.67) for $N = \infty$, in the chiral limit. Since

$$(1 - \frac{p}{PN})^N \to e^{\frac{-p}{P}} \tag{3.70}$$

the wave function

$$\tilde{\psi}(p) = A(1 - \frac{p}{P})^b,\ 0 < p < P \tag{3.71}$$

should be a good ansatz for the ground-state. From the above argument, we expect $b$ to be proportional to $N$. More generally, the Frobenius analysis (3.3.3) shows that

$$\tilde{\psi}(p) = Ap^a(1 - \frac{p}{P})^b,\ 0 < p < P \tag{3.72}$$

should be a good ansatz for the ground-state even for non-zero current quark mass. $A$ is fixed by normalization; $a$ is determined by the minimization of baryon $(mass)^2$ . From the Frobenius analysis we know that for small $\frac{m^2}{\tilde{g}^2}$, $a \to \sqrt{\frac{3}{\pi}}\frac{m}{\tilde{g}}$. $b$ can be eliminated using the momentum sum rule:

$$N\int_0^P p|\tilde{\psi}(p)|^2[dp] = P \tag{3.73}$$

Thus

$$b = \frac{N}{2} - 1 + a(N-1) \ \to\ \frac{N}{2} - 1 + \sqrt{\frac{3}{\pi}}\frac{m}{\tilde{g}}(N-1) \tag{3.74}$$

which is proportional to $N$ as we expected. The valence-quark probability density is $V(\xi) = \frac{1}{2\pi}|\tilde{\psi}(\xi P)|^2,\ 0 \le \xi \le 1$,

$$V(\xi) = \frac{\xi^{2a}(1-\xi)^{2b}}{\mathrm{Beta}(1+2a, 1+2b)}. \tag{3.75}$$

In the chiral limit,

$$V(\xi) = (N-1)(1-\xi)^{N-2} \tag{3.76}$$

which agrees with to the distribution obtained from numerical discretized light cone calculations by Hornbostel et. al. in [58].

## 3.4 Beyond the Separable Ansatz

### 3.4.1 Fixed Rank Submanifolds of Grassmannian

We have been studying a reduced dynamical problem on the separable submanifold of the phase space. By the variational principle, the minimum of energy on a subset of the phase space provides an upper bound for the true minimum. To improve on this estimate, we must allow for a larger set of configurations. We will now give an increasing family of submanifolds of the phase space, whose union is dense in the grassmannian . Each of these is the phase space for a reduced dynamical system. They turn out to correspond to interacting quark models, that go beyond the valence-quark approximation.

Let us generalize the rank-one ansatz to a rank $r$ ansatz ($a, b$ here do not stand for color, the meaning should be clear from the context):

$$M_r = \sum_{a,b=1}^{r} \xi_b^a \psi_a \otimes \psi^{\dagger b}. \tag{3.77}$$

We must pick $\xi_b^a$ and the $\psi_a$ such that the configuration $M_r$ is admissible: $trM_r^2 < \infty$ and satisfies the Pauli exclusion principle: $M_r^\dagger = M_r$, $(M_r + \epsilon)^2 = 1$. The admissibility condition is automatic since $M_r$ is a finite-rank operator. We pick the $\psi_a$ to be a finite number $r$ of orthonormal eigenstates of $\epsilon$, the sign of momentum.

$$\epsilon\psi_a = \epsilon_a\psi_a,\ \epsilon_a = \pm 1. \tag{3.78}$$

Suppose there are $r_+$ positive momentum vectors and $r_- = r - r_+$ negative momentum vectors. Let $\tilde{\epsilon} = \mathrm{Diag}(\epsilon_a)$ be the diagonal $r \times r$ matrix representing the restriction of $\epsilon$ to the sub-space of $\mathcal{H}_- \oplus \mathcal{H}_+$ spanned by the $\psi_a$. This ansatz satisfies the constraints if the $r \times r$ matrix $\xi$ is hermitian and satisfies the constraint

$$(\tilde{\epsilon} + \xi)^2 = 1, \tag{3.79}$$

a mini-version of the constraint on $M_r$. If we let $\tilde{\phi} = \tilde{\epsilon} + \xi$, then the constraints on $\tilde{\phi}$ are

$$\tilde{\phi}^\dagger = \tilde{\phi}, \quad \tilde{\phi}^2 = 1 \tag{3.80}$$

Thus $\tilde{\phi}$ (and hence $\xi$) lies on a finite dimensional grassmannian . The connected component of this finite dimensional grassmannian is fixed by Baryon number: $B = -\frac{1}{2} tr M_r = -\frac{1}{2} tr \xi$.

Specializing to the case of baryon number one, it is easily seen that the simplest solution to the constraints on $\xi$ is $r_+ = 1, r_- = 0$, so that $\xi$ is just a $1 \times 1$ matrix, the number -2. Then $M_1 = -2\psi \otimes \psi^\dagger$. This is just the rank-one separable ansatz that gave rise to the valence-quark model.

To go beyond the valence-quark approximation, we need to find other non-trivial solutions $r_+, r_-, \xi$ which satisfy these constraints. If $r = r_+ + r_- = 2$, the only non-trivial solutions to the constraints reduces to the rank-one solution. To see this we first note that for $r = 2$, $\tilde{\phi}$ is a point on the finite dimensional grassmannian of $\mathbf{C}^2$, the set of subspaces of $\mathbf{C}^2$. This grassmannian has three connected components, corresponding to tr $\tilde{\phi} = -2, 0, 2$. The baryon number constraint implies that $tr\tilde{\phi} = r_+ - r_- - 2$. For $\tilde{\phi}$ to lie on the finite dimensional grassmannian , we must have $r_+ = 1, 2$. The case $r_+ = 1$ gives $\tilde{\phi} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\tilde{\epsilon} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so that $\xi = \tilde{\phi} - \tilde{\epsilon} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$, which is the same as the rank-one solution.

The case $r_+ = 2, r_- = 0$ may also be reduced to the rank-one ansatz. In this case tr $\tilde{\phi} = 0, \tilde{\epsilon} = \mathbf{1}$. Therefore, $\tilde{\phi}$ is a point on $\mathbf{CP}^1$, which may be parameterized as $\tilde{\phi} = \mathbf{1} - 2zz^\dagger$. Here $z \in \mathbf{C}^2, ||z|| = 1$, though $\phi$ is independent of the overall phase of $z$. Then $\xi = \tilde{\phi} - \tilde{\epsilon} = -2zz^\dagger$ and $M_2 = -2(zz^\dagger)^a_b \psi_a \otimes \psi^{b\dagger}$. However, we notice that $M_2$ has a *gauge symmetry*, under the simultaneous action of $U(2)$ on both $z$ and $\psi$. This symmetry may be used to bring $z$ to the standard form $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. It is then clear that $M_2$ reduces to the rank-one ansatz.

Thus, $r = 2$ does not provide any new solutions. The physical reason for this is clear. Roughly speaking, $r = 2$ corresponds to a baryon that contains only valence and sea quarks ($r_+ = 2$) or a baryon that contains only valence and anti-quarks ($r_+ = 1$). However, we know that a consistent description of the baryon is possible only if we have all three components: valence, sea and anti-quarks. This suggests that we should try a rank three configuration. Indeed, a new non-trivial rank three solution exists; we discuss

it in the next section.

### 3.4.2 Rank Three Ansatz

The next simplest possibility is of rank three, with $r_+ = 2, r_- = 1$ so that $\tilde{\epsilon} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let us denote the negative momentum vector $\psi_-$ and the two positive momentum vectors $\psi, \psi_+$. It will turn out that the former represents anti-quarks while the latter represent valence and "sea" quarks. The constraints on $\tilde{\phi} = \tilde{\epsilon} + \xi$ are

$$\tilde{\phi}^\dagger = \tilde{\phi},\ \tilde{\phi}^2 = 1,\ tr\tilde{\phi} = -1 \tag{3.81}$$

Each solution to these constraints determines a one dimensional subspace of $\mathbf{C}^3$, the $-1$ eigenspace of $\tilde{\phi}$; i.e., a point in $\mathbf{CP}^2$. We may parameterize $\mathbf{CP}^2$ by unit vectors $\zeta$ in $\mathbf{C}^3$, whose overall phase is irrelevant:

$$\tilde{\phi} = -\mathbf{1} + 2\zeta \otimes \zeta^\dagger,\quad ||\zeta||^2 = 1,\quad \zeta \sim e^{i\alpha}\zeta. \tag{3.82}$$

Letting $\zeta = \begin{pmatrix} \zeta_- \\ \zeta_0 \\ \zeta_+ \end{pmatrix}$, we get

$$\xi = \tilde{\phi} - \tilde{\epsilon} = \begin{pmatrix} 2|\zeta_-|^2 & 2\zeta_-\zeta_0^* & 2\zeta_-\zeta_+^* \\ 2\zeta_0\zeta_-^* & -2 + 2|\zeta_0|^2 & 2\zeta_0\zeta_+^* \\ 2\zeta_+\zeta_-^* & 2\zeta_+\zeta_0^* & -2 + 2|\zeta_+|^2 \end{pmatrix} \tag{3.83}$$

Then $M_3 = \xi^a_b \psi_a \otimes \psi^{b\dagger}$ has a $U(1) \times SU(2)$ a global *gauge symmetry*. The phase of the negative momentum vector $\psi_-$ may be changed and the two positive momentum vectors $\psi$ and $\psi_+$ may be rotated into each other, provided we transform $\zeta$ in the same way.

Let us use the three degrees of freedom in the $SU(2)$ symmetry to rotate the vector $\begin{pmatrix} \zeta_- \\ \zeta_+ \end{pmatrix}$ to the standard form $\begin{pmatrix} 0 \\ \zeta_+ \end{pmatrix}$, with $\zeta_+$ real and positive. Since $||\zeta|| = 1$, $\zeta_+ = \sqrt{1 - |\zeta_-|^2}$. Then $M_3$ has the form

$$M_3 = -2\psi \otimes \psi^\dagger + 2|\zeta_-|^2[\psi_- \otimes \psi_-^\dagger - \psi_+ \otimes \psi_+^\dagger] + 2[\zeta_-\zeta_+\psi_- \otimes \psi_+^\dagger + \zeta_+\zeta_-^*\psi_+ \otimes \psi_-^\dagger]. \tag{3.84}$$

We see that $M_3$ is still invariant under

$$\psi \to e^{i\alpha}\psi,\ \psi_+ \to e^{i\alpha_+}\psi_+,\ \psi_- \to e^{i\alpha_-}\psi_-,\ \zeta_- \to e^{i(\alpha_+ - \alpha_-)}\zeta_- \tag{3.85}$$

In the case of the separable ansatz, the phase space of rank-one configurations was $\mathcal{P}(\mathcal{H}_+)$. The analogous space of rank three configurations is more involved. $\psi$ and $\psi_+$ determine a point in the flag $\mathcal{F}l_{2,1}(\mathcal{H}_+)$. $\psi_-$ determines a point in the projective space $P(\mathcal{H}_-)$. The total space of rank three configurations is then a bundle over $\mathcal{F}l_{2,1}(\mathcal{H}_+) \times P(\mathcal{H}_-)$, with fibre the unit interval $[0,1]$, parameterized by the absolute value of $\zeta_-$.

We may use the $\zeta_- \to e^{i(\alpha_+ - \alpha_-)}\zeta_-$ freedom to make $\zeta_-$ real. Thus, the rank three configurations which satisfy the constraints are given by

$$M_3 = -2\psi \otimes \psi^\dagger + 2\zeta_-^2[\psi_- \otimes \psi_-^\dagger - \psi_+ \otimes \psi_+^\dagger] + 2\zeta_-\zeta_+[\psi_- \otimes \psi_+^\dagger + \psi_+ \otimes \psi_-^\dagger]. \tag{3.86}$$

Baryon number may be expressed as

$$B = -\frac{1}{2} tr M_3 = \int_0^\infty \left\{ |\tilde{\psi}(p)|^2 + \zeta_-^2 \left[ |\tilde{\psi}_+(p)|^2 - |\tilde{\psi}_-(-p)|^2 \right] \right\} [dp]. \tag{3.87}$$

The total momentum $P = -\frac{1}{2} N \int p \tilde{M}(p,p)[dp]$ becomes

$$\bar{P} = \frac{P}{N} = \int_0^\infty p\{|\tilde{\psi}(p)|^2| + \zeta_-^2[|\tilde{\psi}_-(-p)|^2 + |\tilde{\psi}_+(p)|^2]\}[dp] \tag{3.88}$$

The variational quantities $\psi$, $\psi_-$, $\psi_+$, $\zeta_-$ are determined by minimizing the energy of the baryon. The kinetic energy is simple in momentum space

$$K = \int_0^\infty \frac{1}{2}[p + \frac{\mu^2}{p}]\{|\tilde{\psi}(p)|^2| + \zeta_-^2[|\tilde{\psi}_-(-p)|^2 + |\tilde{\psi}_+(p)|^2]\}[dp]. \tag{3.89}$$

The expression for potential energy is simpler in position space, but is lengthy since it is quadratic in $M(x,y)$. Unlike in the rank-one case, our wave functions also depend on a discrete index. A discrete analog of the Fourier transform, which diagonalizes $\xi$ makes the potential energy simpler. Let,

$$\psi_1 = \frac{1}{\sqrt{2}} \left\{ \sqrt{[1-\zeta_-]}\psi_- - \sqrt{[1+\zeta_-]}\psi_+ \right\} \tag{3.90}$$

$$\psi_2 = \frac{1}{\sqrt{2}} \left\{ \sqrt{[1+\zeta_-]}\psi_- + \sqrt{[1-\zeta_-]}\psi_+ \right\} \tag{3.91}$$

In this basis $\xi = \begin{pmatrix} 2\zeta_- & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2\zeta_- \end{pmatrix}$ is diagonal. However, the new wave functions $\psi_{1,2}$ are no longer eigenstates of $\epsilon$. $M_3$ becomes ($\psi_0 := \psi$):

$$-\frac{1}{2} M_3 = \psi_0 \otimes \psi_0^\dagger + \zeta_-\{\psi_1 \otimes \psi_1^\dagger - \psi_2 \otimes \psi_2^\dagger\} \tag{3.92}$$

And the potential energy is given by

$$\begin{aligned} U &= \frac{\tilde{g}^2}{2}\int\Big\{|\psi_0(x)|^2V_{00}(x)+\zeta_-^2|\psi_1(x)|^2V_{11}(x)+\zeta_-^2|\psi_2(x)|^2V_{22}(x)\Big\}dx \\ &\quad +\frac{\tilde{g}^2}{2}2\ \mathrm{Re}\int\Big\{\zeta_-\psi_0(x)\psi_1^*(x)V_{10}(x)- \\ &\quad \zeta_-\psi_0(x)\psi_2^*(x)V_{20}(x)-\zeta_-^2\psi_1(x)\psi_2^*(x)V_{21}(x)\Big\}dx. \end{aligned} \tag{3.93}$$

The mean-fields $V_{\alpha\beta}$ are defined as

$$V_{\alpha\beta}(x)=\frac{1}{2}\int dy\psi_\alpha(y)|x-y|\psi_\beta^*(y) \tag{3.94}$$

and satisfy the hermiticity condition $V_{\alpha\beta}^*(x)=V_{\beta\alpha}(x)$. They are the solutions to the differential equations

$$V_{\alpha\beta}''(x)=\psi_\alpha(x)\psi_\beta^*(x). \tag{3.95}$$

with the boundary conditions

$$V_{\alpha\beta}(0)=\frac{1}{2}\int dy\psi_\alpha(y)|y|\psi_\beta^*(y) \tag{3.96}$$

and

$$V_{\alpha\beta}'(x)\to\frac{1}{2}\delta_{\alpha\beta},\ \text{for}\ |x|\to\infty. \tag{3.97}$$

Thus we have defined the phase space and hamiltonian of the rank three ansatz. As in the rank-one case, (section 3.2.2), we can work out the Poisson bracket relations among $\psi,\ \psi_\pm$ implied by the Poisson brackets of the $\tilde{M}(p,q)$. We can then quantize this classical dynamical system to obtain an interacting quantum system of quarks and anti-quarks, whose color is not being counted explicitly. These particles would be the rank three analogs of the $N$ bosons we encountered in the rank-one ansatz. However, we will follow a different strategy this time, which clarifies the connection of the rank three ansatz to the quark model.

### 3.4.3 Fock Space Description

In what follows, we will give two Fock space descriptions of the rank three configurations. In the first case, we will generalize the rank-one ($N=\infty$) formula $|M_1>=b_\psi^\dagger|\tilde{0}>$ to the case $M=M_3$. This will give us a description of the baryon in terms of quasi-particles

which are certain colorless combinations of the true quarks and anti-quarks. Next we will generalize the (finite $N$) formula $|V>=q_{\tilde{\psi}}^{1\dagger}\cdots q_{\tilde{\psi}}^{N\dagger}|0>$ in order to get a description in terms of colored particles.

In order to find the Fock space state corresponding to $M_3$, we use an analog of the Plücker embedding [24, 79] of the grassmannian in the Fock space, just as we did in the rank-one case in §3.2.4. The filled states $\chi$ are $-1$ eigenstates of $\Phi_3$

$$\Phi_3\ \chi=(\epsilon+\xi_b^a\psi_a\psi^{b\dagger})\ \chi=\ -\chi \tag{3.98}$$

There are two types of solutions to this eigenvalue problem. First let us look in the orthogonal complement of the subspace spanned by $\{\psi_a\}_{a=1}^r$:

$$\epsilon\chi=-\chi \text{ and } <\psi_b,\chi>=0,\ b=1,\cdots r. \tag{3.99}$$

This means $\chi$ is a negative momentum state, orthogonal to $\psi,\psi_\pm$. Thus, all negative momentum states, orthogonal to $\psi_-$ are filled.

For the other type of solution, we consider those $\chi$ that lie in the span of the $\psi_a$. Let $\chi=\sum_{a=1}^r\tilde{\chi}_a\psi_a$.

$$\epsilon_a\tilde{\chi}_a+\xi_a^b\tilde{\chi}_b=-\tilde{\chi}_a. \tag{3.100}$$

Thus, we are looking for $-1$ eigenvectors of $\tilde{\epsilon}+\xi=\begin{pmatrix}-1+2|\zeta_-|^2 & 0 & 2\zeta_-\zeta_+\\ 0 & -1 & 0\\ 2\zeta_+\zeta_- & 0 & 1-2|\zeta_-|^2\end{pmatrix}$. $\tilde{\chi}^{(1)}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\tilde{\chi}^{(2)}=\begin{pmatrix}-\zeta_+\\0\\\zeta_-\end{pmatrix}$ are the solutions. Thus, the states

$$\chi^{(1)}=\psi \text{ and } \chi^{(2)}=-\zeta_+\psi_-+\zeta_-\psi_+ \tag{3.101}$$

are also filled. Thus the Fock space state corresponding to the rank three ansatz $M_3$ is a semi-infinite wedge product:

$$\cdots f^{2'}\wedge f^{1'}\wedge\psi\wedge(-\zeta_+\psi_-+\zeta_-\psi_+) \tag{3.102}$$

Here $f^{j'}$ is a dual basis in the orthogonal complement of $\psi_-$ in $\mathcal{H}_-$. As before, we focus on the deviation from the vacuum:

$$|\tilde{0}>=\cdots f^{2'}\wedge f^{1'}\wedge\psi_-' \tag{3.103}$$

Then $|M_3>$ is obtained by populating $\psi$ and $(-\zeta_+\psi_- + \zeta_-\psi_+)$ and depopulating $\psi_-$, starting from the vacuum $|\tilde{0}>$.

$$|M_3>= (-\zeta_+ b^\dagger_{\psi_-} + \zeta_- b^\dagger_{\psi_+}) b_{\psi_-} b^\dagger_\psi |\tilde{0}> \tag{3.104}$$

However, the operators $b^\dagger_{\psi_\pm}, b^\dagger_\psi, b_{\psi_\pm}, b_\psi$ are not the creation-annihilation operators of quarks. They create and annihilate the quasi-particles of the rank three ansatz. This point of view is simpler for computations, and avoids the unobservable color quantum number.

However, we are more accustomed to thinking in terms of the Fock space of quarks and anti-quarks, which carry the color quantum number. We now determine the state consisting of "valence", "sea" and anti-quarks, $|VSA>$ that corresponds to the rank three ansatz. We will see that the advantage of this point of view is that it has a dual description in terms of an embellished quark model, which we discuss in the next section.

Recall that the Fock space state $|V>$ corresponding to the rank-one ansatz is

$$|V>= q^{1\dagger}_{\tilde{\psi}} \cdots q^{N\dagger}_{\tilde{\psi}} |0> \tag{3.105}$$

where, $q^{a\dagger}_{\tilde{\psi}}$ creates a quark with color $a$ in the state $\tilde{\psi}$. These operators satisfy canonical anti-commutation relations. As before, the state $|VSA>$ in the Hartree approximation is determined by the condition that the form factor of the quark density matrix n the solitonic state $|VSA>$ should be the classical density matrix in the large-$N$ limit

$$< VSA|\hat{\rho}(p,q)|VSA >= \tilde{\rho}_3(p,q). \tag{3.106}$$

Here $\hat{\rho}(p,q) = \frac{1}{N} q^{a\dagger}(p)\ q_a(q)$ and $\tilde{\rho}_3(p,q) = \frac{1}{2}(2\pi\delta(p-q) - \tilde{M}_3(p,q) - \tilde{\epsilon}(p,q))$. While it was easy to guess the form of $|V>$, we will use an indirect method to find $|VSA>$. We know that $|V>$ satisfies the condition $< V|\hat{\rho}(p,q)|V >= \tilde{\rho}_1(p,q)$. We proceed by finding the operator that transforms $\tilde{\rho}_1$ into $\tilde{\rho}_3$. Then we represent this operator on the Fock space and use it to transform $|V>$ into $|VSA>$. This is feasible since the grassmannian carries a transitive action of the infinite dimensional restricted unitary group (section 2.2). We may obtain the rank three configuration $\Phi_3 = M_3 + \epsilon$ from the separable ansatz $\Phi_1 = M_1 + \epsilon$ by a unitary transformation:

$$U^\dagger(\epsilon - 2\psi\otimes\psi^\dagger)U = \epsilon + \xi^a_b \psi_a \otimes \psi^{\dagger b}. \tag{3.107}$$

(We work with $\Phi = M + \epsilon$ or $\rho$ since they transform homogeneously under unitary transformations, while due to normal ordering, $M$ does not.) The infinite dimensional restricted unitary group is disconnected, unlike its finite dimensional counterparts. However, since both $M_1$ and $M_3$ are configurations of baryon number one, $U$ lies in the connected component of the identity, and is of the form $U = e^{iA}$, for hermitian $A$. From the expressions for $M_1$ and $M_3$,

$$\begin{aligned} M_1 &= -2\ \psi \otimes \psi^\dagger \\ M_3 &= -2\ \psi \otimes \psi^\dagger + 2\zeta_-^2[\psi_- \otimes \psi_-^\dagger - \psi_+ \otimes \psi_+^\dagger] \\ &\quad +2\ \zeta_-\zeta_+[\psi_- \otimes \psi_+^\dagger + \psi_+ \otimes \psi_-^\dagger]. \end{aligned} \tag{3.108}$$

We see that $U$ must be the identity except on the span of $\psi_-$ and $\psi_+$. The restriction of $\Phi_1$ to this subspace is $-\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, while the restriction of $\Phi_3$ is $s\sigma_1 + (r-1)\sigma_3$, where $r = 2\zeta_-^2$ and $s = 2\zeta_-\zeta_+$. Here $\sigma_i$ are the Pauli matrices. Thus, the restriction of $A$ to this subspace satisfies

$$e^{-iA}(-\sigma_3)e^{iA} = s\sigma_1 + (r-1)\sigma_3. \tag{3.109}$$

Therefore $A$ is a $2 \times 2$ traceless hermitian matrix $\sigma.\mathbf{w}$. Here $\mathbf{w}$ is the vector in $\mathbf{R}^3$ about which $(0, 0, -1)$ must be rotated by an angle $2|\mathbf{w}|$ to reach $(s, 0, r-1)$). We get

$$A(p,q) = i \arcsin(\zeta_-)(\psi_-(p)\psi_+^*(q) - \psi_+(p)\psi_-^*(q)) \ \text{ and } \ \tilde{U}(p,q) = e^{iA(p,q)}. \tag{3.110}$$

Let us denote $\theta = \arcsin(\zeta_-)$. Therefore we have ($\rho = \frac{1}{2}(1 - \Phi)$),

$$\int e^{-iA(p,r)}\tilde{\rho}_1(r,s)e^{iA(s,q)}[dr][ds] = \tilde{\rho}_3(p,q). \tag{3.111}$$

Now we look for a representation of $\tilde{U}(p,q)$ as a unitary operator $\hat{U}$ acting on the quark creation annihilation operators. Moreover, suppose we can show that

$$\begin{aligned} \hat{U}^\dagger\hat{q}(q)\hat{U} &= \int [ds]\hat{q}(s)\tilde{U}^\dagger(s,q) \\ \hat{U}^\dagger\hat{q}^\dagger(q)\hat{U} &= \int [dr]\tilde{U}(p,r)\hat{q}^\dagger(r). \end{aligned} \tag{3.112}$$

Then

$$|VSA> = \hat{U}\ |V>\ . \tag{3.113}$$

since for this choice, the expectation value condition is satisfied:

$$\begin{aligned} < VSA|\frac{1}{N}\hat{q}^\dagger(p)\ \hat{q}(q)|VSA > &= < V|\hat{U}^\dagger\hat{q}^\dagger(p)\hat{U}\hat{U}^\dagger\hat{q}(q)\hat{U}|V > \\ &= \int [dr][ds] < V|\tilde{U}(p,r)\hat{q}^\dagger(r)\hat{q}(s)\tilde{U}^\dagger(s,q)|V > \\ &= \int [dr][ds]\tilde{U}(p,r)\tilde{\rho}_1(r,s)\tilde{U}^\dagger(s,q) \\ &= \tilde{\rho}_3(p,q). \end{aligned} \tag{3.114}$$

It only remains to find the second quantized representative $\hat{U}$ and show that the quark creation and annihilation operators transform in the desired manner under it. $\hat{U}$ may be obtained by promoting the wave functions that appear in $U = e^{iA(p,q)}$ to operators:

$$\hat{U} = e^{i\hat{A}} = e^{-\arcsin(\zeta_-)(\hat{q}_{a\psi_-}\hat{q}^{a\dagger}_{\psi_+} - \hat{q}_{a\psi_+}\hat{q}^{a\dagger}_{\psi_-})} \tag{3.115}$$

We sum over colors to produce a color-singlet. We can check that $\hat{q}(p),\ \hat{q}^\dagger(q)$ have the desired transformation properties under $\hat{U}$. For example, we show that $\hat{U}^\dagger\hat{q}^\dagger(p)\hat{U} = \int [dr]\tilde{U}(p,r)\hat{q}^\dagger(r)$. For this we will need the preliminary result

$$[\hat{A},\ \hat{q}^\dagger(p)] = \tilde{\psi}_+(p)\hat{q}^\dagger_{\tilde{\psi}_-} - \tilde{\psi}_-(p)\hat{q}^\dagger_{\tilde{\psi}_+} \tag{3.116}$$

which is easily verified. Then

$$\begin{aligned} \hat{U}^\dagger\hat{q}^\dagger(p)\hat{U} &= e^{-i\hat{A}}\hat{q}^\dagger(p)e^{i\hat{A}} \\ &= (1 - i\hat{A} + O(\theta)^2)\ \hat{q}^\dagger(p)\ (1 + i\hat{A} + O(\theta)^2) \\ &= \hat{q}^\dagger(p) - i[\hat{A},\ \hat{q}^\dagger(p)] + O(\theta)^2 \\ &= \hat{q}^\dagger(p) - i(\tilde{\psi}_+(p)\int [dr]\tilde{\psi}^*_-(r)\hat{q}^\dagger(r) - \tilde{\psi}_-(p)\int [dr]\tilde{\psi}^*_+(r)\hat{q}^\dagger(r)) + O(\theta)^2 \\ &= \hat{q}^\dagger(p) - i\int [dr]\hat{q}^\dagger(r)(\tilde{\psi}_+(p)\tilde{\psi}^*_-(r) - \tilde{\psi}_-(p)\tilde{\psi}^*_+(r)) + O(\theta)^2 \\ &= \int [dr][2\pi\tilde{\delta}(p,q) + i\tilde{A}(p,q) + O(\theta)^2]\hat{q}^\dagger(r) \\ &= \int e^{i\tilde{A}(p,r)}\hat{q}^\dagger(r)[dr] \\ \hat{U}^\dagger\hat{q}^\dagger(p)\hat{U} &= \int [dr]\tilde{U}(p,r)\hat{q}^\dagger(r) \end{aligned} \tag{3.117}$$

as desired. We conclude that

$$|VSA >= e^{-i\hat{A}}|V >= e^{\arcsin\zeta_-[q_{a\psi_-}q^{a\dagger}_{\psi_+} - \text{h.c.}]}|V > . \tag{3.118}$$

$|VSA>$ is the second quantized Fock space state corresponding to the rank three ansatz $M_3$. Here the $q^{\dagger a}, q_a$ create and annihilate colored quarks. We see that $|VSA>$ may be obtained from the valence-quark state by a rotation. In the next section we will show that the same result could have been obtained by a Bogoliubov transformation, applied to the valence-quark model.

### 3.4.4 Valence, Sea and Antiquarks: Bogoliubov Transformation

The interacting valence-quark model we derived earlier (section 3.2.2) ignored the anti and sea-quark degrees of freedom. Now, we show how they can be included by performing a unitary transformation on the Fermionic Fock space. This is like a Bogoliubov transformation that mixes positive and negative momentum states.

In general, the baryon can be in a superposition of states containing $N+\eta$ quarks and $\eta$ anti-quarks for $\eta = 0, 1, \cdots, \infty$. This is because only the difference, baryon number, is a conserved quantum number in the full theory. The valence-quark approximation amounts to ignoring all states for $\eta > 0$ and in the Hartree approximation, corresponds to the state $|V>= q^{1\dagger}_{\tilde{\psi}} \cdots q^{N\dagger}_{\tilde{\psi}}|0>$.

Let us continue to work within a factorized Hartree approximation, ignoring correlations except when required by the Pauli exclusion principle or color-invariance. The next simplest possibility beyond the valence approximation is to consider only states with $\eta \leq N$. In the *ground-state* of the baryon, we find that this is a good approximation.

The quarks cannot all occupy the positive momentum state $\tilde{\psi}(p)$ as in the valence approximation, because they would violate the Pauli exclusion principle: there is no completely anti-symmetric tensor in more than $N$ indices transforming under $SU(N)$. In the ground-state, it is energetically favorable to introduce just one more positive momentum state $\tilde{\psi}_+(p)$ to accommodate the quarks. As for the anti-quarks, they occupy the lowest available negative momentum state $\tilde{\psi}_-(p)$.

To obtain a state containing valence, "sea" and anti-quarks (say $|VSA>$), we apply a unitary transformation to the valence-quark state $|V>$. It will be determined by the variational wave functions $\psi, \psi_+$ and $\psi_-$ in additional to an "angle of rotation" $\theta$. These are to be determined by minimizing the $(mass)^2$ of the baryon. In order to produce a

state $|VSA>$ of norm 1, we require the transformation to be unitary. In analogy with a Bogoliubov transformation, the operator that creates a quark in state $\psi_+$ and an anti-quark in state $\psi_-$ is $a_{b\psi_-}a^{b\dagger}_{\psi_+}$. We have summed over colors to produce a color-singlet state. Our variational ansatz for the state $|VSA>$ is then $e^{\theta[a_{b\psi_-}a^{b\dagger}_{\psi_+}-\text{h.c.}]}|V>$.

We see that this is identical to the state $|VSA>$ we obtained from the rank three ansatz, provided we identify $\zeta_-$ with $\sin\theta$ where $\theta$ is the angle of the Bogoliubov rotation.

This point of view confirms the interpretation of $\psi_-$ as the wave function describing anti-quarks. $\psi$ and $\psi_+$ may be interpreted as "valence" and "sea" quark wave functions. Their orthogonality can be understood as a consequence of the Pauli exclusion principle. $0 \leq \zeta_- \leq 1$ measures the departure from the valence-quark approximation. The probability of finding an anti-quark in the baryon is $\frac{\zeta_-^2}{1+2\zeta_-^2}$.

While the physical meaning of these quantities are best understood in the context of the quark model, actual computations are most easily performed in the context of Hadrondynamics. We turn next to an estimation of the anti-quark content of the two-dimensional baryon.

### 3.4.5 Variational Estimate of Antiquark Content

In §3.3.4 we found that the absolute minimum of energy (baryon mass) in the limit of chiral symmetry occurs for a rank-one configuration, in large-$N$ limit. In this limit, the anti-quark content of the baryon vanishes. We find that for small values of current quark mass ($\frac{m}{\tilde{g}} << 1$), the baryon has a small but non-vanishing anti-quark content. We already saw that the leading effect of finite $N$ was to make the wave functions $\tilde{\psi}_a(p)$ vanish beyond the total baryon momentum $P$.

To determine the rank three configuration, we can derive integral equations for the minimization of $\mathcal{M}$. However, from our exact solution, we expect that the anti-quark content is small for small current quark masses. To estimate $\zeta_-(\frac{m^2}{\tilde{g}^2})$, we do a variational minimization of $\mathcal{M}$, assuming some simple functional forms for $\psi_\pm, \psi$, which are suggested by the form of the exact solution. Moreover, we work in the limit $N = \infty$. As before, we pick

$$\tilde{\psi}(p) = C(\frac{p}{P})^a e^{\frac{-bp}{P}},\ p \geq 0 \tag{3.119}$$

Here $\bar{P} = \frac{P}{N}$ is the mean baryon momentum per color, $C$ is fixed by normalization and $b$ is fixed by the choice of Lorentz reference frame. $\psi_+$ must be orthogonal to $\psi$, so it must have a node:

$$\tilde{\psi}_+(p) = C_+(\frac{p}{\bar{P}})^a(\frac{p}{\bar{P}} - C_1)e^{\frac{-bp}{P}},\ p \geq 0 \tag{3.120}$$

Here $C_1$ is fixed by orthogonality. In the ground-state the anti-quark wave function has no nodes (besides at $p = 0$), and for ease of computation we pick it to be the same function (but for $p \leq 0$) as the valence-quark wave function. (Note: this is *not* required by any symmetry.)

$$\tilde{\psi}_-(p) = \tilde{\psi}(-p),\ p \leq 0 \tag{3.121}$$

Since a Lorentz boost in the longitudinal direction corresponds to a rescaling $p \to \lambda p$ in null coordinates, we pick a reference frame in which $b = 1$, and minimize the Lorentz invariant $\mathcal{M}^2$ in that frame. This determines the Lorentz invariant quantities $\zeta_-(\frac{m^2}{\tilde{g}^2})$ and $a(\frac{m^2}{\tilde{g}^2})$.

The $(mass)^2$ of the baryon is $\mathcal{M}^2 = P(2E - P)$

$$\begin{aligned}\frac{\mathcal{M}^2}{\tilde{g}^2N^2} &= \Big[-\frac{1}{2}\int p\tilde{M}(p,p)[dp]\Big]\Big[-\nu\int\frac{\tilde{M}(p,p)}{2p}[dp] + \frac{1}{\pi}\int\frac{\tilde{M}(p,p)}{2p}[dp] \\ &\quad +\frac{1}{4}\int dxdy|M(x,y)|^2\frac{1}{2}|x-y|\Big]\end{aligned} \tag{3.122}$$

where $(\nu = \frac{m^2}{\tilde{g}^2})$. We abbreviate

$$\frac{\mathcal{M}^2}{\tilde{g}^2N^2} = \bar{P} * (\nu KE + SE + PE). \tag{3.123}$$

Using the rank three ansatz,

$$M_3 = -2\psi\otimes\psi^\dagger + 2\zeta_-^2[\psi_-\otimes\psi_-^\dagger - \psi_+\otimes\psi_+^\dagger] + 2\zeta_-\sqrt{1-\zeta_-^2}[\psi_-\otimes\psi_+^\dagger + \psi_+\otimes\psi_-^\dagger] \tag{3.124}$$

for the above choices of wave functions we find:

$$\bar{P}(\zeta_-,a) = \frac{1}{2} + 2\zeta_-^2 + a(1+2\zeta_-^2),\ KE(\zeta_-,a) = \frac{1+2\zeta_-^2}{a},\ SE(\zeta_-,a) = -\frac{1+2\zeta_-^2}{\pi a} \tag{3.125}$$

$$PE(\zeta_-,a) = (U_0(a) + \zeta_-\sqrt{1-\zeta_-^2}U_1(a) + \zeta_-^2U_2(a) + \zeta_-^3\sqrt{1-\zeta_-^2}U_3(a) + \zeta_-^4U_4(a)) \tag{3.126}$$

The potential energies $U_i(a)$ can be calculated analytically for the most part, and we use the symbolic package Mathematica for this purpose. We must minimize $\mathcal{M}^2$ in the

strip $0 \leq \zeta_- \leq 1, a \geq 0$. We find that for small but non-vanishing $\frac{m^2}{\tilde{g}^2}$, $\mathcal{M}^2$ has a minimum for small but non-zero values of both $\zeta_-$ and $a$, indicating a small anti-quark content. To see that such a non-trivial minimum exists, it suffices to consider the factor $(\nu KE + SE + PE)$, since $\bar{P}$ is monotonic in $\zeta_-$ and $a$. Moreover, since $\zeta_-$ turns out to be small, it is enough to consider only terms in $PE(\zeta_-, a)$ which dominate for small $\zeta_-$

$$PE(\zeta_- \to 0, a) \to (U_0(a) + \zeta_- \sqrt{1-\zeta_-^2} U_1(a) + \zeta_-^2 U_2(a)) \tag{3.127}$$

$U_0(a)$ is the same as in the valence-quark approximation and was obtained in §3.3.4. For small $a$,

$$U_0(a) = \frac{1}{\pi a} + \frac{\pi a}{3} + O(a^2) \tag{3.128}$$

We also find that $U_1(a)$ is negative and goes to zero as $a \to 0$. As for $U_2(a)$, we find,

$$U_2(a) \to \frac{2}{\pi a} + \text{constant}.a + O(a^2) \tag{3.129}$$

Thus, the terms proportional to $\frac{1}{a}$ in PE and SE cancel out exactly, and we get

$$(\nu KE + SE + PE)(\zeta_-, a) = d_1 a + \nu + d_2 a \zeta_-^2 + 2\nu \frac{\zeta_-^2}{a} - d_3 a^{d_4} \zeta_- \sqrt{1-\zeta_-^2} \tag{3.130}$$

for positive constants $d_i,\ i = 1, \cdots, 4$. Isolating the $a$ and $\zeta_-$ dependence below we see that for $\nu > 0$, $(\nu KE + SE + PE)(\zeta_-, a)$, and hence $\mathcal{M}^2$ must have a non-trivial minimum for some $\zeta_- > 0$, $a > 0$:

$$\begin{aligned} (\nu KE + SE + PE)(\zeta_-) &= f_0 - f_1 \zeta_- \sqrt{1-\zeta_-^2} + f_2 \zeta_-^2, \\ (\nu KE + SE + PE)(a) &= \frac{h_{-1}}{a} - h_0 a^{h_2} + h_1 a. \end{aligned} \tag{3.131}$$

Here the $f_i(a), h_j(\zeta_-)$ are positive. Thus we estimate that the minimum of baryon $(mass)^2$ on the rank three sub-manifold occurs for a small but non-trivial anti-quark content, when the parameter $\frac{m^2}{\tilde{g}^2}$ is small. However, as the plots (fig. 3.5 & 3.4) show, in the limit of chiral symmetry, $\zeta_- \to 0,\ a \to 0$, recovering the exact rank-one solution.

To include the leading $\frac{1}{N}$ corrections the wave functions must vanish beyond $p = P$. Setting $\xi = p/P$, we pick

$$\psi(\xi) = C\xi^a (1-\xi)^b, \qquad \psi_+(\xi) = C_+ \xi^a (\xi - C_1)(1-\xi)^b \text{ for } 1 \geq \xi \geq 0$$

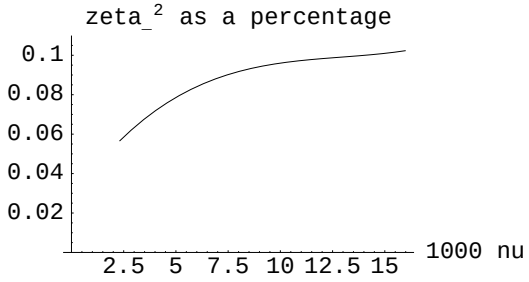


Figure 3.4: $\zeta_-^2$ measuring departure from valence-quark approximation plotted as a percent as a function of $1000\frac{m^2}{\tilde{g}^2}$. The probability of finding an anti-quark in the baryon is $\frac{\zeta_-^2}{1+2\zeta_-^2}$. The valence-quark approximation becomes exact in the chiral limit.

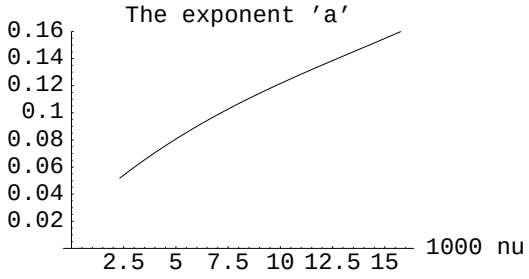


Figure 3.5: The power $p^a$ governing behavior of the wave functions near the origin, plotted as a function of $1000\nu = 1000\frac{m^2}{\tilde{g}^2}$. The wave functions tend to a constant as $p \to 0$ in the chiral limit.

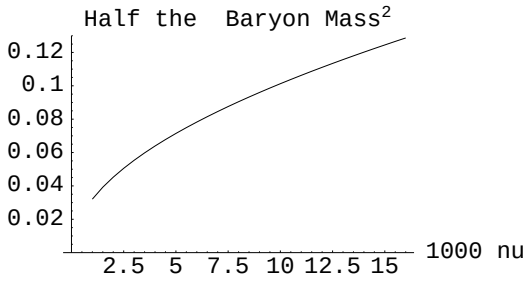


Figure 3.6: Variational upper bound for $\frac{1}{2}$ mass$^2$ of baryon in the rank three approximation plotted as a function of $1000\frac{m^2}{\tilde{g}^2}$. The baryon is massless in the chiral limit.

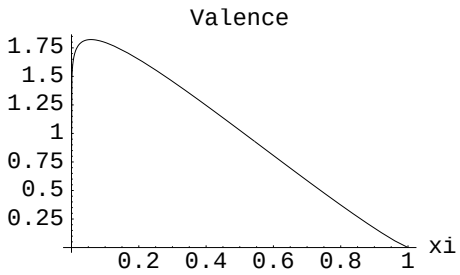


Figure 3.7: Variational approximation for valence-quark density $V(\xi) = 2\pi|\psi(\xi)|^2$ in the rank three ansatz. It is normalized to 1.

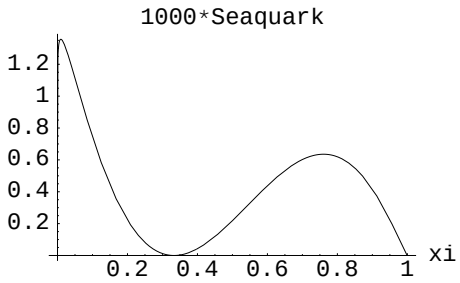


Figure 3.8: Variational approximation for sea quark distribution $2\pi\zeta_-^2|\psi_+(\xi)|^2$ in the rank three ansatz. It is normalized to $\zeta_-^2$.

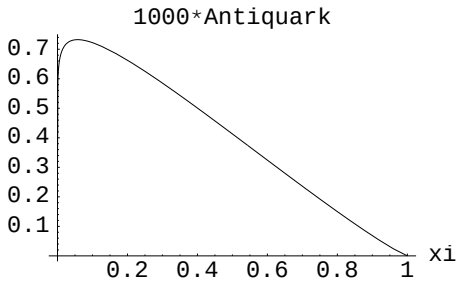


Figure 3.9: Variational approximation for antiquark quark distribution $2\pi\zeta_-^2|\psi(\xi)|^2$ in the rank three ansatz. It is normalized to $\zeta_-^2$.

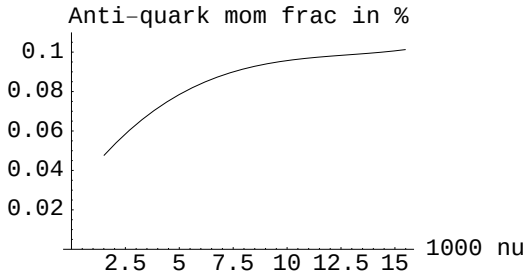


Figure 3.10: Fraction of baryon momentum carried by anti-quarks as a function of $\nu = \frac{m^2}{\tilde{g}^2}$.

$$\text{and} \quad \tilde{\psi}_-(\xi) \quad = \quad \tilde{\psi}(-\xi) \text{ for } -1 \leq \xi \leq 0 \tag{3.132}$$

The behavior of the wave functions for $p \to 0$, is insensitive to the boundary condition for large $p$, so we may use the value of $a$ determined in the large-$N$ limit. Similarly, we ignore the $\frac{1}{N}$ correction to the small parameter $\zeta_-$ (See Fig 3.5 & 3.4). $b$ is determined in terms of $a$ and $\zeta_-$ by the momentum sum rule:

$$\frac{P}{N} = \int_0^P p\{|\tilde{\psi}(p)|^2| + \zeta_-^2[|\tilde{\psi}_-(-p)|^2 + |\tilde{\psi}_+(p)|^2]\}[dp]. \tag{3.133}$$

Ignoring the small difference between the momentum carried by sea and anti-quarks (both of which are suppressed by factors of $\zeta_-^2$ compared to the valence-quark contribution) we find

$$b \sim -1 + N(\frac{1}{2} + \zeta_-^2) + a(-1 + N + 2N\zeta_-^2) \tag{3.134}$$

We plot the valence, sea and anti-quark densities as functions of $\xi = \frac{p}{P}$ for $\frac{m^2}{\tilde{g}^2} = 10^{-3}$, $N = 3$. For these parameters we estimate $\zeta_-^2 = 4 \times 10^{-4}$, $a = 3.5 \times 10^{-2}$ and $b = 0.57$.

## 3.5 Application to Deep Inelastic Scattering

The structure of nucleons (proton and neutron) is measured in Deep Inelastic Scattering (DIS) experiments. Electrons (or muons, neutrinos) are scattered off nucleons by the exchange of a space-like ($q^2 < 0$) virtual photon (or weak gauge boson). Thus, the electro-weak force is used as a probe of the strong force binding hadrons, just as Rutherford used alpha particles to study atomic structure. The DIS experiments of

the late 1960s and early 1970s [38] discovered scaling. The structure of the proton was roughly independent of the wave length of the virtual photon used as a probe: the measured structure functions were independent of $q^2$. The quark-parton model of Feynman [17, 36] explained this phenomenon in terms of point like quarks, which had to be weakly interacting at high energies. Soon after, it was discovered that the only renormalizable four dimensional quantum field theory of quarks with this property of asymptotic freedom at high momentum transfers is a non-abelian gauge theory of quarks coupled to gluons, which we now refer to as QCD. Perturbative QCD predicts small logarithmic violations of scaling. The evolution of structure functions as $Q^2(\equiv -q^2)$ is increased is given by the DGLAP equations [45]. This $Q^2$ dependence has been experimentally verified. However, the measured structure functions also contain information on how quarks and gluons bind to form a relativistic bound state. This is encoded in the momentum fraction $\xi = p/P$ or Bjorken variable $x_B = \frac{Q^2}{2P.q}$ dependence of structure functions, or parton distribution functions. Like atomic wave functions, they tell us the probability of finding a quark with a given momentum $p$ in the proton whose momentum is $P$. It has not been possible to describe the formation of hadronic bound states by perturbing around the limit of non-interacting quarks. What is more, unlike the electrons in an atom, quarks appear to be permanently confined inside hadrons, and have never been isolated. Understanding the bound state structure of hadrons and the phenomenon of confinement remains a challenge in $3+1$ dimensions.

However, in Deep Inelastic Scattering, the transverse momenta of partons are small compared to their longitudinal momenta. Moreover, a perturbative treatment of tansverse momenta with an upper cut off $Q$ leads to the same leading $Q^2$ dependence for structure functions as predicted by the operator product expansion in perturbative QCD. We have determined the bound state structure of quarks in a baryon in $1+1$ dimensions in preceding sections. This leads us to propose a model for the $x_B$ dependence of structure functions at an initial low value of $Q^2 = Q_0^2$ where transverse momenta are ignored. Our quark distribution functions are to be used as initial conditions for $Q^2$ evolution and compared with experimental data at higher values of $Q^2$.

It is an experimental fact that at low $Q^2$, about half the momentum of the proton is

carried by gluons [21]. However, two-dimensional QCD does not contain any dynamical gluon degrees of freedom, which are necessary to find the gluon distribution. Rather than two-dimensional QCD, it is more appropriate to study the dimensional reduction of QCD from four to two dimensions, which includes the transverse polarization states of the gluon field. But understanding the non-perturbative effects of the gluon degrees of freedom is substantially harder. Indeed, the second part of this thesis is motivated by a desire to do just that. For the present, we account for the omission of gluons by merely assuming that the quarks carry only a fraction $f$ of the baryon momentum. Moreover, we found that anti-quarks and sea quarks carry a very small portion of the baryon momentum in two dimensions (see §3.4.5). Phenomenological fits [21, 76] to experiment show that anti and sea quarks carry very small fractions of baryon momentum at low $Q^2$, the bulk of it is carried by valence-quarks and gluons. Thus we make the following ansatz: at an initial low value $Q_0^2$, the valence-quarks carry a fraction $f$ of baryon momentum and their probability distribution $V(\xi) = \frac{P}{2\pi}|\tilde{\psi}(\xi P)|^2$ is determined by our approximate solution of two-dimensional QCD.

Now, under $Q^2$ evolution, the quark, anti-quark and gluon distributions mix, in general. However, the difference between quark and anti-quark distributions $q^V(\xi, Q^2) = q(\xi, Q^2) - \bar{q}(\xi, Q^2)$ evolves independently of the gluon distribution. This difference is referred to as the "valence" quark distribution in the perturbative QCD literature. Moreover, the experimentally measured structure function $F_3(x_B, Q^2) \simeq q^V(\xi, Q^2)$ where the Bjorken variable $x_B$ is approximately equal to the momentum fraction $\xi$ up to corrections suppressed by the ratio of the proton target mass to $Q^2$: $x_B \approx \xi\left(1 - \frac{\xi^2 M_p^2}{Q^2}\right)$ [41]. Here we consider the isospin averaged valence-quark distribution, which corresponds to $F_3$ averaged over neutrino and anti-neutrino DIS on isoscalar targets. Finally, the relation between the valence-quark distribution function and the valence-quark probability density is $q^V(\xi, Q^2) = \nu(Q^2)V(\xi)$ where $\nu(Q^2)$ is a normalization constant, the perturbatively renormalized number of valence-quarks, given by the GLS sum rule [46]. It is determined to high order since it is obtained by integrating the DGLAP equation from

$Q^2 = \infty$ down to $Q_0^2$

$$\int_0^1 q^V(\xi,t)d\xi = \nu(t) = 3\Big[1 - \frac{\alpha_s(t)}{\pi} + \frac{\alpha_s(t)}{\pi}^2(-\frac{55}{12} + \frac{N_f}{3}) + \cdots\Big] \tag{3.135}$$

where $t = \log\left(\frac{Q^2}{\Lambda^2_{QCD}}\right)$, the number of flavors is $N_f = 2$ (up and down) and $\alpha_s(t)$ is the strong coupling constant.

Within the rank-one approximation, the valence-quark wave function is determined in the large-$N$ limit, by minimizing the baryon mass$^2$ (3.28) holding $||\psi||^2 = 1$ subject to the modified momentum sum rule

$$N\int_0^P p|\tilde{\psi}(p)|^2[dp] = fP \tag{3.136}$$

where $f$ is the fraction of baryon momentum carried by valence-quarks. Our variational solution is $\tilde{\psi}(p) = C(\frac{p}{P})^a(1-\frac{p}{P})^b$ where in the chiral limit, $a = 0$ and $b = \frac{N}{2f} - 1$. We work in the chiral limit of massless current quarks since the up and down quarks are essentially massless ($5 - 7$ MeV) compared to the scale of strong interactions ($\Lambda_{QCD} = 200$ MeV). The valence-quark probability density is then $V(\xi) = (\frac{N}{f} - 1)(1-\xi)^{\frac{N}{f}-2}$ where we set the number of colors to its physical value $N = 3$. According to our ansatz, the valence distribution at $Q_0$ is $q^V(\xi, Q_0^2) = \nu(Q_0^2)V(\xi)$.

The two parameters of our model, $Q_0^2$ and $f$ are to be determined by fitting to data. We expect $Q_0$ to be low, since we predict the anti-quark distribution to be small at $Q_0$. And we expect $f$ to be roughly a half since gluons and valence-quarks share baryon momentum roughly equally at low $Q_0$. To compare with experimental data we evolve the valence-quark distribution from $t_0$ to $t > t_0$ according to the leading order DGLAP [45] equation

$$\frac{dq^V(\xi,t)}{dt} = \frac{\alpha_s(t)}{2\pi}C_2\Big[\int_\xi^1 \frac{q^V(\frac{\xi}{\zeta},t)(1+\zeta^2) - 2\zeta q^V(\xi,t)}{\zeta(1-\zeta)}d\zeta + \frac{3}{2}q^V(\xi,t)\Big] \tag{3.137}$$

We use the leading order DGLAP equation since we are evolving only over a small range of $Q_0^2 \le Q^2 \le 13GeV^2$ Here $C_2 = \frac{N^2-1}{2N}$. The evolution is done numerically by discretizing $\xi$ and $t$ with increments of $\delta\xi = .01$ and $\delta t = .025$ respectively. Since the DGLAP equation is linear, we may normalize $q^V(\xi,t)$ to $\nu(t)$ at any value of $t$. But in practice we normalize it at $t_0$ to reduce discretization errors in the region $0 \le \xi \le \delta\xi$

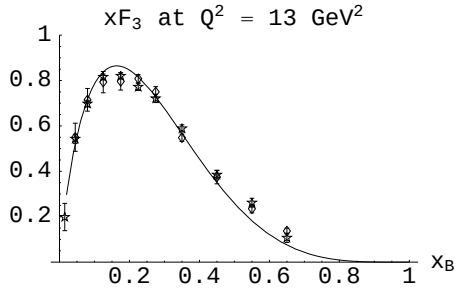


Figure 3.11: Comparison of predicted $xF_3$ at $Q^2 = 13$ GeV$^2$ (solid curve) with measurements by CCFR (star) at 12.6 GeV$^2$ and CDHS (diamond) at $12.05 \leq Q^2 \leq 14.3$ GeV$^2$. $Q_0^2 = 0.4$ GeV$^2$ and $f = \frac{1}{2}$.

which contributes significantly to $\nu(t)$ since $q^V(\xi, t)$ diverges logarithmically for small $\xi$ for $t > t_0$. The loss of knowledge of $q^V(\xi, t)$ for $0 \leq \xi \leq \delta\xi$ for $t > t_0$ does not affect the evolution for $\xi > \delta\xi$ because information flows from higher to lower values of $\xi$ as is evident from the evolution equation.

$F_3$ has been experimentally measured by the CCFR and CDHS collaborations [16] at a variety of values of $Q^2$ and $x_B$. The parameters $f, Q_0^2$ should be determined by a best fit to experimental data. For now, we assign to them the values $f = \frac{1}{2}$ and $Q_0^2 = 0.4$ GeV$^2$ suggested by phenomenological fits to data [21, 76]. We may now compare our prediction with experimental measurements of $F_3$ at higher values of $Q^2$. In Fig.3.11, we show a comparison with $xF_3$ measurements by the CDHS and CCFR collaborations [16] at $Q^2 \sim 13$ GeV$^2$. We have ignored the difference between the Bjorken variable $x_B$ and the momentum fraction $\xi$ which is small at $Q^2 \sim 13$ GeV$^2$. The plot shows that our model gives a zeroth order explanation of the data.

Note: Too much should not be read into the values $f = \frac{1}{2}$ and $Q_0^2 = 0.4$ GeV$^2$ we have picked for these parameters. In a more detailed comparison, and to further constrain the fit, one should substitute $x_B F_3$ which has not been measured for $x_B > 0.7$ with the structure function $F_2$. The two are approximately equal for large momentum fractions, where the contributions of anti-quarks and gluons are small. It appears that our choice for $f$ and $Q_0$ produces a prediction that underestimates $F_3$ for large momentum fractions. Increasing $f$ makes the valence-quark distribution vanish slower as $\xi \to 1$, since the

exponent is $(1-\xi)^{\frac{N}{f}-2}$ at $Q^2 = Q_0^2$. Merely changing $f$ from 50% to 60% would change the exponent from 4 to 3 at $Q^2 = Q_0^2$. Increasing $Q_0$ also has a similar effect: smaller the range of $Q^2$ over which the distributions are evolved, smaller is the momentum that flows from large to small values of $\xi$. We do not discuss this any further. Better still, it should be possible to eliminate these fits to data and predict the values of both $Q_0$ and $f$. This requires a much deeper understanding of renormalization and gluon dynamics respectively. We now turn to the problem of understanding matrix-valued dynamical gluons, which we have ignored so far. We do this within the regularized context of matrix models.

# Part II

# Large-$N$ Matrix Models as Classical systems

## Large-$N$ Matrix Models as Classical systems

We now turn to $N \times N$ hermitian multi-matrix models, which are to gluons what $N$-component vector models are to quarks. Matrix field theories are much harder to deal with since there are of order $N^2$ rather than $N$ degrees of freedom at each point in space-time. Non-perturbative techniques and the necessary mathematics to understand them are only just being developed. Understanding even a finite number of matrix degrees of freedom is a challenge, and there are many conceptually new phenomena to contend with.

We will follow a strategy similar to that in Part I, and study the large-$N$ limit, which turns out to be a classical limit. The analogs of the color-singlet meson variable $M \sim \chi^{\dagger i}(x)\chi_i(y)$, which are unitary invariants of a collection of vectors, are the gluon Green's functions, the traces of products of matrices: cyclically symmetric tensors $G_I = G_{i_1 i_2 \cdots i_n}$. We find that the classical configuration space is the space of non-commutative probability distributions, whose coordinates are the gluon correlations. Non-commutative probability theory (introduced in §4) is the probability theory of operator-valued random variables, and is an outgrowth of the study of von Neumann algebras, especially due to D. Voiculescu [99]. The entropy from confinement of color degrees of freedom manifests itself in the large volume of matrices that correspond to the same set of correlations, a non-commutative entropy.

In the euclidean formalism, matrix models are defined by an action $S$, which is usually the trace of a polynomial in the matrices. Classical equations of motion for the gluon correlation tensors are the factorized Schwinger-Dyson equations, also known as the loop equations. The solution of these classical equations is a point on the configuration space, a particular non-commutative probability distribution. We discover a classical action principle for these loop equations: maximize non-commutative entropy holding the correlations conjugate to the coupling tensors fixed. The main subtlety is that there is a cohomological obstruction to expressing the entropy in terms of the gluon correlations. To circumvent this, we express the configuration space as a coset space, taking inspiration from the Wess-Zumino-Witten model [103]. The automorphism group of the tensor algebra acts as changes of variable on the space of probability distributions.

Indeed, the latter is a coset space of the former, by the isotropy subgroup fixing a given reference distribution. We express the entropy as a function on the automorphism-group invariant under the action of the isotropy subgroup. This allows us to get an explicit formula for the large-$N$ classical action of euclidean matrix models. In other words, we express the correlations of a given matrix model in terms of those of a reference model and the change of variables that relates the two. This also leads to a variational approach to estimating the large-$N$ correlations and partition function. We parameterize the change of variable in terms of variational parameters, and use the extremization of the classical action to find their optimal values. In other words, given a submanifold of the configuration space parameterized by variational parameters, our action principle finds the point on this submanifold that best approximates the true correlations of a given matrix model. The variational estimate for the logarithm of the partition function is then the maximum value of entropy.

Part II of this thesis, is based on our paper [5]. The reader is also referred to the related paper [2] and [1], where we study the large-$N$ limit of hamiltonian matrix models as classical systems. Chapter 4 deals with some mathematical background on tensor algebras (free algebras) and non-commutative probability. Chapter 5 deals with euclidean large-$N$ matrix models as classical systems. Appendix B discusses the case of a single random matrix (one-matrix model). Appendix C summarizes some facts about the simplest of matrix models, the gaussian.

# Chapter 4

# Free Algebras and Free Probability Theory

In this chapter we will cover some mathematical background on tensor algebras, operator-valued random variables and the non-commutative version of probability theory that is relevant to their study. We will follow a point of view that is a synthesis of our own work, previous work in the physics literature [77, 18, 26, 27, 88, 43, 74, 54, 90, 105, 72, 30] and that of Voiculescu [99] in the mathematics literature.

A pair of real-valued random variables are statistically independent if their joint probability distribution factorizes as a product of individual probability distributions. Quantum mechanics can be considered as the ordinary (commutative) probability theory of real-valued random variables such as energy, momentum, position and spin; their expectation values being given by averaging with respect to a weight specified by the wave function. If a pair of quantum systems are independent, then the total wave function is just the tensor product of the individual wave functions. Thus, the notion of statistical independence of classical probability theory is based on the tensor product. When the tensor product is replaced by the free product, the notion of statistical independence is replaced by the mathematical notion of freeness. (Note: freeness must not be confused with free field theory in physics!) The resulting probability theory is called free probability theory (developed by Voiculescu [99]). Of course, not all random variables

in free probability theory need be free of each other, just as not all real-valued random variables are statistically independent. Real-valued random variables commute and are elements of commutative algebras such as the algebra of measurable functions on $\mathrm{R}^M$. The random variables in free probability theory are no longer real-valued, they do not commute in general and are elements of non-abelian algebras of "measurable operators", also known as $W^*$ algebras or von Neumann algebras. Thus the study of von Neumann algebras is just a non-abelian extension of measure theory. Free probability theory is also referred to as non-commutative probability theory. Somewhat counter intuitively, the simplest examples of non-commutative probability theory occur when the algebra of random variables is maximally non-abelian, i.e. the basic operators (generators) satisfy no relations. Then the operator algebra is just a free product of the operator algebra for each of the generators. Examples are (1) the free algebra on hermitian generators (full tensor algebra) which is associated to hermitian large-$N$ matrix models; (2) the group algebra of the free group (the algebra of convolution operators on the free group) which has to do with large-$N$ unitary matrix models; and (3) the Cuntz and Toeplitz algebras of creation-annihilation operators satisfying the Cuntz relations having to do with quantum matrix models in the canonical formalism [72] or the master field formalism for matrix models [43, 30, 99].

The three examples are related. For example, the information in the connected moments (cumulants) of the tensors in the free algebra can also be packaged as the vacuum expectation values of so-called master fields which are thought of as elements of the Cuntz algebra. Similarly, the probability theory on (1) and (2) are related by a kind of exponentiation. We will follow an approach based on example (1), since it is most closely tied to the hermitian matrix models studied in this thesis.

## 4.1 Tensor Algebras and Operator-valued Random Variables

In the usual commutative probability theory of $M$ real-valued random variables $x_1 \cdots x_M$ we are concerned with determining the expectation values of observables which are func-

tions of the random variables: $\langle f(x_1, \cdots, x_M)\rangle$. These expectation values are determined by averaging with respect to a joint probability distribution function (pdf or measure) $\rho(x_1, \cdots, x_M)$ on $\mathrm{R}^M$. The expectation values of functions that can be written as power series in the $x_i$ can be built up from the expectation values of monomials in the random variables: the moments $G_{i_1 \cdots i_n} = \langle x_{i_1} \cdots x_{i_n}\rangle$ which are symmetric tensors since real-valued random variables $x_i$ commute. Thus the algebra of observables of commutative probability theory can be taken to be the commutative algebra of polynomials (more generally formal power series) in $M$ indeterminates. In solving the classical moment problem, nineteenth century mathematicians showed that it is possible to reconstruct the joint pdf from a knowledge of the moments, under mild hypotheses [7]. Indeed, one can formulate almost the entire theory in terms of moments. (But moments do not tell the whole story. They are not adequate to compute expectation values of functions non-analytic in the random variables, for instance. In fact, one of the most important quantities in probability theory, the Boltzmann-Shannon entropy of a probability distribution $-\int \rho(\mathbf{x}) \log \rho(\mathbf{x}) d\mathbf{x}$ cannot be expressed in terms of the moments alone.) For instance, the condition on the positivity of a probability measure $\rho(\mathbf{x}) \geq 0$ translates into the moment inequalities: for $M = 1$, they are the conditions that the expectation value of the square of any polynomial $f(x) = \sum_n f_n x^n$ must be positive, $\sum_{m,n} f_m f_n G_{m+n} \geq 0$. This is the same as the positivity of the Hankel matrix $G_{m,n} := G_{m+n}$. Similarly, the normalization constraint $\langle 1\rangle = 1$ becomes $G_\emptyset = 1$ where $\emptyset$ denotes the absence of any indices.

Now suppose $\xi_1, \cdots, \xi_M$ are $M$ operator-valued random variables. We take them to be hermitian. We can think of the $\xi_i$ as hermitian matrices. What is the analogue of a joint probability distribution? If there was just a single random hermitian operator, the distribution of its eigenvalues may be regarded as the probability distribution. But when we have two or more operators which do not commute, they cannot be simultaneously diagonalized. In this situation, the moments come to the rescue and serve to define the joint probability distribution.

First, what is the algebra of observables? We shall consider random variables which are polynomials (or more generally formal power series) in the basic ones (the generators)

$\xi_i$ :

$$f(\xi) = f^{i_1 \cdots i_n} \xi_{i_1} \cdots \xi_{i_n} = f^I \xi_I \tag{4.1}$$

where $\xi_I := \xi_{i_1 \cdots i_n} := \xi_{i_1} \cdots \xi_{i_n}$ and repeated indices are summed. Thus random variables are determined by their complex-valued coefficient tensors $f^I$. Just like functions, these formal power series may be multiplied by complex scalars, added and multiplied "pointwise"

$$[\alpha f]^I = \alpha f^I; \quad [f+g]^I = f^I + g^I, \quad [fg]^I = \delta^I_{I_1 I_2} f^{I_1} g^{I_2} \tag{4.2}$$

where $\delta^I_{I_1 I_2} = 1$ if $I_1 I_2 = I$ and vanishes otherwise. The multiplication is the associative direct product of tensors. If we think of the string $I$ as a path in the space of indices, then this is the product induced by concatenation of paths. The simplest situation is the one in which the $\xi_i$ satisfy no relations, this can be modelled by $N \times N$ hermitian matrices in the limit of large-$N$, where the constraints imposed by the vanishing of characteristic polynomials may be ignored. In this case, the algebra of observables is the full tensor algebra or the free algebra $\mathcal{T}_M$ on $M$ generators. Note that the tensors are not required to satisfy any symmetry property. $\mathcal{T}_M$ is the free product of $M$ copies of $\mathcal{T}_1$

$$\mathcal{T}_M = \mathcal{T}_1 * \cdots * \mathcal{T}_1 \tag{4.3}$$

The free product of a pair of algebras $\mathcal{A}, \mathcal{B}$ is the linear span of all products of the form $a_1 b_1 a_2 b_2 \cdots a_n b_n$ where $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ are arbitrary. The multiplication is the concatenation product given above. Other associative algebras on hermitian generators can be obtained from this free algebra by quotienting by the ideal generated by the relations satisfied by the generators.

## 4.2 Non-Commutative Probability Distributions

Now, a non-commutative probability distribution is a rule to calculate the expectation values of the random variables in the algebra of observables. They are determined by the moment tensors

$$G_{i_1 \cdots i_n} = \langle \xi_{i_1} \cdots \xi_{i_n} \rangle \tag{4.4}$$

By analogy with commutative probability theory, we define the non-commutative joint probability distribution to be a sequence of moment tensors $G_\emptyset, G_{i_1}, G_{i_1 i_2}, \cdots$ satisfying the normalization, hermiticity and positivity conditions

$$G_\emptyset = 1; \quad G_I^* = G_{\bar{I}}; \quad G_{IJ} f^{\bar{I}*} f^J \geq 0 \tag{4.5}$$

for all polynomials $f(\xi) = f^I \xi_I$. The positivity condition can be regarded as the positivity of the Hankel matrix $G_{I;J} := G_{IJ}$. If $I = i_1 \cdots i_n$, $\bar{I} = i_n i_{n-1} \cdots i_1$. The expectation value of an arbitrary polynomial is then

$$\langle f(\xi) \rangle = f^I G_I \tag{4.6}$$

In the case of relevance to matrix models (pure gluondynamics or closed string theory), the moments are cyclically symmetric tensors since they are modelled by the expectation values of traces of products of matrices. The space of such moment tensors is the space of non-commutative probability distributions in $M$ generators, $\mathcal{P}_M$. More generally, if we were to include the quark degrees of freedom along with the gluons, we would have to deal with open strings of gluons with quarks at the ends (1.4). The corresponding moments would not be cyclically symmetric.

Hermitian large-$N$ matrix models provide examples of non-commutative probability distributions. The moments are given by integrals over large hermitian matrices (see §5.1)

$$G_I = \lim_{N\to\infty} \frac{1}{Z} \int dA e^{-N \text{ tr } S(A)} \frac{\text{tr}}{N} A_I \tag{4.7}$$

More generally, Voiculescu defines a non-commutative probability space as an associative algebra $\mathcal{A}$ over the complex numbers with unit element 1 and linear functional (or "state" in quantum statistical mechanics [48]) $\tau$ such that $\tau(1) = 1$. The elements of the algebra are the random variables. He adds probabilistic content to this algebraic definition by requiring the algebra to be a weakly closed operator algebra with the adjoint operation $a \mapsto a^*, (a^*)^* = a$ ($W^*$ algebra or von Neumann algebra). The state must be real $\tau(a^*) = \tau^*(a)$ and positive elements must have positive expectation values $\tau(a^* a) \geq 0$. The case of cyclically symmetric moment tensors above corresponds to a

"tracial" state $\tau(ab) = \tau(ba)$. Other examples of operator algebras that admit a normalizable tracial state $\tau$ include Type I algebras: finite dimensional matrix algebras and finite type II factor algebras such as the von Neumann algebra of operators in the left regular representation of a group with infinite conjugacy classes (e.g. group algebras of free groups).

The precise definition of freeness can be given within the algebraic context. A family of subalgebras $\mathcal{A}_i$ of a non-commutative probability space are a free family if

$$\tau(a_1 \cdots a_n) = 0 \tag{4.8}$$

whenever $a_j \in \mathcal{A}_{i_j}$ with $i_j \neq i_{j+1}$ for $1 \leq j < n$ and $\tau(a_j) = 0$ for $1 \leq j \leq n$. In other words, a family of subalgebras are free of each other if the expectation value of a product of random variables vanishes if they each have zero mean and adjacent random variables are from distinct subalgebras. Random variables are free of each other if they are from subalgebras that are free of each other.

For example, consider a matrix model with action $S(A_1, A_2) = S_1(A_1)+S_2(A_2)$. Then random variables $A_1$ and $A_2$ are free of each other in the large-$N$ limit. In commutative probability theory, the joint pdfs of a pair of statistically independent random variables is determined by their individual pdfs. Similarly, the freeness property can be used to find all the mixed moments such as $G_{1122}$ once the pure moments $G_{11}, G_{22}$ etc are known:

$$\begin{aligned} 0 &= \langle(\xi_1\xi_1 - \langle\xi_1\xi_1\rangle)(\xi_2\xi_2 - \langle\xi_2\xi_2\rangle)\rangle \\ \Rightarrow \quad & G_{1122} - G_{11}G_{22} - G_{11}G_{22} + G_{11}G_{22} = 0 \\ \Rightarrow \quad & G_{1122} = G_{11}G_{22} \end{aligned} \tag{4.9}$$

The most famous of non-commutative probability distributions is the Wigner distribution. We discuss it in Appendix C.

## 4.3 Automorphisms and Derivations of Tensor Algebras

We will be interested in how one non-commutative probability distribution can be transformed into another. We can use the group of effective transformations as a way to

parameterize the space of probability distributions. This group plays a key role in the theory to be developed in §5.

Let us consider transformations $\phi$ of the algebra of observables that preserve its structure $\phi(fg) = \phi(f)\phi(g)$. These are the automorphisms of the tensor algebra $\mathcal{T}_M$

$$\xi_i \mapsto \phi(\xi)_i = \phi_i^I \xi_I \tag{4.10}$$

provided they are invertible. The composition of a pair of automorphisms is

$$[(\psi \circ \phi)]_i^J = \delta^J_{K_1 \cdots K_n} \psi_i^{k_1 \cdots k_n} \phi_{k_1}^{K_1} \cdots \phi_{k_n}^{K_n} \tag{4.11}$$

The inverse, $(\phi^{-1})_i(\xi) \equiv \psi_i(\xi)$, is determined by the conditions

$$[(\psi \circ \phi)_i]^K = \delta^K_{P_1 \cdots P_n} \psi_i^{j_1 \cdots j_n} \phi_{j_1}^{P_1} \cdots \phi_{j_n}^{P_n} = \delta_i^K. \tag{4.12}$$

They can be solved recursively for $\psi_J^i$:

$$\begin{aligned} \psi_j^i &= (\phi^{-1})_j^i \\ \psi_{j_1 j_2}^i &= -\psi_{j_1}^{k_1} \psi_{j_2}^{k_2} \psi_{l_1}^i \phi_{k_1 k_2}^{l_1} \\ &\cdots \\ \psi_{j_1 \cdots j_n}^i &= -\sum_{m<n} \delta^{P_1 \cdots P_m}_{k_1 \cdots k_n} \psi_{j_1}^{k_1} \cdots \psi_{j_n}^{k_n} \psi^i_{l_1 \cdots l_m} \phi_{P_1}^{l_1} \cdots \phi_{P_m}^{l_m}. \end{aligned} \tag{4.13}$$

A necessary condition for an automorphism to be invertible is that its linear part $\phi_j^i$ be invertible: $\det \phi_j^i \neq 0$. Even if the linear part is invertible, the above recursive procedure may not converge in a finite number of steps. Indeed, we would not be able to find inverses of polynomials within the world of polynomials. This forces us to work with the "completed" tensor algebra of formal power series in the generators $\xi_i$, rather than just non-commutative polynomials. This is why we allowed our random variables to be formal power series in the basic ones. (Note that the composition rule given above makes sense for two formal power series since the operations involve only finite sums.) However, we pay a price for using formal power series instead of polynomials for our algebra of observables. It may not be possible to evaluate the expectation value of an arbitrary formal power series in the generators. However, it turns out that in the large-$N$ limit, many of the interesting probability distributions are in a sense compactly supported so

that moments do not grow too fast (see for example the Wigner semi-circular distribution given in Appendix C; an even stronger restriction on the growth of moments is the Wilson area law conjecture (Section §1.2)) and it is possible to evaluate the expectation value of many interesting observables.

We will further restrict to the "orientation preserving" automorphisms, $\det \phi^i_j > 0$. Thus, the automorphisms of the tensor algebra form a group $\mathcal{G}_M = \text{Aut } \mathcal{T}_M$. For $M = 1$ it is the diffeomorphism group of the real line, where the real line is thought of in terms of the commutative algebra of formal power series $\mathcal{T}_1$ on it.

$$\mathcal{G}_1 = \left\{ \phi(x) = \sum_{n=1}^{\infty} \phi_n x^n \middle| \phi_1 > 0 \right\} \tag{4.14}$$

with the composition law

$$[\psi \circ \phi]_n = \sum_{k=1}^{n} \psi_k \sum_{l_1+\cdots+l_k=n} \phi_{l_1} \cdots \phi_{l_k} \tag{4.15}$$

and inverses given by Lagrange's recursion relations

$$\phi^{-1} = \psi; \quad \psi_1 \phi_1 = 1; \quad \psi_n = -\frac{1}{\psi_1^n} \sum_{k=1}^{n-1} \psi_k \sum_{l_1+\cdots+l_k=n} \phi_{l_1} \cdots \phi_{l_k}. \tag{4.16}$$

For $M > 1$ the free algebra $\mathcal{T}_M$ is non-abelian and can at best be considered as the algebra of functions on a non-commutative manifold. Then $\mathcal{G}_M$ is a non-commutative analogue of the diffeomorphism group.

Finally, the action of the automorphism group on probability distributions is given by the transformation rule for moments. If $\Gamma_I = \langle \xi_I \rangle$ and $\xi_i \mapsto \phi(\xi)_i$, then

$$G_I = \langle \phi(\xi)_{i_1} \cdots \phi(\xi)_{i_n} \rangle = [\phi_* \Gamma]_{i_1 \cdots i_n} = \phi^{J_1}_{i_1} \cdots \phi^{J_n}_{i_n} \Gamma_{J_1 \cdots J_n} \tag{4.17}$$

However, only a subgroup of the automorphism group preserves the positivity of probability distributions

$$\tilde{\mathcal{G}}_M = \{ \phi \in \mathcal{G}_M | [\phi_* \Gamma]_{IJ} f^{\bar{I}*} f^J \geq 0 \ \forall \text{ polynomials } f(\xi) \} \tag{4.18}$$

Let us now characterize the infinitesimal version of $\mathcal{G}_M$, the Lie algebra $\underline{\mathcal{G}_M}$.

An automorphism $\phi(A)_i = A_i + v^I_i A_I$ that deviates infinitesimally from the identity is a vector field, a derivation of the tensor algebra. Such a derivation can be expressed

as a linear combination of the basis vector fields $L_v = v_i^I L_I^i$ where the action of the basis vector fields is

$$[L_I^i \xi]_j = \delta_j^i \xi_I \tag{4.19}$$

and by the chain rule,

$$L_I^i \xi_J = \delta_J^{J_1 i J_2} \xi_{J_1 I J_2} \tag{4.20}$$

Taking expectation values, we get the action on moments

$$\begin{aligned} [L_I^i G]_J &= \delta_J^{J_1 i J_2} \xi_{J_1 I J_2} \\ L_I^i &= G_{J_1 I J_2} \frac{\partial}{\partial G_{J_1 i J_2}} \end{aligned} \tag{4.21}$$

The Lie algebra $\underline{\mathcal{G}_M}$ of these vector fields can be worked out

$$[L_I^i, L_J^j] = \delta_J^{J_1 i J_2} L_{J_1 I J_2}^j - \delta_I^{I_1 j I_2} L_{I_1 J I_2}^i \tag{4.22}$$

For $M = 1$ it reduces to the Lie algebra of polynomial vector fields on the real line. Consider the infinitesimal changes of variable $\phi(x) = x + \epsilon x^{k+1},\ k = 0, 1, 2 \ldots$. (translations, corresponding to $k = -1$ are not allowed since they are not invertible.) Thus a basis for the Lie algebra $\underline{\mathcal{G}_1}$ is

$$L_k = x^{k+1} \frac{\partial}{\partial x} \equiv x^k D,\ k = 1, 2 \ldots \tag{4.23}$$

and they satisfy the commutation relations of the Virasoro algebra $[L_p, L_q] = (q-p)L_{p+q}$ (see also Appendix B.1). So far, we have been thinking of the Lie algebra as acting on the tensor algebra. We can also think of the Lie algebra as acting on the group itself. By taking either $\psi$ or $\phi$ to be infinitesimal ($x + \epsilon x^{k+1}$) in the composition law $\psi \circ \phi$ for a pair of group elements, we get the left and right actions of the Lie algebra on the group $\mathcal{G}_1$. The left action is $(\mathcal{L}^k \phi)(x) = \phi^k(x)$:

$$\mathcal{L}^k \phi_n = \sum_{l_1 + \cdots + l_k = n} \phi_{l_1} \cdots \phi_{l_k}. \tag{4.24}$$

For a physical application of this see (B.11). The right action is: $\mathcal{R}^k \phi(x) = x^k D\phi(x)$. They both satisfy the same commutation relations as the $L_k$.

The reason we studied how probability distributions transform under a change of variable is that it gives us a novel way of understanding the entropy of non-commutative

probability theory. It is also essential to understand the classical action principle of large-$N$ matrix models (see §5.2).

## 4.4 Free Entropy of Non-Commutative Probability Theory

In §5.3 entropy of operator-valued random variables is discussed from the point of view of a variational principle for matrix models and the cohomology of the automorphism group of the free algebra. Here, we present Voiculescu's precise definition [99] in terms of matricial microstates. The free entropy $\chi$ of a collection of self adjoint random variables $X_1, \cdots, X_n$ in a $W^*$ probability space $(\mathcal{A}, \tau)$ is the logarithm of volume of a set of hermitian $N \times N$ matrices $A_i$, the matricial microstates. The set of matricial approximants are restricted by requiring all their $N \to \infty$ moments to equal those of the random variables $X_i$. The volume measure on the space of hermitian matrices is the Lebesgue measure $\lambda$ corresponding to the Hilbert-Schmidt (HS) norm $\sqrt{\mathrm{tr}\,(A_1^2 + \cdots + A_n^2)}$. Let $\Gamma_R(X_1, \cdots, x_n; m, N, \epsilon)$ be the set of $n$-tuples $(A_1, \cdots, A_n)$ with

$$\begin{aligned} ||A_j||_{HS} &\leq R \\ |\frac{\mathrm{tr}}{N} A_I - \tau(X_I)| &< \epsilon \quad \forall\ |I| \leq m \end{aligned} \tag{4.25}$$

Successively, the regulators are removed $N \to \infty, m \to \infty, \epsilon \to 0, R \to \infty$

$$\begin{aligned} \chi_R(X_1, \cdots, X_n; m, N, \epsilon) &= \log \lambda(\Gamma_R(X_1, \cdots, x_n; m, N, \epsilon)) \\ \chi_R(X_1, \cdots, X_n; m, \epsilon) &= \limsup_{N \to \infty} (\frac{1}{N^2} \chi_R(X_1, \cdots, X_n; m, N, \epsilon) + \frac{n}{2} \log N) \\ \chi_R(X_1, \cdots, X_n) &= \inf\{\chi_R(X_1, \cdots, X_n; m, \epsilon) | m \in \mathcal{N}, \epsilon > 0\} \\ \chi(X_1, \cdots, X_n) &= \sup_{R>0} \chi_R(X_1, \cdots, X_n) \end{aligned} \tag{4.26}$$

The cut off $R$ plays a minor role, $\chi_R$ is independent of $R$ once $R$ exceeds the HS norms of all the $X_i$. Some properties of free entropy $\chi$ that Voiculescu establishes are listed below. See [99] for additional results.

1. The entropy of a single semi-circular variable $S$ with $\tau(S^2) = 1$ is $\chi(S) = \frac{1}{2} \log 2\pi$.

2. $\chi(X_1, \cdots, X_n) \leq \frac{n}{2} \log (\frac{2\pi C^2}{n})$ where $C^2 = \tau(\sum_{i=1}^n X_i^2)$. The RHS is the entropy of the standard Wigner distribution.

3. **n = 1:** If $X$ has distribution $\mu$, then $\chi(X)$ is the logarithmic energy

$$\chi(X) = \int \log|s-t| d\mu(s) d\mu(t) + \frac{3}{4} + \frac{1}{2}\log 2\pi \tag{4.27}$$

4. **Semi-circular maximum:** If $\tau(X_i^2) = 1 \; \forall \; i$ then $\chi(X_1, \cdots, X_n)$ is maximal iff the $X_i$ are semi-circular and freely independent. Thus the semi-circular distribution plays the role of the gaussian of ordinary probability theory.

5. **Additivity under Freeness:** If $X_i$ are pairwise freely independent,

$$\chi(X_1, \cdots, X_n) = \sum_{i=1}^{n} \chi(X_i) \tag{4.28}$$

   This is the analog of the additivity of entropy of statistically independent random variables in usual probability theory. Statistical independence is replaced by freeness here. This explains the choice of the logarithm of the volume rather than any other function of volume in defining entropy.

# Chapter 5

# Euclidean Large-$N$ Matrix Models

We will consider matrix models, whose random variables are a collection of $M$ $N \times N$ hermitian matrices, $A_i,\ i = 1, \ldots, M$. (Appendix B contains a rapid summary of some facts about the $M = 1$ matrix model.) We can think of them as the gluon field at a finite number of points in space-time labelled by $i$. The $i$ could also label links of a lattice as in lattice gauge theory, in which case it is more convenient to consider unitary matrices. The matrix elements are distributed according to

$$e^{-N \text{ tr } S(A)} \ \frac{dA}{Z(S)} \tag{5.1}$$

$dA$ stands for the usual Lebesgue measure on the matrix elements of the $M$ hermitian matrices. The theory is defined by the action $S(A)$, which we take to be a polynomial

$$S(A) = \sum_n S^{i_1 \cdots i_n} A_{i_1} \cdots A_{i_n} \equiv S^I A_I \tag{5.2}$$

with cyclically symmetric tensors $S^I$, the coupling constants. (Note: We will use capital letters to denote strings of indices $A_{i_1} \cdots A_{i_n} \equiv A_{i_1 \cdots i_n} \equiv A_I$ and sum repeated indices.) The partition function is

$$Z(S) = \int e^{-N \text{ tr } S(A)} \ dA \tag{5.3}$$

and the free energy is defined as

$$F(S) = -\frac{1}{N^2} \log Z(S) \tag{5.4}$$

The expectation value of any function of the matrices is

$$\langle f(A)\rangle = \frac{1}{Z(S)}\int f(A)e^{-N \text{ tr } S(A)}dA \tag{5.5}$$

The observables we are interested in are those that are invariant under the simultaneous action of $U(N)$, $A_i \to UA_iU^\dagger,\ i = 1,\ldots,M$. We can think of these as the global gauge transformations that remain after local gauge fixing. The simplest examples of invariants are $\Phi_I = \frac{\text{tr}}{N}A_I$ known as loop variables. The factor of $\frac{1}{N}$ is inserted with hindsight to ensure a good large-$N$ limit. Also of interest, are the Wilson loop-like observables $\frac{\text{tr}}{N}e^{A_I}$, though these can be built out of the loop variables by expanding the exponential in a power series.

## 5.1 Loop Equations, Classical Configuration Space

The loop variables satisfy Schwinger-Dyson equations

$$S^{J_1iJ_2}\langle\Phi_{J_1IJ_2}\rangle = \delta_I^{I_1iI_2}\langle\Phi_{I_1}\Phi_{I_2}\rangle \tag{5.6}$$

which are the conditions for the invariance of the partition function under an infinitesimal but nonlinear change of integration variable $A_i \mapsto A_i + v_i^I A_I$. In other words, under the action of the vector field $L_v = v_i^I L_I^i$. The basis vector fields act as $L_I^i A_j = \delta_j^i A_I$ and by the Leibnitz rule, $L_I^i A_J = \delta_J^{J_1iJ_2}A_{J_1IJ_2}$. There are contributions from both the change in the action and the change in the measure.

$$\begin{aligned} Z(S) &= \int dA\left|\det\left(\frac{\partial(A_i + v_i^I A_I)_b^a}{\partial(A_j)_q^p}\right)\right|e^{-N \text{ tr } S(A)}\left[1 - Nv_i^I \text{ tr } L_I^i S(A)\right] \\ &= \int dA\left(1 + v_i^I\delta_I^{I_1iI_2} \text{ tr } A_{I_1} \text{ tr } A_{I_2}\right)\left(1 - Nv_i^I S^{J_1iJ_2} \text{ tr } A_{J_1IJ_2}\right)e^{-N \text{ tr } S(A)} \end{aligned} \tag{5.7}$$

where we have used $\det(1 + vT) \simeq 1 + v \text{ tr } T$. Comparing coefficients of $v_i^I$ and dividing by $Z(S)N^2$ gives the advertised Schwinger Dyson equations. The latter are not a closed system of equations for the expectation values of the loop variables. They are related to those of a product of two loop variables. Migdal and Makeenko [74] noticed that in the large-$N$ limit of 't Hooft [55] , holding the coupling constants $S^I$ fixed, the loop variables satisfy a closed system of equations, known as the loop equations or factorized

Schwinger-Dyson equations. This is because, in the large-$N$ limit, the loop variables do not have any fluctuations, and their expectation values factorize

$$\langle \frac{\mathrm{tr}}{N} A_{I_1} \frac{\mathrm{tr}}{N} A_{I_2} \rangle = \langle \frac{\mathrm{tr}}{N} A_{I_1} \rangle \langle \frac{\mathrm{tr}}{N} A_{I_2} \rangle + O(\frac{1}{N^2}) \tag{5.8}$$

This property can be proven perturbatively. Thus the large-$N$ limit is a classical limit and it suffices to restrict attention to the gluon correlations or moments

$$G_I = \lim_{N\to\infty} \langle \frac{1}{N} \mathrm{tr}\ A_I \rangle \tag{5.9}$$

which are cyclically symmetric tensors. In terms of them, the factorized Schwinger-Dyson equations are

$$S^{J_1 i J_2} G_{J_1 I J_2} = \eta^i_I \equiv \delta^{I_1 i I_2}_I G_{I_1} G_{I_2}. \tag{5.10}$$

What is the classical configuration space? It is the space of allowed moments. From the definition of $Z(S)$ we see that $G_\emptyset =< \frac{\mathrm{tr}}{N} 1 >= 1$. Hermiticity of $A_i$ implies $G^*_I = G_{\bar{I}}$, where $\bar{I} = i_n \cdots i_1$ if $I = i_1 \cdots i_n$. Moreover, if $f(A) = f^I A_I$ is a polynomial, then the expectation value of its absolute square should be positive, $\langle \frac{\mathrm{tr}}{N} f^\dagger(A) f(A) \rangle \geq 0$ provided the integrals converge. In other words, $G_{IJ} f^{\bar{I}*} f^J \geq 0$ which is the same as the positivity of the Hankel matrix $G_{I;J} \equiv G_{IJ}$. This imposes certain inequalities on the moments, which are known as the moment inequalities. We see that the configuration space of large-$N$ matrix models is the space of non-commutative probability distributions in $M$ generators $\mathcal{P}_M$, as defined in §4.2. The moments $G_I$ are coordinates on this configuration manifold. Thus the moments of large-$N$ matrix models provide an example of non-commutative probability distributions. If we associate with each matrix $A_i$ a generator of the free algebra $\xi_i$, then $G_I =< \xi_I >$.

In commutative probability theory of real-valued random variables, the moments $G_k = \int x^k \rho(x) dx$ determine the probability distribution $\rho(x)$ up to technicalities [7]. But there could still be observables whose expectation values are not easily expressed in terms of the moments; an example is entropy $\int \rho(x) \log \rho(x) dx$. In matrix models, the large-$N$ limit of the free energy $F = \lim_{N\to\infty} -\frac{1}{N^2} \log Z$ is of much physical interest. Though in principle it is determined by the moments, we will see that there is no simple

formula for it in terms of the moments. Indeed, this subtlety is quite central to our entire discussion and we will return to it.

The loop equations (5.10) are quadratically non-linear in the moments and in general couple all the moments to each other. The LHS, $S^{J_1 i J_2} G_{J_1 I J_2}$ comes from the expectation value of the variation of the action $S(A)$. The RHS $\eta_I^i$ is an anomaly in the sense that it comes from the change in integration measure [39]. It is a universal term since it is independent of the coupling constants $S^I$. We can regard these as the classical equations of motion. For a single matrix ($M = 1$) these equations are recursion relations and can be solved (see Appendix B). There are a few special choices of action, such as the gaussian, where $S$ is a quadratic polynomial (see Appendix C and Ref. [66, 78, 97, 31]) for which the partition function or some special classes of moments can be obtained exactly. But in general, even after the simplifications of the large-$N$ limit, the loop equations of a multi-matrix model have not been solved exactly. The situation is not out of the ordinary, since the generic classical system is not exactly solvable, it is understood by approximation methods such as perturbation theory, mean-field theory, variational principles etc. Perturbation theory around the gaussian leads to the expansion in terms of planar Feynman diagrams [55]. In the case of finite matrix models, the perturbative series is expected to have a finite radius of convergence in the coupling constant and one might hope for a purely perturbative construction of the theory [52]. But there are limitations to this approach due to the renormalon singularities upon passage to full-fledged field theories [15]. Moreover, as we saw in two-dimensional QCD, there are many interesting phenomena that are either inaccessible to perturbation theory or more easily understood by other methods. Understanding the large-$N$ limit of matrix models as bona-fide classical mechanical systems will permit us to adapt the methods of classical mechanics to this new class of dynamical systems.

## 5.2 Classical Action Principle

### 5.2.1 Anomaly as a Cohomology Class

One peculiar feature of our derivation of the classical equations of motion is that we got them sans any knowledge of the classical action! The original action $S(A)$ will not do since its variation will not give the anomalous term $\eta_I^i$ which is independent of coupling constants and came from a change in the measure. We look for a classical action $\Omega(G)$, the conditions for whose stationarity under the vector fields $L_I^i$ are the loop equations

$$L_I^i \Omega(G) = -S^{J_1 i J_2} G_{J_1 I J_2} + \delta_I^{I_1 i I_2} G_{I_1} G_{I_2} \tag{5.11}$$

The action of these vector fields on moments is given by $[L_I^i G]_J = \delta_J^{J_1 i J_2} G_{J_1 I J_2}$. The Lie algebra (which we denote $\underline{\mathcal{G}_M}$) of these vector fields is (see §4)

$$[L_I^i, L_J^j] = \delta_J^{J_1 i J_2} L_{J_1 I J_2}^j - \delta_I^{I_1 j I_2} L_{I_1 J I_2}^i \tag{5.12}$$

Equivalently, we can think of them as first order differential operators on the configuration space

$$L_I^i = G_{J_1 I J_2} \frac{\partial}{\partial G_{J_1 i J_2}} \tag{5.13}$$

The action dependent term is clearly the variation of ($\frac{1}{N}$ times) the expectation value of the original action $S(A)$

$$L_I^i (S^J G_J) = S^{J_1 i J_2} G_{J_1 I J_2}. \tag{5.14}$$

So let $\Omega(G) = -S^J G_J + \chi(G)$ where

$$L_I^i \chi(G) = \eta_I^i = \delta_I^{I_1 i I_2} G_{I_1} G_{I_2} \tag{5.15}$$

This is a system of inhomogeneous first order linear PDEs on the configuration space. Necessary integrability conditions (though not sufficient) for a solution $\chi(G)$ are:

$$L_I^i \eta_J^j - L_J^j \eta_I^i - \delta_J^{J_1 i J_2} \eta_{J_1 I J_2}^j + \delta_I^{I_1 j I_2} \eta_{I_1 J I_2}^i = 0. \tag{5.16}$$

These complicated looking conditions simply state that $\eta_I^i$ are the components of a closed one-form. They look complicated because the $L_I^i$ are a non-commuting basis for vector

fields unlike the coordinate basis $\frac{\partial}{\partial G_{iI}}$. Define the 1-form $\eta$ such that evaluated on the vector field $L^i_I$ it gives $\eta(L^i_I) = \eta^i_I$. Then the condition for $\eta$ to be closed ($d\eta = 0$) is

$$(d\eta)(L^i_I, L^j_J) = L^i_I \eta(L^j_J) - L^j_J \eta(L^i_I) - \eta([L^i_I, L^j_J]) = 0. \tag{5.17}$$

When written in components we get (5.16). We show in Appendix D that the integrability conditions are satisfied. So $\eta$ is indeed a closed 1-form!

In the same vein, (5.15) becomes $\eta \stackrel{?}{=} d\chi$. Is $\eta$ an exact one-form? The answer to a question on exactness depends on the class of functions on $\mathcal{P}_M$ from which $\chi(G)$ can be chosen. We could not find any formal power series in the moments whose exterior derivative is $\eta$. Formal power series are natural since the other term in the classical action ($S^I G_I$) is such a function. To see why $\chi(G)$ is not a power series in the moments, it suffices to consider the one-matrix ($M = 1$) case where $\chi = \int \log|x-y|\rho(x)\rho(y)dxdy$ cannot be expressed as a series in the moments $G_k = \int x^k \rho(x)\ dx$ on account of the non-analyticity of the logarithm at the origin (see Appendix B). Unfortunately, there is no straightforward generalization of this one-matrix formula for $\chi$, to $M > 1$. The notion of a probability density $\rho(x)$ makes sense as the density of eigenvalues of a single matrix. But when we have several matrices that do not commute, there is no meaning to their eigenvalues since there is no common basis in which they are all diagonal. The basis independent notion of a joint spectrum is then the collection of moments $G_I$.

We conclude that the anomaly $\eta$ is a closed but not exact one-form, in other words an element of the first cohomology of the Lie algebra of vector fields $\underline{\mathcal{G}_M}$ valued in the space of power series in the moments. But $\eta$ is just the infinitesimal version of $\chi(G)$ ($\eta = d\chi$). It is therefore natural to expect $\chi(G)$ to be an element of the first cohomology of the group $\mathcal{G}$ whose Lie algebra is $\underline{\mathcal{G}_M}$. Now $\underline{\mathcal{G}_M}$ is spanned by the vector fields $L^i_I$ which are just infinitesimal automorphisms of the tensor algebra on $M$ generators (or free algebra) $\mathcal{T}_M$ (see §4). Therefore $\mathcal{G}_M$ must be the group of automorphisms of the tensor algebra $\mathrm{Aut}\mathcal{T}_M$.

### 5.2.2 Configuration Space $\mathcal{P}$ as a Coset Space $\mathcal{G}/\mathcal{SG}$

More concretely, to find a formula for $\chi(G)$, we must find some other way of describing functions on the configuration space $\mathcal{P}_M$, since power series in $G_I$ were inadequate. To do this it helped us recall that $\eta$ was the change in the matrix integration measure under an infinitesimal change of variable. This suggests another way of parameterizing $\mathcal{P}_M$ – in terms of the finite (invertible) change of variables $\phi$ that maps the moments of a reference probability distribution $\Gamma_I$ to those of the probability distribution of interest $G_I = [\phi_*\Gamma]_I$. Moreover, $\chi(G)$ should be the change in measure under a finite change of variables.

But such a change of variables is nothing but an automorphism of the free algebra $\mathcal{T}_M$! We see that the configuration space $\mathcal{P}_M$ carries an action of $\mathcal{G}_M =$ Aut $\mathcal{T}_M$ and this action is transitive if we restrict to those distributions which can be obtained by such a change of variable starting from $\Gamma$. However, there may be more than one automorphism that maps $\Gamma$ to $G$. Suppose $\mathcal{SG}_M$ is the isotropy subgroup for this action, the subgroup of measure-preserving automorphisms which map $\Gamma$ to itself. Then the space of probability distributions $\mathcal{P}_M$ is the coset space $\mathcal{G}_M/\mathcal{SG}_M$.

Returning to $\chi(G)$, how does this coset space description of the configuration space get around the cohomological obstruction to expressing $\chi(G)$ as a power series in the moments? We have a new way of thinking of functions on $\mathcal{P}_M$: functions on $\mathcal{G}_M$ invariant under the action of $\mathcal{SG}_M$! In particular, power series in the $G_I$ can always be expressed in terms of $\phi_i^I$ by substituting $G_I = [\phi_*\Gamma]_I$. But there are power series in the $\phi_i^I$ (functions on $\mathcal{G}_M$) invariant under $\mathcal{SG}_M$ which cannot be expressed as power series in the $G_I$. Indeed, we find that $\chi(G)$ is one such function!

Let us explain our implementation of the rough ideas outlined in the last few paragraphs. Suppose $\phi$ is a transformation of the matrices

$$[A_i]_b^a \mapsto \phi_i(A) = \phi_i^{i_1\cdots i_n}[A_{i_1}]_{a_1}^a[A_{i_2}]_{a_2}^{a_1}\cdots[A_{i_n}]_b^{a_{n-1}} \tag{5.18}$$

which is the finite version of the infinitesimal transformation $A_i \mapsto A_i + v_i^I A_I$ associated with the vector fields $L_i^I$ (see §5.1). $\phi$ defines an automorphism of the free algebra

$$\xi_i \mapsto \phi_i^I \xi_I \tag{5.19}$$

provided its linear part $\phi_i^j$ is an invertible $M \times M$ matrix. Let $\Gamma_I$ be the moments for the standard action $S_0(A)$ and $G_I$ those for action $S(A)$. Suppose they are related by the change of variables $S_0(A) = S(\phi(A))$. Under such an automorphism, the moments and free energy $F(S) = -\lim_{N\to\infty} \frac{1}{N^2} \log Z(S)$ transform as

$$\begin{aligned} G_{i_1\cdots i_n} &= [\phi_*\Gamma]_{i_1\cdots i_n} = \phi_{i_1}^{J_1}\cdots\phi_{i_n}^{J_n}\Gamma_{J_1\cdots J_n} \\ F(S) &= F(S_0) + \lim_{N\to\infty} < -\frac{1}{N^2} \text{ tr } \log J >_{S_0} \end{aligned} \tag{5.20}$$

where $J = \frac{\partial\phi(A)}{\partial A}$ is the jacobian matrix for the transformation and $<>_{S_0}$ denotes expectation value with respect to the action $S_0$. To see this, consider

$$G_I = \lim_{N\to\infty} \left\{ \int dA e^{-N \text{ tr } S(A)} \frac{\text{tr}}{N} A_I \right\} / \left\{ \int dA e^{-N \text{ tr } S(A)} \right\} \tag{5.21}$$

Make the change of variables $A_i = \phi_i(\tilde{A})$

$$\begin{aligned} G_I &= \lim_{N\to\infty} \frac{\int d\tilde{A} \det(J) e^{-N \text{ tr } S_0(\tilde{A})} \frac{\text{tr}}{N} \phi_{i_1}(\tilde{A}) \cdots \phi_{i_n}(\tilde{A})}{\int d\tilde{A} \det(J) e^{-N \text{ tr } S_0(\tilde{A})}} \\ &= \lim_{N\to\infty} \frac{\int d\tilde{A} e^{-N \text{ tr } S_0(\tilde{A})} \frac{\text{tr}}{N} \phi_{i_1}(\tilde{A}) \cdots \phi_{i_n}(\tilde{A})}{\int d\tilde{A} e^{-N \text{ tr } S_0(\tilde{A})}} \\ &= \phi_{i_1}^{J_1}\cdots\phi_{i_n}^{J_n}\Gamma_{J_1}\cdots\Gamma_{J_n} \end{aligned} \tag{5.22}$$

where we have used the factorization of correlations in the large-$N$ limit to cancel the jacobian determinants. Similarly,

$$\begin{aligned} Z(S) = \int dA e^{-N \text{ tr } S(A)} &= \int d\tilde{A} \det(J) e^{-N \text{ tr } S_0(\tilde{A})} \\ \frac{1}{N^2} \log\{Z(S)/Z(S_0)\} &= \frac{1}{N^2} \log < e^{\text{ tr } \log(J)} >_{S_0} \\ F(S) - F(S_0) &= -\lim_{N\to\infty} < \frac{1}{N^2} \text{ tr } \log J >_{S_0} \end{aligned} \tag{5.23}$$

where in the last step we have used large-$N$ factorization to reverse the order of the logarithm and expectation value.

However, not all automorphisms preserve the positivity of the Hankel matrix $\Gamma_{I;J} = \Gamma_{IJ}$. The subset that do is $\tilde{\mathcal{G}}_M = \left\{ \phi \in \mathcal{G}_M \mid [\phi_*\Gamma]_{I_1 I_2} \text{ a positive matrix} \right\}$ We will largely ignore the somewhat technical difference between $\tilde{\mathcal{G}}$ and $\mathcal{G}$. (Does $\mathcal{G}_M$ have the same Lie algebra as $\tilde{\mathcal{G}}_M$?)

Now we will think of any probability distribution of interest, $G_I$ as being obtained from a fixed reference distribution $\Gamma_I$ by the action of a change of variable $\phi \in \tilde{\mathcal{G}}_M$.

Can every probability distribution be obtained in such a manner? Here we show that any distribution in the infinitesimal neighborhood of a given $\Gamma_I$ can be obtained by an infinitesimal automorphism, a derivation $v = v_i^I L_I^i$. To see this, let $G_I = \Gamma_I + \delta\Gamma_I$ be a distribution that differs infinitesimally from our reference $\Gamma_I$. Suppose $\Gamma$ satisfies the strict moment inequalities, i.e. $\Gamma_{I;J} = \Gamma_{IJ}$ is a positive Hankel matrix so that it is invertible $\Gamma^{I;J}\Gamma_{J;K} = \delta_K^I$. $\delta\Gamma_I$ are cyclically symmetric tensors (preserving the moment inequalities). The question is, can we find a vector field $L_v = v_i^I L_I^i$ such that $[L_v\Gamma]_I = \delta\Gamma_I$? Let $I = i_1 \cdots i_n$. Then

$$\begin{aligned}
[L_v\Gamma]_I &= v_j^J L_J^j \Gamma_I = v_j^J \delta_I^{I_1 j I_2} \Gamma_{I_1 J I_2} \\
&= v_{i_i}^J \Gamma_{J i_2 \cdots i_n} + v_{i_2}^J \Gamma_{i_1 J i_3 \cdots i_n} + \cdots v_{i_n}^J \Gamma_{i_1 \cdots i_{n-1} J} \\
&= v_{i_1}^J \Gamma_{J i_2 \cdots i_n} + v_{i_2}^J \Gamma_{J i_3 \cdots i_n i_1} + \cdots + v_{i_n}^J \Gamma_{J i_1 \cdots i_{n-1}}
\end{aligned} \tag{5.24}$$

Given $\delta\Gamma$, pick any tensor $w$ whose cyclic part is $\delta\Gamma$,

$$w_{i_1 \cdots i_n} + w_{i_2 \cdots i_n i_1} + \cdots w_{i_n i_1 \cdots i_{n-1}} = \delta\Gamma_{i_1 \cdots i_n} \tag{5.25}$$

and put $w_I = v_i^J \Gamma_{JI}$. Since $\Gamma_{I;J}$ is positive, we can invert it and solve for $v_i^J$

$$\begin{aligned}
w_I \Gamma^{I;K} &= v_i^J \Gamma_{JI} \Gamma^{I;K} = v_i^J \delta_J^K \\
\Rightarrow v_i^K &= w_I \Gamma^{I;K}
\end{aligned} \tag{5.26}$$

It follows that $[L_v\Gamma]_I = \delta\Gamma_I$. Thus, we have found an infinitesimal change of variable $v$ which allows us to transform from our reference $\Gamma$ to any given distribution in its infinitesimal neighborhood. In other words $\underline{\mathcal{G}_M}$ acts transitively on the tangent space to the classical configuration space at $\Gamma$.

We have not investigated whether this action of the Lie algebra can be exponentiated to a transitive action of the automorphism group on $\mathcal{P}_M$. Though it seems likely, the necessary change of variable may turn out to be singular if the probability distribution $G$ is qualitatively very different from $\Gamma$, this may be the sign of a "phase transition". In that case we will need one reference distribution for each "universality class" of matrix models.

What is more, for $M > 1$ there are many changes of variable that transform $\Gamma$ into $G$. They differ by those automorphisms that fix $\Gamma$, the isotropy subgroup of *measure preserving* automorphisms $\mathcal{SG}$. Infinitesimally, the isotropy Lie algebra consists of derivations $v$ that leave $\Gamma$ invariant

$$\begin{aligned} [L_v\Gamma]_I &= 0 \text{ for all } I \\ \Rightarrow \qquad 0 &= v^J_{i_1}\Gamma_{Ji_2\cdots i_n} + v^J_{i_2}\Gamma_{Ji_3\cdots i_n i_1} + \cdots + v^J_{i_n}\Gamma_{Ji_1\cdots i_{n-1}} \end{aligned} \tag{5.27}$$

A somewhat more explicit characterization of this isotropy Lie algebra when the reference $\Gamma$ is the multi-variate Wigner distribution is given in Appendix C.

The upshot is, we can regard the configuration space of large-$N$ matrix models as a coset space of the automorphism group of the free algebra: $\mathcal{P}_M = \tilde{\mathcal{G}}_M/\mathcal{SG}_M$.

### 5.2.3 Finite Change of Variable: Formula for $\chi$ on $\mathcal{G}/\mathcal{SG}$

We now use the coset space characterization of the configuration space to solve for $\chi(G)$ such that $L^i_I\chi(G) = \eta^i_I$. Suppose $\Gamma$ is a reference probability distribution and $\phi$ is a change of variable that maps $\Gamma$ to the distribution $G$. We shall express

$$\chi(G) - \chi(\Gamma) = c(\phi, \Gamma) \tag{5.28}$$

where $c(\phi, \Gamma)$ is a function on the group $\mathcal{G}$. But the LHS is independent of the "path" $\phi$ that connects $\Gamma$ to $G$. So to make this a sensible equation, we show that $c(\phi, \Gamma)$ is invariant under the action of the isotropy subgroup $\mathcal{SG}$. Then it will really be a function on the quotient $\mathcal{P} = \mathcal{G}/\mathcal{SG}$, which is the same as the space of moments.

Recall (§5.2.1) that $\eta$ is an element of the first cohomology of the Lie algebra $\underline{\mathcal{G}}$ valued in the space of power series in the moments. By naturality, $\chi(G) - \chi(\Gamma)$ must be in the first cohomology of the group $\mathcal{G}$, valued in the space of formal power series in the moments (See Appendix E on group and Lie algebra cohomology). In other words $c(\phi, \Gamma)$ must be a non-trivial one cocycle. The analogue of $d\eta = 0$ is the cocycle condition in group cohomology. Let $\phi$ and $\psi$ be two automorphisms. The cocycle condition is

$$c(\phi\psi, \Gamma) = c(\phi, \psi_*(\Gamma)) + c(\psi, \Gamma) \tag{5.29}$$

Our answer for $c(\phi,\Gamma)$ must satisfy this transformation rule.

Recall that $\eta$ is the expectation value of the jacobian for an infinitesimal change of variable $\phi_i(\xi)=\xi_i+v_i^I\xi_I$. Since $\eta=L_I^i(\chi(G)-\chi(\Gamma))$ is the infinitesimal version of $\chi$ we guess that $c(\phi,\Gamma)$ is the expectation value of the logarithm of the jacobian determinant for a finite change of variable $\phi$. Why logarithm? For, the addition rule for the logarithm of a product will translate into the cocycle condition.

The jacobian matrix for a change of variables $\phi_i^I A_I$ (5.18), is $J(\phi,A)=\frac{\partial\phi(A)}{\partial A}$. Under two successive changes of variable, the jacobians multiply

$$J(\phi\psi,A)=J(\phi,\psi(A))J(\psi,A) \tag{5.30}$$

Then $\sigma(\phi,A)=\frac{1}{N^2}\log\det J(\phi,A)$ satisfies $\sigma(\phi\psi,A)=\sigma(\phi,\psi(A))+\sigma(\psi,A)$. Taking the expectation value with respect to the reference distribution $\Gamma$ we define

$$c(\phi,\Gamma)=<\sigma(\phi,A)>=<\frac{1}{N^2}\log\det J> \tag{5.31}$$

and the cocycle condition (5.29) is automatically satisfied.

We will get a formula for $c(\phi,\Gamma)$ as a power series in $\phi_i^I$ and $\Gamma_J$. The simplest possibility is a linear change of variable, $[\phi(A)]_i=[\phi_1]_i^jA_j$. In this case, $\sigma(\phi,J)=\log\det\phi_1$. Since $\phi$ is invertible, this term is always non-vanishing, and we factor it out

$$[\phi(A)]_i=[\phi_1]_i^j\,[\tilde\phi(A)]_j,\quad \tilde\phi(A)_i=A_i+\sum_{|I|\geq 2}\tilde\phi_i^IA_I,\quad \tilde\phi_i^I=[\phi_1^{-1}]_i^j\phi_j^I \tag{5.32}$$

We show in Appendix F that $\sigma(\phi,A)$ can be written in terms of the loop variables $\Phi_I=\frac{\mathrm{tr}}{N}A_I$

$$\sigma(\phi,A)=\log\det\phi_1+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\tilde\phi_{i_1}^{K_1i_2L_1}\tilde\phi_{i_2}^{K_2i_3L_2}\cdots\tilde\phi_{i_n}^{K_ni_1L_n}\Phi_{K_1\cdots K_n}\Phi_{L_n\cdots L_1}. \tag{5.33}$$

Taking expectation values and using factorization in the large-$N$ limit

$$c(\phi,\Gamma)=\log\det\phi_1+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\tilde\phi_{i_1}^{K_1i_2L_1}\tilde\phi_{i_2}^{K_2i_3L_2}\cdots\tilde\phi_{i_n}^{K_ni_1L_n}\Gamma_{K_1\cdots K_n}\Gamma_{L_n\cdots L_1}. \tag{5.34}$$

Now let $\phi$ be an infinitesimal automorphism, $\phi_i(A)=A_i+v_i^IA_I$ where $v_i^I$ are small. Then $[\phi_1]_i^j=\delta_i^j+v_i^j$, $\tilde\phi_i^J=v_i^J$ and only the $n=0,1$ terms in the infinite series for $c$

survive

$$c(A_i + v_i^I A_I, \Gamma) \simeq v_i^i + \sum_{|J|+|K|>0} \phi_i^{JiK} \Gamma_J \Gamma_K = \sum_{|I|>0} v_i^I \delta_I^{I_1 i I_2} \Gamma_{I_1} \Gamma_{I_2} = v_i^I \eta_I^i \tag{5.35}$$

So $c(\phi, \Gamma)$ reduces to $\eta$ for infinitesimal $\phi$, so if we define $\chi(G) = \chi(\Gamma) + c(\phi, \Gamma)$, we are guaranteed that $L_I^i \chi = \eta_I^i$ which is what we wanted all along!

To complete our construction of a classical action, we need to show that $c(\phi, \Gamma)$ is a function of $\Gamma$ and $G = \phi_* \Gamma$ alone and is independent of the choice of $\phi$ that maps one to the other. In other words, that $c(\phi, \Gamma)$ is actually a function on $\mathcal{P} = \mathcal{G}/\mathcal{SG}$. Is $c(\phi, \Gamma)$ invariant under the right action of the isotropy subgroup? Suppose $\psi \in \mathcal{SG}$, i.e. $\psi_* \Gamma = \Gamma$ Then we need to show

$$c(\phi\psi, \Gamma) \stackrel{?}{=} c(\phi, \Gamma) \tag{5.36}$$

But this follows from the cocycle condition (5.29) if we can show that

$$c(\psi, \Gamma) = 0 \quad \text{if} \quad \psi_* \Gamma = \Gamma \tag{5.37}$$

We establish this at the level of Lie algebras. In other words, suppose $\psi_i(A) = A_i + v_i^I A_I$ is infinitesimal, and fixes the distribution $\Gamma$. Then $v_i^I L_I^i \Gamma_J = 0 \ \ \forall J$. On the other hand,

$$\begin{aligned} c(\psi, \Gamma) &= v_i^I \eta_I^i \\ &= v_i^I L_I^i (S_0^J \Gamma_J) \\ &= S_0^J (v_i^I L_I^i \Gamma_J) \\ &= S_0^J(0) = 0 \end{aligned} \tag{5.38}$$

In the second equality we have used the factorized Schwinger Dyson (5.10) satisfied by the reference action $S_0$ and reference moments $\Gamma$. Thus $c(\phi, \Gamma)$ is invariant under the infinitesimal action of $\mathcal{SG}$. It should be possible to exponentiate this to an invariance under the full isotropy subgroup though we do not address this issue here. Thus we regard $c(\phi, \Gamma)$ as a function on the coset space $\mathcal{P} = \mathcal{G}/\mathcal{SG}$ and

$$\chi(G) = \chi(\Gamma) + \log\det \phi_1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \tilde{\phi}_{i_1}^{K_1 i_2 L_1} \tilde{\phi}_{i_2}^{K_2 i_3 L_2} \cdots \tilde{\phi}_{i_n}^{K_n i_1 L_n} \Gamma_{K_1 \cdots K_n} \Gamma_{L_n \cdots L_1}. \tag{5.39}$$

We have finally solved our problem of finding a classical action for large-$N$ multi-matrix models! Suppose the distribution $G$ is obtained by a change of variables $\phi$ applied to the reference distribution $\Gamma$. Then extremization of $\Omega(G) = -S^I G_I + \chi(G)$:

$$\begin{aligned}\Omega(\phi,\Gamma) &= -\sum_{n=1}^{\infty} S^{i_1\cdots i_n}\phi^{J_1}_{i_1}\cdots\phi^{J_n}_{i_n}\Gamma_{J_1\cdots J_n} + \chi(\Gamma) + \log\det\phi^i_{1j} \\ &\quad + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\tilde{\phi}^{K_1 i_1 L_1}_{i_2}\tilde{\phi}^{K_2 i_2 L_2}_{i_3}\cdots\tilde{\phi}^{K_n i_n L_n}_{i_1}\Gamma_{K_n\cdots K_1}\Gamma_{L_1\cdots L_n} \qquad (5.40)\end{aligned}$$

gives the factorized Schwinger Dyson equations (5.10). This variational problem is equivalent to computing the partial Legendre transform of $\chi(G)$: extremize $\chi$ holding fixed the moments $G_I$ which are conjugate to the couplings $S^I$ that appear in the action. The coupling constants $S^I$ play the role of Lagrange multipliers. Furthermore, $\Omega(\phi,\Gamma) = \Omega(G)$ actually lives on the quotient $\mathcal{P} = \mathcal{G}/\mathcal{S}\mathcal{G}$. On account of $\eta$ not being an exact one-form, we could not find a power series in the moments $G_I$ for $\chi$. But we have found auxiliary variables $(\phi,\Gamma)$ that allow us to accomplish the task of finding a classical action.

In principle, the problem of solving a large-$N$ matrix model, given the coupling constants $S^I$ in its action, reduces to the extremization of $\Omega$ with respect to the change of variable $\phi$ for fixed reference moments $\Gamma_I$. Once the optimal $\phi$ is determined, the moments are given by

$$G_{i_1\cdots i_n} = \phi^{J_1}_{i_1}\cdots\phi^{J_n}_{i_n}\Gamma_{J_1\cdots J_n} \qquad (5.41)$$

Furthermore, recalling the transformation rule for the free energy under an automorphism (5.20), the large-$N$ free energy in terms of the optimal $\phi$ is

$$\begin{aligned}F(S) - F(S_0) &= -\,c(\phi_{optimal},\Gamma) \\ \Rightarrow F(S) &= -\text{ constrained extremum of } \chi(G) \qquad (5.42)\end{aligned}$$

where the constrained extremum is with respect to all moments except the ones conjugate to the couplings $S^I$. Also, $S_0(A)$ is the reference action with moments $\Gamma_I$. If $\mathrm{tr}\ S(A)$ is an action that grows as a polynomial as $\mathrm{tr}\ A^2 \to \infty$ so that the matrix integrals converge, then the extremum of $\Omega$ is actually a maximum and the free energy is a minimum.

A byproduct of this is a variational approximation method for large-$N$ matrix models. Rather than extremizing $\Omega$ with respect to all possible $\phi \in \tilde{\mathcal{G}}$, if we choose an ansatz

for $\phi$ with a few variational parameters and maximize $\Omega$, we will get a variational upper bound on the free energy and a variational estimate for the moments! As an application we use this method to solve a two matrix model approximately in §5.4. But before that, we explain the physical meaning of $\chi$. It is the entropy that comes from our lack of knowledge of matrix elements when only the invariants $\Phi_I = \frac{\mathrm{tr}}{N} A_I$ are observable.

## 5.3 $\chi$ as an Entropy

The appearance of the Legendre transform, the logarithm of the jacobian for a change in volume measure and the restriction of observables to $U(N)$ invariants should all remind us of entropy in thermodynamics. The non-exactness of $\eta$ is reminiscent of the heat one-form in thermodynamics [23].

In thermodynamics, entropy is a measure of the number of microstates that have a given value for the macroscopic observables such as energy, pressure or volume. Entropy arises when we restrict the set of allowed observables of a physical system. We should expect there to be an entropy due to color confinement in the strong interactions: quarks and gluons are confined and the only observable particles are their hadronic bound states.

In a matrix model, which is a watered-down version of gluon dynamics, we have restricted the class of observables from matrix elements $[A_i]^a_b$ to $U(N)$ invariants $\Phi_I$. There should be an associated entropy. Indeed it is the term $\chi$ in the classical action we have just obtained. It is the average of the logarithm of the volume of the space of matrices that have a given set of moments. It can also be regarded as the expectation value of a matrix model analogue of the Fadeev-Popov determinant of gauge theories. Despite the analogies with statistical mechanics, there are a few differences. The concept of thermal equilibrium is not relevant to matrix models. What is more, we have an infinite number of *macroscopic* observables $G_I$ while there were only a finite number such as energy, pressure, density, volume, and chemical potential in thermodynamics. Furthermore, the microscopic dynamical variables, the matrices, do not commute as opposed to the real-valued positions of particles in a gas which commute. As a consequence, the entropy in matrix models is that of non-commutative probability theory rather than the

usual Boltzmann-Shannon entropy of commutative probability theory. Nevertheless, the Boltzmann notion that the entropy should be the logarithm of the volume of microstates corresponding to a given macrostate continues to makes sense.

The interpretation of $\chi$ as the entropy is most transparent for the case of a single matrix. The integral over the matrix elements of a hermitian matrix $A = UDU^\dagger$ can be factored into an integral over its eigenvalues $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ and the unitary transformation $U$ that diagonalizes it. The jacobian determinant for this change of variables is the volume of the space of hermitian matrices which share a common spectrum $\lambda_1 < \lambda_2 < \cdots \lambda_N$ where the eigenvalues are generically distinct. The well known result [77] is

$$dA = \prod_a dA^a_a \prod_{a<b} d\ \mathrm{Re}A^a_b\ d\ \mathrm{Im}A^a_b = Vol(U(N))\ \Delta^2\ \prod_a d\lambda_a \tag{5.43}$$

where $Vol(U(N))$ is the volume of the unitary group and $\Delta$ is the Van der Monde determinant $\Delta = \prod_{a<b}(\lambda_a - \lambda_b)$. Then the logarithm of volume of matrix elements corresponding to a given spectrum is $\chi = \log \Delta^2$, up to the additive constant $\log Vol(U(N))$ which we ignore since it is independent of the eigenvalues. If $\rho(x) = \frac{1}{N}\sum_a \delta(x - \lambda_a)$ is the eigenvalue density, then the entropy is

$$\begin{aligned} \chi &= 2\sum_{a<b} \log|\lambda_a - \lambda_b| \\ &= \mathcal{P}\int \rho(x)\rho(y)\log|x-y|\ dx\ dy \end{aligned} \tag{5.44}$$

where we have used the principal value in anticipation of the passage to the large-$N$ limit where we have a continuous distribution of eigenvalues. Notice that this formula for entropy is different from the Boltzmann entropy $-\int\ \rho(x)\ \log\rho(x)\ dx$ of a real-valued random variable with density $\rho$.

Now suppose $G_k = \int \rho_G(x)\ x^k\ dx$ are the moments of the probability density $\rho_G$ and $\Gamma_k$ are the moments of a reference distribution $\rho_\Gamma$. Furthermore, let $\phi$ be the change of variable that relates the two,

$$\begin{aligned} \rho_\Gamma(x) &= \rho_G(\phi(x))\phi'(x) \\ G_k &= \int \rho_\Gamma(y)\phi^k(y)dy \end{aligned} \tag{5.45}$$

then the above formula for entropy becomes

$$\begin{aligned}\chi(G) &= \mathcal{P}\int dxdy\ \rho_G(x)\rho_G(y)\log|x-y| \\ \chi(G) &= \chi(\Gamma)+\mathcal{P}\int dxdy\ \rho_\Gamma(x)\rho_\Gamma(y)\log\left|\frac{\phi(x)-\phi(y)}{x-y}\right| \end{aligned} \tag{5.46}$$

where we have multiplied and divided the argument of the logarithm by $|x-y|$ so that we may isolate $\chi(\Gamma)$. The second term on the right has the physical meaning of the relative entropy of $G$ with respect to $\Gamma$. Moreover, if we assume the change of variable is an invertible power series

$$\phi(x)=\phi_1[x+\tilde{\phi}(x)];\quad \phi_1>0;\quad \tilde{\phi}(x)=\sum_{k=2}^{\infty}\tilde{\phi}_k x^k \tag{5.47}$$

the entropy becomes

$$\chi(G)=\chi(\Gamma)+\log\phi_1+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{k_i+l_i>0}\tilde{\phi}_{k_1+1+l_1}\cdots\tilde{\phi}_{k_n+1+l_n}\Gamma_{k_1+\cdots k_n}\Gamma_{l_1+\cdots+l_n} \tag{5.48}$$

We see that this is just the $M=1$ case of the formula for $\chi$ we obtained earlier (5.39) in looking for a classical action for large-$N$ matrix models. Formula (5.46) for entropy in terms of an eigenvalue density has no simple generalization to several non-commuting matrices. There is no common basis in which the matrices are diagonal. However, the joint spectrum of non-commuting matrices makes sense if we think in terms of the basis independent moments. The series in Taylor coefficients of $\phi$ for entropy continues to make sense. This is what we obtained in (5.39). Thus $c(\phi,\Gamma)=\chi(G)-\chi(\Gamma)$ has the physical meaning of relative entropy of $G$ with respect to $\Gamma$. For more on entropy of non-commutative random variables, see §4.4.

In light of this, our variational principle from §5.2.3 has a simple physical interpretation. The moments of the large-$N$ classical limit of a matrix model are determined by maximizing entropy holding the moments $G_I$ conjugate to the coupling tensors $S^I$ fixed using Lagrange multipliers $S^I$. In looking for a variational principle for matrix models we have naturally been lead to the cohomology of automorphisms of the free algebra and to the entropy of non-commuting random variables! Let us now return to our original problem of determining the moments and free energy of large-$N$ matrix models.

## 5.4 Variational Approximations

We next discuss the use of our variational principle to obtain approximate solutions of large-$N$ matrix models. (For other approximation methods in the literature, see for example [82]). A linear automorphism with the wignerian reference leads to an analogue of mean-field theory for matrix models. We closely follow our presentation in [5, 69]. Consider a quartic multi-matrix model with action

$$S(M) = \text{ tr } [\frac{1}{2}K^{ij}A_{ij} + \frac{1}{4}g^{ijkl}A_{ijkl}]. \tag{5.49}$$

Let us use the standard multivariate Wigner distribution (Appendix C) with action $S_0 = \frac{1}{2}\delta^{ij}A_iA_j$ as the reference distribution. The reference moments are $\Gamma_{ij} = \delta_{ij}, \Gamma_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{li}\delta_{jk}$ etc. Let $E = F(S) - F(S_0)$ denote the free energy of $S$ relative to $S_0$. For a linear change of variables $\phi_i(A) = \phi_i^j A_j$. We must maximize

$$\begin{aligned} \Omega[\phi] &= \text{ tr } \log[\phi_i^j] - \frac{1}{2}K^{ij}G_{ij} - \frac{1}{4}g^{ijkl}G_{ijkl}. \\ G_{ij} &= \phi_i^k\phi_j^k; \quad G_{ijkl} = G_{ij}G_{kl} + G_{il}G_{jk} \end{aligned} \tag{5.50}$$

Thus $G_{ij}$ may be regarded as variational parameters and the condition for an extremum is

$$\frac{1}{2}K^{pq} + \frac{1}{4}[g^{pqkl}G_{kl} + g^{ijpq}G_{ij} + g^{pjqk}G_{jk} + g^{ipql}G_{il}] = \frac{1}{2}[G^{-1}]^{pq} \tag{5.51}$$

This non-linear equation for the variational matrix $G$, is reminiscent of the self consistent equation for a mean-field. We will use the terms mean-field theory, gaussian ansatz and wignerian ansatz interchangeably for this variational approximation. To test it, we consider a two matrix model for which some exact results are known from the work of Mehta [78].

### 5.4.1 Two Matrix Model with Interaction $\frac{g}{4}(A^4 + B^4) - 2cAB$

We will study the two matrix model whose action is

$$S(A,B) = -\text{ tr } [\frac{1}{2}(A^2 + B^2 - cAB - cBA) + \frac{g}{4}(A^4 + B^4)]. \tag{5.52}$$

Kazakov relates this model to the Ising model on the collection of all planar lattices with coordination number four [65]. The partition function of this model has been

calculated exactly by Mehta [78]. This is a special case of the chain matrix models $S(A_1, \cdots A_n) = \sum_{k=1}^{n} \text{ tr } V(A_k) - \sum_{k=1}^{n-1} c_k \text{ tr } A_k A_{k+1}$ which also share this integrable property. Indeed, as $k \to \infty$, we can replace it by a continuous index. When the continuous index is thought of as time, we get the quantum mechanics of a single matrix whose ground-state was obtained by Brezin et. al. [18] by mapping it onto a system of free fermions.

We restrict to $|c| < 1$, where $K^{ij}$ is a positive matrix. Since $S(A, B) = S(B, A)$ and $G_{AB} = G^*_{BA}$ we may take

$$G_{ij} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \tag{5.53}$$

with $\alpha, \beta$ real. For $g > 0$, $\Omega$ is bounded above if $G_{ij}$ is positive. Its maximum occurs at $(\alpha, \beta)$ determined by $\beta = \frac{c\alpha}{1+2g\alpha}$ and

$$4g^2\alpha^3 + 4g\alpha^2 + (1 - c^2 - 2g)\alpha - 1 = 0. \tag{5.54}$$

We must pick the real root $\alpha(g, c)$ that lies in the physical region $\alpha \geq 0$. Thus, the gaussian ansatz determines the vacuum energy ($E(g, c) = -\frac{1}{2}\log(\alpha^2 - \beta^2)$) and *all* the Green's functions (e.g. $G_{AA} = \alpha,\ G_{AB} = \beta,\ G_{A^4} = 2\alpha^2$ e.t.c) approximately.

By contrast, only a few observables of this model have been calculated exactly. Mehta [78] obtains the exact vacuum energy $E^{ex}(g, c)$ implicitly, as the solution of a quintic equation. $G^{ex}_{AB}$ and $G^{ex}_{A^4}$ may be obtained by differentiation. (Generating series for some other special classes of moments are also accessible (see [97, 31]).) As an illustration, we compare with Mehta's results in the weak and strong coupling regions. For small $g$ and $c = \frac{1}{2}$:

$$E^{ex}(g, \frac{1}{2}) = -.144 + 1.78g - 8.74g^2 + \cdots \qquad E^{var}(g, \frac{1}{2}) = -.144 + 3.56g - 23.7g^2 + \cdots$$
$$G^{ex}_{AB}(g, \frac{1}{2}) = \frac{2}{3} - 4.74g + 53.33g^2 + \cdots \qquad G^{var}_{AB}(g, \frac{1}{2}) = \frac{2}{3} - 4.74g + 48.46g^2 + \cdots$$
$$G^{ex}_{AAAA}(g, \frac{1}{2}) = \frac{32}{9} - 34.96g + \cdots \qquad G^{var}_{AAAA}(g, \frac{1}{2}) = \frac{32}{9} - 31.61g + 368.02g^2 + \cdots$$

For strong coupling and arbitrary $c$ :

$$E^{ex}(g, c) = \frac{1}{2}\log g + \frac{1}{2}\log 3 - \frac{3}{4} + \cdots \qquad E^{var}(g, c) = \frac{1}{2}\log g + \frac{1}{2}\log 2 + \frac{1}{\sqrt{8g}} + \mathcal{O}(\frac{1}{g})$$
$$G^{ex}_{AB}(g, c) \to 0 \text{ as } g \to \infty \qquad G^{var}_{AB}(g, c) = \frac{c}{2g} - \frac{c}{(2g)^{\frac{3}{2}}} + \mathcal{O}(\frac{1}{g^2})$$

$$G^{ex}_{AAAA}(g,c) = \frac{1}{g} + \cdots \qquad G^{var}_{AAAA}(g,c) = \frac{1}{g} - \frac{2}{(2g)^{\frac{3}{2}}} + \mathcal{O}(\frac{1}{g^2}) \qquad (5.55)$$

The linear change of variable from a wignerian reference gives a crude first approximation for both weak and strong coupling. It fails near singularities of the free energy (phase transitions). As $|c| \to 1^-$, the energy $E^{ex}$ diverges; this is not captured well by the gaussian ansatz. This reinforces our view that the gaussian variational ansatz is the analogue of mean-field theory.

### 5.4.2 Two Matrix Model with $A_1^2 + A_2^2 - [A_1, A_2]^2$ Interaction

Next, consider the two matrix model with Yang-Mills + gaussian type of action ($m^2 > 0$)

$$S(A_1, A_2) = \left[\frac{m^2}{2}(A_1^2 + A_2^2) - \frac{g^2}{4}[A_1, A_2]^2\right] \qquad (5.56)$$

Certain observables in the large-$N$ limit of the gaussian + commutator squared model (such as the partition function, though not $G_{ABAB}$ – this can be seen via the planar diagram expansion) are the same as in the gaussian + anti-commutator squared two matrix model. Its partition function has been studied analytically in [35, 71] in relation to the three color problem on a random lattice. The partition function of this model also arises in the study of the $\mathcal{N} = 1^*$ supersymmetric gauge theory, the mass deformation of the $\mathcal{N} = 4$ supersymmetric gauge theory upon integrating out one of the three adjoint chiral matter fields [28]. We can also regard this matrix model as a simple model for Yang-Mills theory.

It has been shown [10] that the integrals over matrices for the pure commutator squared action for a two matrix model are not convergent. To see this, consider the partition function, and go to the basis in which $A_1$ is diagonal. In this basis, the integrand is independent of the diagonal elements of $A_2$, and therefore diverges. The divergence is even worse if we consider the expectation value of the trace of a polynomial involving $A_2$, since then the integrand grows for large values of the diagonal elements of $A_2$. Moreover, from the work of [35, 71], it is expected that the free energy develops a singularity for sufficiently large $g^2$ (or small $m^2$). Since there is only one independent coupling constant we will set $g = 1$.

In [69] we did a small-scale Monte-Carlo simulation to test our wignerian variational ansatz for this model. This ansatz does a good job of estimating 2 and 4 point correlations for $m^2 > 1$ but not in the strongly-coupled region of small $m^2$. It does not capture the divergence in free energy for small $m^2$, as we would expect of a mean-field type approximation. Our numerical simulation is also not expected to be accurate for small $m^2$. We quote here the results from [69].

**Polynomial Moments**

For a linear automorphism with standard wignerian reference $S_0 = \frac{1}{2}\delta^{ij}A_iA_j$

$$\Omega[G] = \frac{1}{2}\log\det[G_{ij}] - \frac{m^2}{2}(G_{11}+G_{22}) + \frac{1}{2}(G_{1212}-G_{1221}) \tag{5.57}$$

As before the variational matrix of second moments is

$$G_{ij} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \tag{5.58}$$

Since $< \frac{\mathrm{tr}}{N}A_1^2 > \ \geq 0$ and $< \frac{\mathrm{tr}}{N}(A_1-A_2)^2 > \ \geq 0$, we must maximize

$$\Omega(\alpha,\beta) = \frac{1}{2}\log(\alpha^2-\beta^2) - m^2\alpha + \frac{1}{2}(\beta^2-\alpha^2) \tag{5.59}$$

in the region $\alpha \geq 0$ and $\alpha \geq \beta$. We get

$$G_{11} = G_{22} = \alpha = \sqrt{1+\frac{m^4}{4}} - \frac{m^2}{2}, \quad G_{12} = G_{21} = \beta = 0 \tag{5.60}$$

Figures 5.1 and 5.2 compare the variational two point correlations with Monte-Carlo measurements for a range of values of $m^2$. All other correlations can be expressed in terms of these. For example, the 4-point correlations are (the rest are determined by cyclic symmetry and $A_1 \leftrightarrow A_2$ exchange symmetry) $G_{1111} = 2\alpha^2; G_{1212} = 0; G_{1221} = \alpha^2; G_{1112} = 0$. Figures 5.3, 5.4 and 5.5 compare variational estimates (solid lines) and Monte-Carlo measurements (dots) of $G_{1111}$, $G_{1212}$ and $G_{1122}$ for $10^{-3} \leq m^2 \leq 10^3$. The $n$ point pure $A_1$ (or $A_2$) correlation is given by the Catalan numbers

$$G_{111\cdots1} = G_{222\cdots2} \equiv G_{(n)} = \left\{ \begin{array}{ll} C_{\frac{n}{2}}\alpha^{\frac{n}{2}} = \frac{n!}{(\frac{n}{2})!(\frac{n}{2}+1)!}\alpha^{\frac{n}{2}}, & \text{if n is even;} \\ 0, & \text{if n is odd.} \end{array} \right\}. \tag{5.61}$$

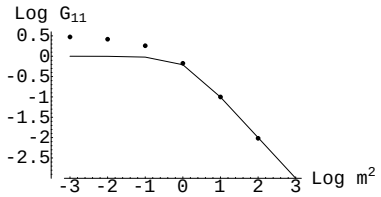


Figure 5.1: $\log_{10}[G_{11}]$ versus $\log_{10}[m^2]$ for $g = 1$. Solid line is variational estimate, dots are Monte-Carlo measurements. The approximation becomes poor for small values of $m^2$

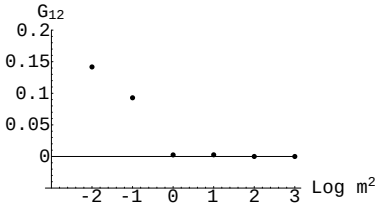


Figure 5.2: $G_{12}$ versus $\log_{10}[m^2]$ for $g = 1$. Solid line is variational estimate, dots are Monte-Carlo measurements.

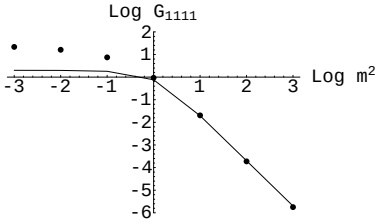


Figure 5.3: $\log_{10}[G_{1111}]$ versus $\log_{10}[m^2]$ for $g = 1$. Solid is line variational estimate, dots are Monte-Carlo measurements.

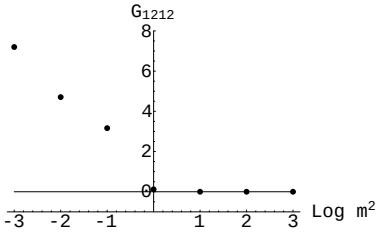


Figure 5.4: $G_{1212}$ versus $\log_{10}[m^2]$ for $g = 1$. Solid line is variational estimate, dots are Monte-Carlo measurements.

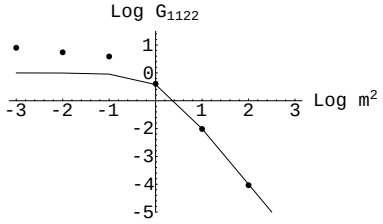


Figure 5.5: $\log_{10}[G_{1122}]$ versus $\log_{10}[m^2]$ for $g = 1$. Solid line is variational estimate, dots are Monte-Carlo measurements.

More generally,

$$\begin{aligned} G_{11\cdots122\cdots2} &\equiv G_{(n_1)(n_2)} = G_{(n_1)}G_{(n_2)} \\ G_{11\cdots122\cdots211\cdots122\cdots2} &\equiv G_{(n_1)(n_2)(n_3)(n_4)} = G_{(n_1+n_3)}G_{(n_2)}G_{(n_4)} \\ &\quad + G_{(n_1)}G_{(n_3)}G_{(n_2+n_4)} - G_{(n_1)}G_{(n_2)}G_{(n_3)}G_{(n_4)} \end{aligned} \tag{5.62}$$

We mention these since they are useful in estimating expectation values of Wilson loop-like operators. The variational estimate for free energy is

$$E_{var} = F(S) - F(S_0) = -\log\alpha = -\log\left[\sqrt{1+\frac{m^4}{4}} - \frac{m^2}{2}\right] \tag{5.63}$$

**Wilson Loop-like Operators**

It is also interesting to see what the wignerian ansatz says about the 2-matrix analogue of the expectation of the Wilson loop in the large-$N$ limit. See [29, 6] for a calculation of these observables without any approximations in the one-matrix models.

**Wilson Line:** The simplest analogue is a "Wilson line" the analogue of the parallel transport along a line of length $l$ in the $A_1$ direction

$$W_{line}(l) \equiv \lim_{N\to\infty} < \frac{\mathrm{tr}}{N} e^{ilA_1} > \tag{5.64}$$

For the wignerian ansatz, we get (using (5.61))

$$\begin{aligned} W_{line}(l) &= \sum_{n=0}^{\infty} \frac{(il)^{2k}}{(2k)!} c_k \alpha^k \\ &= \frac{1}{l\sqrt{\alpha}} J_1(2l\sqrt{\alpha}) \sim \frac{1}{\sqrt{\pi}(l\sqrt{\alpha})^{\frac{3}{2}}} \cos\left(\frac{3\pi}{4} - 2l\sqrt{\alpha}\right) \text{ as } l\to\infty \end{aligned} \tag{5.65}$$

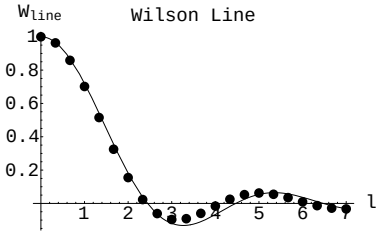


Figure 5.6: $W_{line}(l)$ for $m^2 = 1,\ g = 1$. Dots are numerical and solid line variational estimate.

Here $J_n(z)$ is the Bessel function of the first kind. $W_{line}(l)$ is a real-valued function of real $l$ since the odd order correlations vanish. Thus, for the wignerian ansatz, the expectation value of the "Wilson line" is oscillatory but decays as a power $l^{-3/2}$. For small $l$, $W_{line}(l) \to 1 - \frac{1}{2}\alpha l^2 + \frac{\alpha^2 l^4}{12} - \cdots$. Figure 5.6 compares this ansatz with Monte-Carlo measurements for $m^2 = 1$. The behavior for small and moderate values of $l$ is well captured by our ansatz.

**L shaped Wilson Line:** For an $L$ shaped curve, we define

$$W_L(l) = \lim_{N\to\infty} < \frac{\mathrm{tr}}{N} e^{ilA_1} e^{ilA_2} > \tag{5.66}$$

For the wignerian ansatz (use eq. (5.62); ${}_1F_2(a, \mathbf{b}; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b_1)_n (b_2)_n} \frac{z^n}{n!}$ is a generalized Hypergeometric function with $(a)_n$ the Pochhammer symbol),

$$\begin{aligned} W_L(l) &= \sum_{n_1,n_2=0}^{\infty} \frac{(il)^{n_1+n_2}}{n_1! n_2!} \lim_{N\to\infty} < \frac{\mathrm{tr}}{N} A_1^{n_1} A_2^{n_2} > \\ &= \sum_{k_1,k_2=0}^{\infty} \frac{(-l^2\alpha)^{k_1+k_2}}{k_1!(k_1+1)!k_2!(k_2+1)!} \\ &= \sum_{n=0}^{\infty} (-l^2\alpha)^n \frac{4^{n+1}\Gamma(n+\frac{3}{2})}{\sqrt{\pi}\ \Gamma(n+1)\Gamma(n+2)\Gamma(n+3)} \\ W_L(l) &= {}_1F_2(\frac{3}{2}; \{2,3\}; -4l^2\alpha) \end{aligned} \tag{5.67}$$

As before $W_L(l)$ is real for real $l$. For small $l$, $W_L(l) \to 1 - \alpha l^2 + \frac{5\alpha^2 l^4}{12} - \cdots$. This is compared with the numerical calculation in fig. 5.7 for $m^2 = 1$.

**Wilson Square:** The analogue of the parallel transport around a square of side $l$ in the $A_1 - A_2$ plane is

$$W_{square}(l) = \lim_{N\to\infty} < \frac{\mathrm{tr}}{N} e^{ilA_1} e^{ilA_2} e^{-ilA_1} e^{-ilA_2} > \tag{5.68}$$

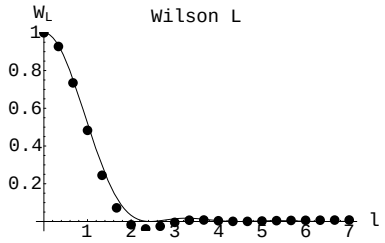


Figure 5.7: $W_L(l)$ for $m^2 = 1,\ g = 1$. Solid line is variational estimate, dots are Monte-Carlo measurements.

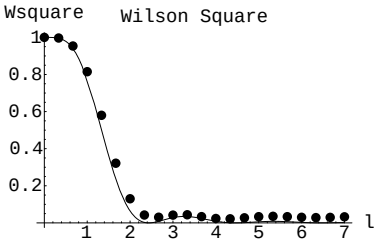


Figure 5.8: $W_{square}(l)$ for $m^2 = 1,\ g = 1$. Solid line is variational estimate, dots are the Monte-Carlo measurements.

In the wignerian variational approximation, $W_{square}(l)$ is real-valued since odd order correlations vanish. Using eq.(5.62) we get

$$W_{square}(l) = \sum_{n=0}^{\infty}(-l^2\alpha)^n T_1(n) + 2(l^2\alpha)\sum_{n=0}^{\infty}(-l^2\alpha)^n T_2(n) \tag{5.69}$$

$$\begin{aligned} \text{where}\ \ T_1(n) &= \sum_{k_i\geq 0, k_1+\cdots+k_4=n} \frac{2c_{k_1+k_3}c_{k_2}c_{k_4} - \Pi_{i=1}^4 c_{k_i}}{\Pi_{i=1}^4 (2k_i)!} \\ T_2(n) &= \sum_{k_i\geq 0, k_1+\cdots+k_4=n} \frac{c_{k_1+k_3+1}c_{k_2}c_{k_4}}{(2k_1+1)!(2k_2)!(2k_3+1)!(2k_4)!} \end{aligned} \tag{5.70}$$

For small $l$, $W_{square}(l) \to 1 - l^4\alpha^2 + \frac{5l^6\alpha^3}{6} - \cdots$. The expectation value of the Wilson loop is a rapidly decaying function for large values of $l$. It would be interesting to find the asymptotic rate of decay. It is oscillatory but a positive function, unlike $W_{line}$. These variational predictions are confirmed by the numerics, in fig. 5.8 for $m^2 = 1,\ g = 1$. Our crude mean-field variational ansatz does a very good job of estimating the Wilson loop averages over a range of small and moderate values of $l$ studied, for $m^2$ of order unity or more and $g = 1$.

# Chapter 6

# Summary

In this thesis, we studied rather simplified models of a non-abelian gauge theory in the limit of a large number of colors ($N$). The emphasis has been on the classical nature of this limit, where, unlike in the usual $\hbar \to 0$ classical limit, where all variables stop fluctuating, fluctuations vanish only in gauge-invariant variables.

Part I focused on the baryon in two-dimensional QCD, and Part II on euclidean multi-matrix models.

Chapter 1 contained historical motivations, and an introduction to QCD, matrix models, and the large-$N$ limit. We illustrated a classical limit, different from the $\hbar \to 0$ limit, in the context of the large-dimension limit for rotation-invariant variables in atomic physics, which provides a "classical" explanation for the stability of the atom. The $1/d$ expansion is an accurate way of calculating atomic energy levels in many electron atoms, as shown by Herschbach and collaborators [50].

Chapter 2 summarizes Rajeev's reformulation of the large-$N$ limit of two-dimensional QCD in terms of color-singlet quark bilinears in the null gauge: two-dimensional Hadrondynamics. This is an interacting bilocal theory of mesons, since the quark density matrix is a projection operator. 't Hooft's linear integral equation for the meson spectrum in two dimensions does not account for baryons. In Hadrondynamics, free mesons are small fluctuations around the vacuum, and baryons arise as topological solitons on the disconnected infinite grassmannian phase space with connected components labelled by baryon

number.

In Chapt. 3, we studied the structure of baryons in two-dimensional Hadrondynamics. A version of steepest descent for the curved grassmannian phase space was provided to determine the ground-state of the baryon. Then, the ground-state of the baryon was determined variationally by minimizing its energy on submanifolds of the phase space. A succession of finite but increasing rank submanifolds of the phase space was found. The reduced dynamical systems on these submanifolds were shown to correspond to interacting-parton models. We thus derived the parton picture as a variational approximation to the soliton description. The rank-one separable ansatz corresponds to a Hartree-type approximation for valence-quarks. We determined the exact two-dimensional baryon form factor in the large-$N$ and chiral limits. The rank-three ansatz allowed us to estimate the sea and anti-quark content of the baryon, which turned out to be small. To model structure functions measured in Deep Inelastic Scattering, we used the two-dimensional valence-quark distribution as the initial condition for perturbative $Q^2$ evolution. This allowed us to model the structure function $F_3$ measured in neutrino-nucleon deep inelastic scattering. We have two free parameters, the initial momentum transfer $Q_0^2$, and the fraction of baryon momentum carried by valence-quarks.

Chapter 4 provided background on free algebras and probability theory for operator-valued random variables. The tensor algebra, its automorphisms, and derivations were introduced in the context of non-commutative probability theory. Voiculescu's notion of non-commutative entropy was defined.

In Chapter 5, we studied euclidean large-$N$ multi-matrix models as toy models for gluon dynamics. We formulated them as classical mechanical systems for $U(N)$ invariant variables. We found that the classical configuration space is a space of non-commutative probability distributions. The classical equations of motion are the factorized Schwinger-Dyson equations for gluon correlations. There is an anomalous term in the factorized Schwinger-Dyson equations arising from the change in measure of integration. We showed that this term is a closed but not exact one-form. This universal term is an element of the Lie algebra cohomology of the automorphism group of the tensor algebra. This cohomological obstruction prevented us from finding a classical action principle as a power

series in gluon correlations. We showed that the configuration space can be expressed as a quotient of the automorphism group of the tensor algebra by the isotropy subgroup of measure-preserving automorphisms. Probability distributions are parameterized by the change of variable that maps them to a standard reference distribution. This provides a classical action on the group that is invariant under the action of the isotropy subgroup. The action principle was interpreted as the maximization of non-commutative entropy, while holding fixed the correlations conjugate to the coupling tensors. The restriction of observables from matrix elements to $U(N)$ invariants is the source of this entropy. The free energy is the negative of the maximum value of this entropy. By treating the change of variable as a variational quantity, we obtain variational approximations for the correlations and free energy of large-$N$ matrix models. A linear automorphism is the simplest variational ansatz, and leads to an analog of mean-field theory. This approximation method gives reasonable estimates for the observables of one and two-matrix models, away from divergences of the free energy (phase transitions).

# Appendix A

# Definition of $\frac{1}{p^2}$ Finite Part Integrals

Here we discuss the definition of the $\frac{1}{p^2}$ singular integrals defining the potential energy in momentum space. In position space integrals this singularity is manifested in the linearly rising $|x-y|$ potential. Let us consider the rank-one case for example:

$$PE = \frac{1}{2}\mathcal{FP}\int [dp]\int \frac{dr}{2\pi}\tilde{\psi}(p)\tilde{\psi}^*(p+r)\tilde{V}(r) \tag{A.1}$$

where $[dp] = dp/2\pi$. In this appendix, we desire to give a meaning to the above "finite part" integral. We call it a "finite part" integral since similar *"part finie"* integrals appear in the work of Hadamard [49]. The self-consistent potential $V(x)$ satisfies Poisson's equation $V''(x) = |\psi(x)|^2$ along with a pair of boundary conditions,

$$V(0) = \int_{-\infty}^{\infty} |\psi(y)|^2 \frac{|y|}{2} dy \text{ and } V'(0) = \frac{1}{2}\int_{-\infty}^{\infty} |\psi(-y)|^2 \text{ sgn } (y) dy. \tag{A.2}$$

In momentum space Poisson's equation implies that

$$\tilde{V}(p) = \mathcal{FP}(\frac{-1}{p^2}\int [dq]\tilde{\psi}(p+q)\tilde{\psi}^*(q)) \equiv \mathcal{FP}(-\frac{1}{p^2}\tilde{W}(p)). \tag{A.3}$$

The two boundary conditions take the form

$$-\mathcal{FP}\int_{-P}^{P}\frac{1}{p^2}\tilde{W}(p)[dp] = \int |\psi(y)|^2\frac{|y|}{2}dy \tag{A.4}$$

$$-i\mathcal{FP}\int_{-P}^{P}\frac{1}{p}\tilde{W}(p)[dp]=\frac{1}{2}\int_{0}^{\infty}\{-|\psi(y)|^2+|\psi(-y)|^2\}dy. \tag{A.5}$$

In effect, the boundary conditions are rules for integrating non-singular functions with respect to the singular measures $\frac{dp}{2\pi p^{1,2}}$. We will use these conditions to define the "finite part" integral in terms of Riemann integrals.

For simplicity, let us consider the case where the wave function $\tilde{\psi}(p)$ is real. Then $\psi(-x)=\psi^*(x)$, and the self-consistent potential is even: $\tilde{V}(-p)=\tilde{V}(p)$. Then the second boundary condition (A.5) becomes

$$\mathcal{FP}\int_{-P}^{P}\frac{-1}{p}\tilde{W}(p)[dp]=0. \tag{A.6}$$

Since the integrand is odd, this condition is automatically satisfied. Let us restrict attention to wave functions $\tilde{\psi}(p)$ such that

$$\tilde{\psi}(p)\sim p^a,\ \text{as } p\to 0,\ \text{for some } a>0. \tag{A.7}$$

From the Frobenius analysis (section 3.3.3), we know that this is equivalent to assuming a non-zero current quark mass, so that superficial infrared divergences are avoided.

Our aim is to define $-\mathcal{FP}\int_{-P}^{P}\frac{1}{s^2}\tilde{W}(s)[ds]$ so as to satisfy the first boundary condition. Moreover, the definition should reduce to the usual one, when this quantity is finite to begin with.

**Claim:** If

$$\mathcal{FP}\int_{-P}^{P}\frac{1}{s^2}\tilde{W}(s)[ds]:=\int_{-P}^{P}\frac{\tilde{W}(s)-\tilde{W}(0)}{s^2}[ds]-\frac{\tilde{W}(0)}{\pi P}. \tag{A.8}$$

Then

$$\mathcal{FP}\int_{-P}^{P}\frac{1}{s^2}\tilde{W}(s)[ds]=-\int_{-\infty}^{\infty}|\psi(x)|^2\frac{|x|}{2}dx. \tag{A.9}$$

provided $\tilde{W}'(0)=0$, which is ensured if $\tilde{\psi}(p)$ is real and $C^a$ from the right at $p=0$ for some $a>0$.

**Proof:** The motivation for this definition is clear: we have essentially subtracted out the divergent terms and analytically continued the result that we would have got if $\tilde{W}(s)$ vanished sufficiently fast at the origin (i.e. like $s^{1+\epsilon},\ \epsilon>0$) to make the integral converge. However, there is always the danger of mistakenly adding/subtracting some

finite quantity along with the divergent one. We show here that this definition actually satisfies the boundary conditions.

Recall that $\tilde{W}(s)$ is the Fourier transform of the charge density:

$$\tilde{W}(s) = \int_{-\infty}^{\infty} |\psi(x)|^2 e^{-isx} dx \tag{A.10}$$

so that

$$\tilde{W}(s) - \tilde{W}(0) = \int_{-\infty}^{\infty} dx |\psi(x)|^2 (e^{-isx} - 1) \tag{A.11}$$

Moreover, $\tilde{W}'(0)$ vanishes, since

$$\tilde{W}'(0) = -i \int_{-\infty}^{\infty} x |\psi(x)|^2 dx \tag{A.12}$$

and the integrand is odd. Therefore, $\tilde{W}(s) - \tilde{W}(0)$ vanishes at least as fast as $s^{1+\epsilon}$, $\epsilon > 0$ as $s \to 0$. E.g. for $\tilde{\psi}(p) = Ap^a e^{-p}$, $\tilde{W}(s) = 1 - \text{ const } s^{2a+1} + O(s^2)$. Therefore,

$$\int_{-P}^{P} \frac{\tilde{W}(s) - \tilde{W}(0)}{s^2} [ds] \tag{A.13}$$

exists as a Riemann integral. Since the integrand is even it suffices to consider the case $s > 0$:

$$\int_0^P \frac{\tilde{W}(s) - \tilde{W}(0)}{s^2} [ds] = \int_0^P \frac{ds}{2\pi s^2} \int_{-\infty}^{\infty} dx |\psi(x)|^2 (e^{-isx} - 1) \tag{A.14}$$

Using the fact that only the even part of $(e^{-isx} - 1)$ contributes to the integral over $x$, we change the order of integration,

$$\int_0^P \frac{\tilde{W}(s) - \tilde{W}(0)}{s^2} [ds] = \int_{-\infty}^{\infty} dx |\psi(x)|^2 \int_0^P \frac{ds}{2\pi s^2} (\cos sx - 1). \tag{A.15}$$

The inner integral can be performed in terms of the sine integral function $Si$. Assuming that $\psi(x)$ is normalized to one, we have

$$\int_0^P \frac{\tilde{W}(s) - \tilde{W}(0)}{s^2} [ds] - \frac{\tilde{W}(0)}{2\pi P} = -\int_{-\infty}^{\infty} dx |\psi(x)|^2 v(x) \tag{A.16}$$

Where,

$$v(x) = \frac{1}{2\pi P} (\cos(Px) + PxSi(Px)) \tag{A.17}$$

The asymptotic expansion of $Si(t)$ for large $t$ is

$$Si(t) \sim \frac{\pi}{2} + \cos t (\frac{-1}{t} + O(\frac{1}{t^3})) + \sin t (\frac{-1}{t^2} + O(\frac{1}{t^4})) \tag{A.18}$$

Since the sine integral is odd, we have

$$v(x) = \frac{|x|}{4} + \frac{1}{2\pi P}(PxSi(Px) - \frac{P|x|\pi}{2} + \cos(Px)) = \frac{|x|}{4} + \frac{\text{Remainder}(Px)}{2\pi P} \quad \text{(A.19)}$$

We have the desired result, except for the remainder term:

$$\begin{aligned}\int_{-P}^{P} \frac{\tilde{W}(s) - \tilde{W}(0)}{s^2}[ds] - \frac{\tilde{W}(0)}{\pi P} &= -\int_{-\infty}^{\infty} |\psi(x)|^2 \frac{|x|}{2} dx \\ &\quad -\frac{1}{\pi P}\int_{-\infty}^{\infty} |\psi(x)|^2 \text{ Remainder}(Px)dx. \quad \text{(A.20)}\end{aligned}$$

In the $P = \infty$ limit, the remainder term vanishes since $-1 \leq \text{Remainder}(t) = tSi(t) - \frac{\pi|t|}{2} \leq 1$. For finite $P$, $\text{Remainder}(t) \sim \frac{-\sin t}{t}, |t| \to \infty$ is an oscillatory function, and we expect the remainder term to be small due to cancellations. In fact, it is zero. Consider

$$\begin{aligned}&\int_{-\infty}^{\infty} dx|\psi(x)|^2 \text{ Remainder}(Px) \\ &= \int_0^P [dq] \int_{-q}^{P-q} [dr]\tilde{\psi}(q+r)\tilde{\psi}^*(q) \int_{-\infty}^{\infty} dx e^{irx} \text{ Remainder}(Px). \quad \text{(A.21)}\end{aligned}$$

Here Remainder(t) is an even function and

$$\int_{-\infty}^{\infty} dx e^{irx} \text{ Remainder}(Px) = 2\int_0^{\infty} dx \cos(rx) \text{ Remainder}(Px) = 0, \quad \text{(A.22)}$$

from the properties of the sine integral, provided $|r| < P$, which is the region of interest. Thus the "Remainder" term vanishes, and we have shown that our definition of the "finite part" integral satisfies the boundary conditions. This justifies our definition

$$\mathcal{FP}\int_{-P}^{P} \frac{\tilde{W}(s)}{s^2}[ds] := \int_{-P}^{P} \frac{\tilde{W}(s) - \tilde{W}(0)}{s^2}[ds] - \frac{\tilde{W}(0)}{\pi P} \quad \text{(A.23)}$$

when $\tilde{W}'(0) = 0$.

# Appendix B

# One Matrix Model

## B.1 Equations of Motion and Classical Action

The observables of the single matrix model are the $U(N)$ invariants $\Phi_n = \frac{\mathrm{tr}}{N} A^n$ of a single $N \times N$ hermitian random matrix $A$. Their expectation values are obtained by averaging with respect to the Boltzmann measure $\frac{dA}{Z} e^{-N \mathrm{\ tr\ } S(A)}$ for a polynomial action $S(A) = \sum_{n=1}^{m} S_n A^n$. The large-$N$ limit is a classical limit where the fluctuations vanish $\langle \Phi_n \Phi_m \rangle = \langle \Phi_n \rangle \langle \Phi_m \rangle + \mathcal{O}(\frac{1}{N^2})$. In this limit, the large-$N$ moments $G_n$ are a complete set of observables.

$$\begin{aligned} G_n &= \lim_{N\to\infty} \langle \frac{\mathrm{tr}}{N} A^n \rangle = \lim_{N\to\infty} \frac{1}{Z_N(S)} \int dA e^{-N \mathrm{\ tr\ } S(A)} \frac{\mathrm{tr}}{N} A^n \\ Z_N(S) &= \int dA e^{-N \mathrm{\ tr\ } S(A)}; \quad F(S) = -\lim_{N\to\infty} \frac{1}{N^2} Z_N(S) \end{aligned} \tag{B.1}$$

$G_n$ parameterize the configuration space of the theory, which is the space of non-commutative probability distributions in a single operator-valued random variable: $\mathcal{P}_1$. The classical equations of motion are the factorized Schwinger Dyson or loop equations

$$\sum_l l \ S_l G_{k+l} = \sum_{p+q=k} G_p G_q := \eta_k, \quad k = 0, 1, 2, \cdots \tag{B.2}$$

These are recursion relations for the higher moments $G_n,\ n > m$ given the moments, $G_1, G_2, \cdots G_m$ conjugate to the coupling constants $S_1, \cdots S_m$. The factorized loop equations are obtained by requiring the partition function to be invariant under the infinitesimal non-linear change of variables $L_n : A \mapsto A + \epsilon A^{k+1},\ k = 0, 1, \cdots$. (The case $k = -1$

also leads to a valid equation, the RHS of (B.2) vanishes in this case, from the translation invariance of the measure.) The LHS comes from the change in $S(A)$ and the RHS $\eta_k$ is an anomaly coming from the change in the measure. The vector fields $L_k$ act on the moments

$$L_k G_l = l G_{k+l}; \qquad L_k = \sum_p p G_{k+p} \frac{\partial}{\partial G_p} \tag{B.3}$$

and satisfy commutation relations of the Virasoro algebra $[L_p, L_q] = (q-p)L_{p+q}$

Since $\langle \frac{\mathrm{tr}}{N} f^\dagger(A) f(A) \rangle \geq 0$ for any polynomial $f(A) = f_n A^n$, the Hankel matrix $G_{p,q} = G_{p+q}$ must be non-negative. This ensures that there is a probability density on the real line $\rho(x)$ whose moments are $G_n = \int \rho(x)\ x^n\ dx$. In fact, $\rho(x)$ is the density of eigenvalues of the random matrix $A$, as discussed in §5.3. The factorized loop equations turn into a linear integral equation for $\rho(x)$, which we call the Mehta-Dyson equation

$$S'(z) = 2\mathcal{P} \int \frac{\rho(x)}{z-x}. \tag{B.4}$$

The factorized loop equations can be obtained by multiplying either side by $z^{k+1}\rho(z)$ and integrating with respect to $z$. The Mehta Dyson equation follows from maximizing the action

$$\Omega(\rho) = \chi - \int \rho(x) S(x) dx; \quad \chi = \mathcal{P} \int \log|x-y| \rho(x)\rho(y) dx dy \tag{B.5}$$

with respect to $\rho$. As explained in §5.3, $\chi$ is the entropy due to our lack of knowledge of matrix elements of $A$ when we restrict measurements to the $U(N)$ invariants of $A$. The simplest possibility is a quadratic action $S(A) = \frac{1}{2}A^2$ which leads to the Wigner semicircular distribution $\rho(x) = \frac{1}{2\pi}\sqrt{[4-x^2]}\theta(|x|<2)$ discussed further in Appendix C. The maximization of $\Omega$ involves a balance: $\chi$ tends to spread the eigenvalues out due to the logarithmic repulsion, while $-\int \rho(x)S(x)dx$ is maximized if the eigenvalues are clustered near the minima of $S(x)$. For polynomial actions such that $S(x) \to \infty$ as $|x| \to \infty$, the eigenvalue distribution is supported over a finite number of intervals on the real line, located around the minima of $S(x)$. These intervals are the branch cuts of the moment generating function $G(z) = \sum_{n\geq 0} G_n z^{-n-1}$. $\rho(x)$ is the discontinuity of $-\frac{1}{\pi}G(z)$ across its branch cuts.

Can $\Omega(\rho)$ be expressed in terms of the moments in order to serve as a classical action principle for the factorized loop equations (B.2)? i.e. $L_k\Omega(G) = \eta_k - \sum_{l\geq 0} lS_lG_{k+l}$? Clearly, $L_k\Big(\sum_{l\geq 1} S_lG_l\Big) = \sum_{l\geq 1} lS_lG_{k+l}$. However, there is no power series in the moments $\chi(G)$ such that $L_k\chi(G) = \eta_k(G)$ . This is because $\log|x-y|$ cannot be expressed as a power series in both $x$ and $y$. This is a symptom of a cohomological obstruction, $\eta(G)$ is not exact, but is a closed one-form valued in the space of power series in the moments. The components of the one-form $\eta$ are obtained by acting on the basis vector fields $\eta_k = \eta(L_k)$. Now,

$$\begin{aligned} d\eta(L_p, L_q) &= L_p\eta(L_q) - L_q\eta(L_p) - \eta([L_p, L_q]) \\ &= L_p\eta_q - L_q\eta_p - (q-p)\eta_{p+q} \end{aligned} \tag{B.6}$$

and it is a straightforward calculation to check that $d\eta = 0$.

We can get around this obstruction by expressing both the moments $G_k$ and $\chi$ in terms of auxiliary variables. Suppose $\rho_\Gamma$ is a reference probability distribution and $\phi$ an invertible change of variables that maps $\rho_\Gamma$ to $\rho_G$

$$\begin{aligned} \rho_\Gamma(x) &= \rho_G(x)\phi'(x) \\ G_k &= \int \rho_\Gamma(y)\phi^k(y)dy \\ \chi(G) &= \chi(\Gamma) + \mathcal{P}\int dxdy\rho_\Gamma(x)\rho_\Gamma(y)\log\left|\frac{\phi(x)-\phi(y)}{x-y}\right| \\ &:= \chi(\Gamma) + \left\langle \log\left|\frac{\phi(x)-\phi(y)}{x-y}\right|\right\rangle_\Gamma \end{aligned} \tag{B.7}$$

If $\phi$ is a formal power series

$$\begin{aligned} \phi(x) &= \sum_{n=1}^{\infty}\phi_n x^n; \phi_1 > 0 \\ \phi(x) &= \phi_1[x + \tilde{\phi}(x)]; \tilde{\phi}(x) = \sum_{k=2}^{\infty}\tilde{\phi}_k x^k \end{aligned} \tag{B.8}$$

Then,

$$\begin{aligned} \chi(G) &= \chi(\Gamma) + \log\phi_1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{k_i+l_i>0}\tilde{\phi}_{k_1+1+l_1}\cdots\tilde{\phi}_{k_n+1+l_n}\Gamma_{k_1+\cdots k_n}\Gamma_{l_1+\cdots+l_n} \\ G_k &= \sum_{p=k}^{\infty}\Gamma_p\sum_{n_1+\cdots+n_k=p,\ n_i\geq 1}\phi_{n_1}\cdots\phi_{n_k} \end{aligned}$$

$$\Omega(G) \quad = \quad \Omega(\phi,\Gamma) = \chi(G) - \sum_{k\geq 1} S_k G_k \tag{B.9}$$

What we have done is to parameterize the configuration space $\mathcal{P}_1$ by automorphisms of the algebra of power series in one variable $\mathcal{G}_1 = \mathrm{Aut}\mathcal{T}_1$ which transform between probability distributions. Actually, we should restrict to the subgroup $\tilde{\mathcal{G}}_1$ of automorphisms that preserve the moment inequalities, i.e. the positivity of the probability distributions. In the case of a single generator that we are considering, the only measure-preserving automorphism is the identity, i.e. the isotropy subgroup $\mathcal{SG}_1$ is trivial. So we may identify $\mathcal{P}_1$ with $\tilde{\mathcal{G}}_1$. In other words, the moments $G_k$ and the Taylor coefficients of the automorphism $\phi_n$ both serve as coordinates on the classical configuration space. The left action of the vector fields $L_k$ on $\phi_n$ is (see §4.3)

$$\begin{aligned} \mathcal{L}_k\phi(x) &= \phi(x)^{k+1} \\ [\mathcal{L}_k\phi]_n &= \sum_{l_1+l_2+\cdots+l_{k+1}} \phi_{l_1}\cdots\phi_{l_{k+1}} \end{aligned} \tag{B.10}$$

Now we can check explicitly that $L_k\chi(G) = \eta_k(G)$

$$\begin{aligned} L_k\chi &= \sum_n [\mathcal{L}_k\phi]_n \frac{\partial\chi}{\partial\phi_n} = \sum_n [\mathcal{L}_k\phi]_n \frac{\partial}{\partial\phi_n}\left\langle \log\left|\frac{\phi(x)-\phi(y)}{x-y}\right|\right\rangle_\Gamma \\ &= \sum_n [\mathcal{L}_k\phi]_n \left\langle \frac{x^n-y^n}{\phi(x)-\phi(y)}\right\rangle_\Gamma = \sum_n \sum_{l_1+\cdots+l_{k+1}=n} \phi_{l_1}\cdots\phi_{l_{k+1}} \left\langle \frac{x^n-y^n}{\phi(x)-\phi(y)}\right\rangle_\Gamma \\ &= \sum_n \left\langle \frac{\phi^{k+1}(x)-\phi^{k+1}(y)}{\phi(x)-\phi(y)}\right\rangle_\Gamma = \sum_{p+q=k} \langle\phi^p(x)\phi^q(y)\rangle_\Gamma \\ &= \sum_{p+q=k} G_pG_q = \eta_k \end{aligned} \tag{B.11}$$

Thus, we have a classical action $\Omega(\phi,\Gamma) = \chi(G) - \sum_n S_nG_n$ whose extremization leads to the factorized loop equations. This leads to a variational principle to solve the classical equations of motion: pick a reference distribution $\Gamma$ such as the Wigner distribution (see Appendix C); maximize entropy $\chi$ with respect to the automorphism $\phi$, while holding the moments $G_n$ conjugate to the coupling constants fixed (via the Lagrange multipliers $S_n$). The free energy $F$ is the negative of the maximum value of entropy. As a variational ansatz, we may pick $\phi$ to be a polynomial, its Taylor coefficients are the variational parameters. They parameterize a finite dimensional space of probability distributions in the neighborhood of $\Gamma$. The extremization of $\Omega$ will give the probability distribution in

this finite dimensional family that best approximates the exact moments of the action $S(A)$. We give below a variational approximation for the quartic one-matrix model.

## B.2 Mean Field Theory for Quartic One Matrix Model

The exact moments and free energy for the quartic one-matrix model were obtained by Brezin et. al. [18]. We use this as a way of calibrating our variational approach

$$\begin{aligned} Z(g) &= \int dA e^{-N \text{ tr } [\frac{1}{2}A^2 + gA^4]} \\ E(g) &= F(g) - F(0) = -\lim_{N\to\infty} \frac{1}{N^2} \log \frac{Z(g)}{Z(0)} \end{aligned} \tag{B.12}$$

We take the Wigner distribution with $\Gamma_2 = 1$ as our reference (see Appendix C). Pick a linear automorphism $\phi(x) = \phi_1 x$, which merely scales the width of the Wigner distribution. The $\phi_1$ that maximizes $\Omega$ represents the Wigner distribution that best approximates the quartic matrix model.

$$\Omega(\phi_1) = \log \phi_1 - \frac{1}{2} G_2(\phi_1) - g G_4(\phi_1) \tag{B.13}$$

Here $G_{2k} = \phi_1^{2k} \Gamma_{2k}$. Let $\alpha = \phi_1^2$. $\Omega(\alpha) = \frac{1}{2}\log\alpha - \frac{\alpha}{2} - 2g\alpha^2$ is bounded above only for $g \geq 0$. The maximum occurs at $\alpha(g) = \frac{-1+\sqrt{1+32g}}{16g}$. Notice that $\alpha$ is determined by a non-linear equation. The linear change of variable ansatz from a wignerian reference is an analogue of mean-field theory for large-$N$ matrix models. Our variational estimates are ($C_k$ are Catalan numbers, see Appendix C):

$$\begin{aligned} E(g) &= -\frac{1}{2} \log \frac{-1+\sqrt{1+32g}}{16g} \\ G_{2k}(g) &= (\frac{-1+\sqrt{1+32g}}{16g})^k C_k. \end{aligned} \tag{B.14}$$

The exact results from [18] are:

$$\begin{aligned} E_{ex}(g) &= \frac{1}{24}(a^2(g)-1)(9-a^2(g)) - \frac{1}{2}\log(a^2(g)) \\ G_{ex}^{2k}(g) &= \frac{(2k)!}{k!(k+2)!} a^{2k}(g)[2k+2-ka^2(g)]. \end{aligned} \tag{B.15}$$

where $a^2(g) = \frac{1}{24g}[-1+\sqrt{1+48g}]$. In both cases, the free energy is analytic at $g = 0$ with a square root branch point at a negative critical coupling. The mean-field critical

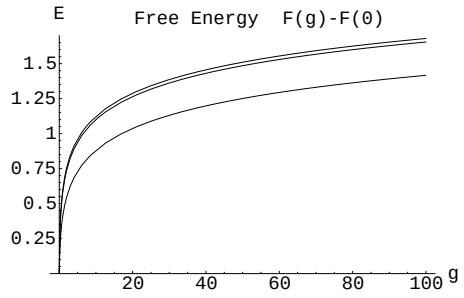


Figure B.1: Free energy $E$ versus $g$. Curves top to bottom are: mean-field ansatz, $\phi(x) = \phi_1 x + \phi_3 x^3$ ansatz calculated to first order beyond mean-field theory, and the exact energy.

coupling $g_c^{MF} = -\frac{1}{32}$ is 50% more than the exact value $g_c^{ex} = -\frac{1}{48}$. The variational and exact free energies are compared in figure B.1. In the weak coupling region

$$\begin{aligned} E_{ex}(g) &= 2g - 18g^2 + \dots & E(g) &= 4g - 48g^2 + \dots \\ G^2_{ex}(g) &= 1 - 8g + 144g^2 + \dots & G^2(g) &= 1 - 8g + 128g^2 + \dots \\ G^4_{ex}(g) &= 2 - 36g + \dots & G^4(g) &= 2 - 32g + \dots \end{aligned} \tag{B.16}$$

and asymptotically, as $g \to \infty$

$$\begin{aligned} E_{ex}(g) &\sim \frac{1}{4}\log g + \frac{\log(144) - 3}{8} - \frac{1}{3\sqrt{3g}} + \dots & E(g) &= \frac{1}{4}\log g + \frac{1}{4}\log 8 + \frac{1}{8\sqrt{2g}} + \dots \\ G^2_{ex}(g) &\sim \frac{2}{3\sqrt{3g}} - \frac{1}{12g} + \dots & G^2(g) &= \frac{1}{\sqrt{8g}} - \frac{1}{16g} + \dots \\ G^4_{ex}(g) &\sim \frac{1}{4g} - \frac{1}{6\sqrt{3}}\frac{1}{g^{\frac{3}{2}}} + \dots & G^4(g) &= \frac{1}{4g} - \frac{1}{8\sqrt{2}}\frac{1}{g^{\frac{3}{2}}} + \dots \end{aligned} \tag{B.17}$$

The distribution of eigenvalues of the best wignerian approximation is given by $\rho_G(x) = \phi_1^{-1}\rho_\Gamma(\phi_1^{-1}x)$ where $\rho_\Gamma(x) = \frac{1}{2\pi}\sqrt{4 - x^2},\ |x| \le 2$ is the reference Wigner distribution. The exact distribution

$$\rho_{ex}(x, g) = \frac{1}{\pi}(\frac{1}{2} + 4ga^2(g) + 2gx^2)\sqrt{4a^2(g) - x^2},\ |x| \le 2a(g). \tag{B.18}$$

is compared with the best wignerian approximation in figure B.2. The latter does not capture the bimodal property of the former. The wignerian ansatz is like mean-field theory, but is not restricted to small values of the coupling $g$. To improve on this, get a non-trivial estimate for the higher cumulants (connected moments) and capture the bimodal distribution of eigenvalues, we need to make a non-linear change of variable.

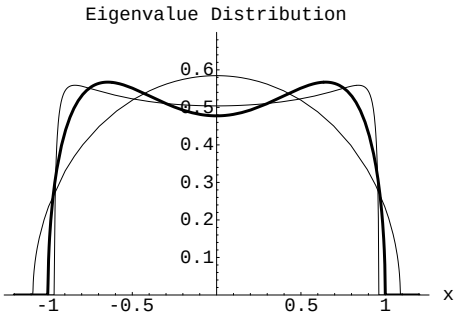


Figure B.2: Eigenvalue Distribution. Dark curve is exact, semicircle is mean-field and bi-modal light curve is cubic ansatz at 1-loop.

## B.3 Beyond Mean Field Theory: Non-linear Ansatz

To go beyond mean-field theory while sticking to the standard Wigner distribution as the reference, we must make a cubic change of variable. $\phi(x) = \phi_1 x + \phi_3 x^3$. A quadratic ansatz will not increase $\Omega(\phi)$ since $S(A)$ is even. $\phi_{1,3}$ are determined by the condition that $\Omega$ is maximal

$$\begin{aligned}
\Omega[\phi] &= \chi - \frac{1}{2}G_2 - gG_4 \\
\chi[\phi] &= \int dxdy\rho_\Gamma(x)\rho_\Gamma(y)\log\left|\frac{\phi(x)-\phi(y)}{x-y}\right| \\
G_4 &= \int dx\rho_\Gamma(x)\phi^4(x) = \phi_1^4\Gamma^4 + 4\phi_1^3\phi_3\Gamma_6 + 6\phi_1^2\phi_3^2\Gamma_8 + 4\phi_1\phi_3^3\Gamma_10 + \phi_3^4\Gamma_{12} \\
G_2 &= \int dx\rho_\Gamma(x)\phi^2(x) = \phi_1^2\Gamma_2 + 2\phi_1\phi_3\Gamma_4 + \phi_3^2\Gamma_6
\end{aligned} \tag{B.19}$$

We were not able to evaluate the integral for $\chi$ exactly. For multi-matrix models, the problem is worse, since we do not have such a closed form expression for $\chi$ and have to rely on the power series (5.39). However, considering the success of the linear change of variable, we expect the deviations of $\phi_{1,3}$ from their mean-field values $(\sqrt{\alpha}, 0)$ to be small. We evaluate $\Omega$ using the power series (B.9) and solve for the corrections to $\phi_{1,3}$ by linearizing around their mean-field values. This may be regarded as a kind of 1-loop correction beyond mean-field theory, though it is not restricted to small values of the coupling $g$. Within this approximation, we get (with $\alpha = \frac{-1+\sqrt{1+32g}}{16g}$)

$$\phi_1 = \sqrt{\alpha} - \frac{\sqrt{\alpha}(-3+2\alpha+(1-32g)\alpha^2+48g\alpha^3+144g^2\alpha^4)}{3+4\alpha+(1+96g)\alpha^2+48g\alpha^3+432g^2\alpha^4}$$

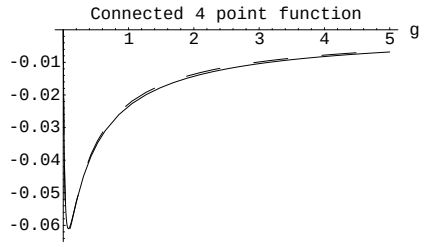


Figure B.3: Connected $4^{\text{th}}$ moment (cumulant). Dashed curve is estimate from cubic ansatz. Solid curve is exact. Mean field estimate was identically zero

$$\phi_3 \quad = \quad \frac{8g\alpha^{\frac{5}{2}}(-2+\alpha)}{3+4\alpha+(1+96g)\alpha^2+48g\alpha^3+432g^2\alpha^4}. \tag{B.20}$$

from which we calculate the variational moments and free energy. Comparing with the exact results of [18], we find the following qualitative improvements over the mean-field ansatz.

In addition to the mean-field branch cut from $-\infty$ to $g_c^{MF}$, the vacuum energy now has a double pole at $g_c^{MF} < g_c^{1-\text{loop}} = \frac{-346-25\sqrt{22}}{15138} < g_c^{ex}$. We can understand this double pole as a sort of Padé approximation to a branch cut that runs all the way up to $g_c^{ex}$. The vacuum energy variational estimate is lowered for all $g$.

Figure B.2 demonstrates that the cubic ansatz is able to capture the bimodal nature of the exact eigenvalue distribution. If $\psi(x) = \phi^{-1}(x)$, then $\rho(x) = \rho_0(\psi(x))\psi'(x)$, where $\rho_0(x) = \frac{1}{2\pi}\sqrt{4-x^2},\ |x| \leq 2$.

The moments $G_2, G_4$ are now within a percent of their exact values, for all $g$. More significantly, the connected 4-point function $G_4^c = G_4 - 2(G_2)^2$ which vanished for the wignerian ansatz, is non-trivial, and within 10% of its exact value, across all values of $g$ (figure B.3).

# Appendix C

# Single and Multivariate Wigner Distribution

The analogue of the gaussian for operator-valued random variables is the Wigner semi-circular distribution. Indeed, the Wigner distribution of matrices is realized when the matrix elements are distributed as independent gaussians. The Wigner distribution derives its importance from the fact that it maximizes entropy (see §4.4, 5.3) among all non-commutative probability distributions with given variance. We collect some facts about the Wigner distribution here. This section relies on some results on matrix models which can be found in Appendix B and Chapter 5.

Let us first consider the case of a single hermitian $N \times N$ matrix $A$ whose elements are gaussian distributed with zero mean and unit variance. The moments of $A$ in the large-$N$ limit are

$$\Gamma_n = \lim_{N\to\infty} \frac{1}{Z_N} \int dA e^{-N \text{ tr } \frac{1}{2}A^2} \frac{\text{tr}}{N} A^n; \quad Z_N = \int dA e^{-N \text{ tr } \frac{1}{2}A^2} \tag{C.1}$$

The Wigner moments satisfy the so-called factorized loop equations (Appendix B) which are recursion relations

$$\Gamma_{n+1} = \sum_{p+1+q=n} \Gamma_p \Gamma_q \tag{C.2}$$

They may be solved in terms of Catalan numbers $C_n$

$$\Gamma_{2n} = C_n = \frac{(2n)!}{n!(n+1)!}; \quad \Gamma_{2n+1} = 0; \quad n = 0, 1, 2 \cdots \tag{C.3}$$

The first few are $\Gamma_0 = 1,\ \Gamma_2 = 1,\ \Gamma_4 = 2,\ \Gamma_6 = 5$. The Wigner moments are realized by the semicircular distribution of eigenvalues $\Gamma_k = \int \rho(x) x^k dx$. Maximizing non-commutative entropy (§4.4, 5.3) holding the variance fixed

$$\begin{aligned} 0 &= \delta\Big( \int dxdy\rho(x)\rho(y)\log|x-y| - \frac{1}{2}\int dx\rho(x)x^2 \Big) \\ \Rightarrow\ z &= 2\mathcal{P}\int_{\mathrm{R}} \frac{\rho(x)}{z-x}dx \end{aligned} \tag{C.4}$$

gives the Wigner semi-circular distribution $\rho(x) = \theta(|x|<2)\frac{1}{2\pi}\sqrt{4-x^2}$ [18].

The analogue of the gaussian process and heat equation of real-valued random variables are the semi-circular process and complex Burger equation in non-commutative probability theory [99]. A semi-circular process starting at $X$ is a time dependent operator-valued random variable $Y(t) = X + \sqrt{t}S$ where $S$ is semicircular $< S^n >= \Gamma_n$. Let $G(z,t) = \sum_{n\geq 0} \frac{<Y(t)^n>}{z^{n+1}}$ be the moment generating function of $Y(t)$. Then, $G(z,t)$ satisfies the analogue of the heat equation, the complex Burger equation:

$$\frac{\partial G}{\partial t} + G\frac{\partial G}{\partial z} = 0 \tag{C.5}$$

The standard multivariate Wigner distribution corresponds to a gaussian $M$ matrix model with action $S_0(A) = \frac{1}{2}\sum_{i=1}^{M} A_i^2$ in the large-$N$ limit

$$\begin{aligned} \Gamma_I &= \lim_{N\to\infty} \int dA e^{-\frac{1}{2}N\delta^{ij}\ \mathrm{tr}\ A_iA_j}\frac{dA}{Z_N} \\ Z_N &= \int dA e^{-\frac{1}{2}N\delta^{ij}\ \mathrm{tr}\ A_iA_j} \end{aligned} \tag{C.6}$$

The odd moments vanish as before and the second moment is

$$\Gamma_{ij} = \delta_{ij} \tag{C.7}$$

The factorized loop equations (5.10) are again recursion relations

$$\Gamma_{jI} = \Gamma_{ji}\delta_I^{I_1 i I_2}\Gamma_{I_1}\Gamma_{I_2} \tag{C.8}$$

involving the order-preserving partition of the original string $I$ into sub-strings. By iterating, we may reduce the evaluation of an arbitrary moment to a non-crossing sum over products of the second moment

$$\Gamma_{i_1\cdots i_n} = \sum_{\pi\in \mathrm{NCP}_2(i_1\cdots i_n)} \prod_{\{i_a,i_b\}\in\pi} \Gamma_{i_a i_b} \tag{C.9}$$

This is Wick's theorem for Feynman diagrams of planar topology. $\mathrm{NCP}_2(i_1, \cdots, i_n)$ is the set of non-crossing partitions of the cyclically symmetric set of indices $(i_1 \cdots i_n)$ into pairs. For example

$$\begin{aligned} \Gamma_{ij} &= \delta_{ij} \\ \Gamma_{ijkl} &= \Gamma_{ij}\Gamma_{kl} + \Gamma_{il}\Gamma_{jk} \\ \Gamma_{ijklmn} &= \Gamma_{ij}\Gamma_{kl}\Gamma_{mn} + \Gamma_{ni}\Gamma_{jk}\Gamma_{lm} + \Gamma_{il}\Gamma_{jk}\Gamma_{mn} + \Gamma_{ij}\Gamma_{kn}\Gamma_{lm} + \Gamma_{jm}\Gamma_{ni}\Gamma_{kl} \end{aligned} \tag{C.10}$$

The Wigner distribution serves as a candidate for a reference distribution from which other distributions may be obtained by a change of variable. For example, a Wigner distribution with positive covariance $G_{ij} = [K^{-1}]_{ij}$ which arises from the action $S(A) = \frac{1}{2}K^{ij}A_iA_j$ can be obtained from the standard one. Use the invertible linear transformation $A_i \mapsto \phi_i^j A_j$ where $G_{ij} = \Gamma_{kl}\phi_i^k\phi_j^l$. By the polar decomposition, the space of Wigner distributions is the coset space $GL(M)/O(M)$.

We can use a similar change of variable along with our variational principle to find the best wignerian variational approximation to a matrix model. It is a kind of mean-field theory, as discussed in §5.4.

# Appendix D

# Anomaly is a Closed 1-form

The conditions for $\eta$ to be a closed 1-form are (5.16)

$$L_i^I \eta_j^J - L_j^J \eta_i^I = \delta^J_{J_1 i J_2} \eta_j^{J_1 I J_2} - \delta^I_{I_1 j I_2} \eta_i^{I_1 J I_2}. \tag{D.1}$$

where $\eta_i^I = \delta^I_{I_1 i I_2} G^{I_1} G^{I_2}$. Let us check that these are satisfied. The RHS of (5.16) is:

$$\begin{aligned}
\text{RHS} &= G^{K_1} G^{K_2} \Big[ \delta^J_{J_1 i J_2} \delta^{J_1 I J_2}_{K_1 j K_2} - (i \leftrightarrow j, I \leftrightarrow J) \Big] \\
&= G^{K_1} G^{K_2} \Big[ \delta^J_{J_1 i J_2} (\delta^{J_1}_{K_1 j P} \delta^{P I J_2}_{K_2} + \delta^I_{P j Q} \delta^{J_1 P}_{K_1} \delta^{Q J_2}_{K_2} + \delta^{J_2}_{P j K_2} \delta^{J_1 I P}_{K_1}) - (i \leftrightarrow j, I \leftrightarrow J) \Big] \\
&= G^{K_1} G^{K_2} \Big[ \delta^J_{K_1 j J_1 i J_2} \delta^{J_1 I J_2}_{K_2} + \delta^J_{J_1 i J_2 j K_2} \delta^{J_1 I J_2}_{K_1} - (i \leftrightarrow j, I \leftrightarrow J) \Big]
\end{aligned} \tag{D.2}$$

We cancelled the middle terms using acyclicity of $G^K$. On the other hand,

$$\begin{aligned}
L_i^I \eta_j^J &= L_i^I \delta^J_{J_1 j J_2} G^{J_1} G^{J_2} \\
&= \delta^J_{J_1 j J_2} [\delta^{J_1}_{K_1 i K_2} G^{K_1 I K_2} G^{J_2} + \delta^{J_2}_{K_1 i K_2} G^{J_1} G^{K_1 I K_2}] \\
&= \delta^J_{J_1 i J_2 j J_3} G^{J_1 I J_2} G^{J_3} + \delta^J_{J_1 j J_2 i J_3} G^{J_1} G^{J_2 I J_3} \\
&= G^{J_1 I J_2} G^{J_3} [\delta^J_{J_1 i J_2 j J_3} + \delta^J_{J_3 j J_1 i J_2}] \\
&= G^{K_1} G^{K_2} [\delta^J_{J_1 i J_2 j K_2} \delta^{J_1 I J_2}_{K_1} + \delta^J_{K_1 j J_1 i J_2} \delta^{J_1 i J_2}_{K_2}]
\end{aligned} \tag{D.3}$$

Thus the LHS of (5.16) is:

$$L_i^I \eta_j^J - L_j^J \eta_i^I = G^{K_1} G^{K_2} [\delta^J_{J_1 i J_2 j K_2} \delta^{J_1 I J_2}_{K_1} + \delta^J_{K_1 j J_1 i J_2} \delta^{J_1 i J_2}_{K_2} - (i \leftrightarrow j, I \leftrightarrow J)] \tag{D.4}$$

Comparing (D.4) and (D.2), we see that the integrability condition (5.16) is satisfied. Thus, $\eta_i^I$ is a closed one-form.

# Appendix E

# Group Cohomology

It is natural in mathematics to study the cohomology of a group twisted by a representation [79, 33]. Given a group $G$ and a representation of $G$ on a vector space $V$ ($V$ is called a $G$ module), we can define a cohomology theory. We will sometimes call this the cohomology of $G$ valued in $V$. The $r$-cochains are functions

$$c : G^r \to V. \tag{E.1}$$

The coboundary $d$ is defined on $r$-cochains as

$$\begin{aligned} dc(g_1, g_2, \cdots, g_{r+1}) &= g_1 c(g_2, \cdots, g_{r+1}) \\ &\quad + \sum_{s=1}^{r} (-1)^s c(g_1, g_2, \cdots, g_{s-1}, g_s g_{s+1}, g_{s+2}, \cdots, g_{r+1}) \\ &\quad + (-1)^{r+1} c(g_1, \cdots, g_r). \end{aligned} \tag{E.2}$$

and satisfies $d^2 = 0$. A closed cochain ($dc = 0$) is called a cocycle. An exact cochain ($b = dc$ for some $c$) is called a coboundary. The $r^{\text{th}}$ cohomology of $G$ twisted by the representation of $G$ on $V$, $H^r(G, V)$ is the space of closed $r$-cochains modulo exact $r$-cochains.

For example, $H^0(G, V)$ is the space of all $G$-invariant elements of $V$, i.e. the space of $v \in V$ satisfying $gv - v = 0$ for all $g \in G$. A 1-cocycle is a function $c : G \to V$ that is closed,

$$dc(g_1, g_2) = g_1 c(g_2) - c(g_1 g_2) + c(g_1) = 0$$

$$\text{i.e.} \qquad c(g_1 g_2) \;=\; g_1 c(g_2) + c(g_1). \tag{E.3}$$

The space of solutions to this equation modulo 1-coboundaries (which are of the form $b(g) = gv - v$ for some $v \in V$) is the first cohomology $H^1(G, V)$. If $G$ acts trivially on $V$, a cocycle is just a homomorphism from $G$ to the additive group of $V$: $c(g_1 g_2) = c(g_2) + c(g_1)$.

As an example consider the loop group $G = S^1 G' = \{g : S^1 \to G'\}$ of a Lie group $G'$. And let $V = S^1 \underline{G'}$ be the loop of the Lie algebra $\underline{G}$. Then there is the adjoint representation of $G$ on $V$: $\mathrm{ad}_g v = gvg^{-1}$. A non-trivial 1-cocycle is $c : G \to V$, $c(g) = gdg^{-1}$. It satisfies the co-cycle condition (E.3)

$$c(gh) = ghd(h^{-1}g^{-1}) = g[hdh^{-1}]g^{-1} + ghh^{-1}dg^{-1} = \mathrm{ad}_g c(h) + c(g). \tag{E.4}$$

# Appendix F

# Formula for Entropy

Here we show that $\sigma(\phi, A) = \frac{1}{N^2} \log\det J(\phi, A)$ can be expressed in terms of the loop variables $\Phi_I$ and the coefficients of the automorphism $\phi_i^I$. The jacobian matrix of the change of variable

$$\begin{aligned}
\phi(A)_i &= [\phi_1]_i^j \tilde{\phi}(A)_j; \quad \tilde{\phi}(A)_i = A_i + \sum_{|I|\geq 2} \tilde{\phi}_i^I A_I \\
\text{is} \quad J_{ib\ c}^{\ a\ jd}(\phi, A) &= \frac{\partial \phi(A)_{ib}^{\ a}}{\partial A_{jd}^{\ c}} = [\phi_1]_i^k \frac{\partial [\tilde{\phi}(A)]_{kb}^{\ a}}{\partial A_{jd}^{\ c}} \\
&= [\phi_1]_i^k \left\{ \delta_k^j \delta_c^a \delta_b^d + \sum_{m+n\geq 1} \tilde{\phi}_k^{i_1 \cdots i_m j j_1 \cdots j_n} [A_{i_1 \cdots i_m}]_c^a [A_{j_1 \cdots j_n}]_b^d \right\} \\
&= [\phi_1]_i^k \left\{ \delta_k^j \delta_c^a \delta_b^d + + K_{kc\ b}^{\ ajd}(A) \right\} \qquad \text{(F.1)}
\end{aligned}$$

Or suppressing color indices,

$$J_i^j(\phi, A) = [\phi_1]_i^k \left\{ \delta_k^j 1 \otimes 1 + \sum_{|I|+|J|\geq 1} \tilde{\phi}_k^{IjJ} A_I \otimes A_J \right\} \equiv [\phi_1]_i^k \left\{ \delta_k^j 1 \otimes 1 + K_k^j(A) \right\} \qquad \text{(F.2)}$$

Then

$$\begin{aligned}
\sigma(\phi, A) &= \frac{1}{N^2} \log\det[\phi_1(1 + K(A))] \\
&= \log\det \phi_1 + \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \frac{1}{N^2} \text{ tr } K^n \qquad \text{(F.3)}
\end{aligned}$$

So we calculate

$$\frac{1}{N^2} \text{ tr } K^n(A) = \frac{1}{N^2} K_{i_1 b_1\ a_2}^{a_1\ i_2 b_2} K_{i_2 b_2\ a_2}^{a_2\ i_3 b_3} \cdots K_{i_n b_n\ a_1}^{a_n\ i_1 b_1}$$

$$
\begin{aligned}
& = \tilde{\phi}_{i_1}^{K_1 i_2 L_1} \tilde{\phi}_{i_2}^{K_2 i_3 L_2} \cdots \tilde{\phi}_{i_n}^{K_n i_1 L_n} \frac{1}{N} [A_{K_1}]_{a_2}^{a_1} \cdots [A_{K_n}]_{a_1}^{a_n} \frac{1}{N} [A_{L_1}]_{b_1}^{b_2} \cdots [A_{L_n}]_{b_n}^{b_1} \\
\frac{1}{N^2} \text{ tr } K^n(A) & = \tilde{\phi}_{i_1}^{K_1 i_2 L_1} \tilde{\phi}_{i_2}^{K_2 i_3 L_2} \cdots \tilde{\phi}_{i_n}^{K_n i_1 L_n} \Phi_{K_1 \cdots K_n} \Phi_{L_n \cdots L_1} \qquad \text{(F.4)}
\end{aligned}
$$

Thus

$$
\sigma(\phi, A) = \log \det \phi_1 + \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \tilde{\phi}_{i_1}^{K_1 i_2 L_1} \tilde{\phi}_{i_2}^{K_2 i_3 L_2} \cdots \tilde{\phi}_{i_n}^{K_n i_1 L_n} \Phi_{K_1 \cdots K_n} \Phi_{L_n \cdots L_1} \qquad \text{(F.5)}
$$

# Bibliography

[1] A. Agarwal, L. Akant, G. S. Krishnaswami and S. G. Rajeev, *Collective potential for large-N Hamiltonian matrix models and free Fisher information,* Int. J. Mod. Phys. A **18**, 917 (2003) [arXiv:hep-th/0207200].

[2] A. Agarwal and S. G. Rajeev, *A cohomological interpretation of the Migdal-Makeenko equations,* Mod. Phys. Lett. A **17**, 481 (2002) [arXiv:hep-th/0202095].

[3] A. Agarwal and S. G. Rajeev, *The dilatation operator of N = 4 SYM and classical limits of spin chains and matrix models,* arXiv:hep-th/0405116.

[4] L. Akant, *Entropy of Large Dimensional Random Matrices*, Ph.D. Thesis, University of Rochester 2003.

[5] L. Akant, G. S. Krishnaswami and S. G. Rajeev, *Entropy of operator-valued random variables: A variational principle for large N matrix models,* Int. J. Mod. Phys. A **17**, 2413 (2002) [arXiv:hep-th/0111263].

[6] G. Akemann and P. H. Damgaard, *Wilson loops in N = 4 supersymmetric Yang-Mills theory from random matrix theory,* Phys. Lett. B **513**, 179 (2001) [Erratum-ibid. B **524**, 400 (2002)] [arXiv:hep-th/0101225].

[7] N. I. Akhiezer, *The Classical Moment Problem*, Hafner Publishing Company, New York (1965).

[8] J. Alfaro, *The Large N limit of the two Hermitian matrix model by the hidden BRST method,* Phys. Rev. D **47**, 4714 (1993) [arXiv:hep-th/9207097].

[9] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York (1989).

[10] P. Austing and J. F. Wheater, *The convergence of Yang-Mills integrals,* JHEP **0102**, 028 (2001) [arXiv:hep-th/0101071].

[11] A. P. Balachandran, *Skyrmions,* SU-4222-314 *Lectures given at Theoretical Advanced Study Inst. in Elementary Particle Physics, New Haven, CT, Jun 9 - Jul 5, 1985* ed. M. J. Bowick and F. Gursey, World Scientific, Singapore (1986).

[12] A. P. Balachandran, V. P. Nair, S. G. Rajeev and A. Stern, *Exotic Levels From Topology In The QCD Effective Lagrangian,* Phys. Rev. Lett. **49**, 1124 (1982) [Erratum-ibid. **50**, 1630 (1983)]. G. S. Adkins, C. R. Nappi and E. Witten, *Static Properties Of Nucleons In The Skyrme Model,* Nucl. Phys. B **228**, 552 (1983).

[13] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *M theory as a matrix model: A conjecture,* Phys. Rev. D **55**, 5112 (1997) [arXiv:hep-th/9610043].

[14] P. F. Bedaque, I. Horvath and S. G. Rajeev, *Two-Dimensional Baryons In The Large N Limit,* Mod. Phys. Lett. A **7**, 3347 (1992) [arXiv:hep-th/9209027].

[15] M. Beneke, *Renormalons,* Phys. Rept. **317**, 1 (1999) [arXiv:hep-ph/9807443].

[16] J. P. Berge *. et. al.* Z. Phys. C49 (1991) 187-223; W. C. Leung et. al., Phys.Lett.B317 (1993) 655; P. Z. Quintas et. al., Phys.Rev.Lett. 71 (1993) 1307.

[17] J. D. Bjorken and E. A. Paschos, *Inelastic Electron Proton And Gamma Proton Scattering, And The Structure Of The Nucleon,* Phys. Rev. **185**, 1975 (1969).

[18] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, *Planar Diagrams,* Commun. Math. Phys. **59**, 35 (1978).

[19] E. Brezin and S. R. . Wadia, *The Large N Expansion In Quantum Field Theory And Statistical Physics: From Spin Systems To Two-Dimensional Gravity,* World Scientific, Singapore (1993).

[20] E. Brezin and A. Zee, *Universality of the correlations between eigenvalues of large random matrices,* Nucl. Phys. B **402**, 613 (1993).

[21] R. Brock *et al.* [CTEQ Collaboration], *Handbook of perturbative QCD: Version 1.0,* Rev. Mod. Phys. **67**, 157 (1995).

[22] F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, *Chiral rings and anomalies in supersymmetric gauge theory,* JHEP **0212**, 071 (2002) [arXiv:hep-th/0211170].

[23] S. Chandrasekhar *An Introduction to the Study of Stellar Structure* Dover 1958.

[24] S. S. Chern, *Complex Manifolds without Potential Theory*, Springer-Verlag, New York (1979).

[25] M. Creutz, *Monte Carlo Study Of Quantized SU(2) Gauge Theory,* Phys. Rev. D **21**, 2308 (1980).

[26] P. Cvitanovic, *Planar Perturbation Expansion,* Phys. Lett. B **99**, 49 (1981).

[27] P. Cvitanovic, P. G. Lauwers and P. N. Scharbach, *The Planar Sector Of Field Theories,* Nucl. Phys. B **203**, 385 (1982).

[28] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings, and supersymmetric gauge theories,* Nucl. Phys. B **644**, 3 (2002), [arXiv:hep-th/0206255]; *On geometry and matrix models,* Nucl. Phys. B **644**, 21 (2002), [arXiv:hep-th/0207106]; *A perturbative window into non-perturbative physics,* [arXiv:hep-th/0208048].

[29] N. Drukker and D. J. Gross, *An exact prediction of N = 4 SUSYM theory for string theory,* J. Math. Phys. **42**, 2896 (2001) [arXiv:hep-th/0010274].

[30] M. R. Douglas, *Stochastic master fields,* Phys. Lett. B **344**, 117 (1995);

[31] M. R. Douglas and M. Li, "Free variables and the two matrix model," Phys. Lett. B **348**, 360 (1995) [hep-th/9412203].

[32] T. Eguchi and H. Kawai, *Reduction Of Dynamical Degrees Of Freedom In The Large N Gauge Theory,* Phys. Rev. Lett. **48**, 1063 (1982).

[33] L. Evans, *The Cohomology of Groups*, Oxford (1992).

[34] B. Eynard, *Large N expansion of the 2-matrix model,* JHEP **0301**, 051 (2003) [arXiv:hep-th/0210047].

[35] B. Eynard and C. Kristjansen, *An Iterative Solution Of The Three-Colour Problem On A Random Lattice,* Nucl. Phys. B **516**, 529 (1998) [arXiv:cond-mat/9710199].

[36] R. P. Feynman *Photon Hadron Interactions* Benjamin, Reading MA (1972).

[37] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, *2-D Gravity and random matrices,* Phys. Rept. **254**, 1 (1995).

[38] J. I. Friedman and H. W. Kendall, *Deep Inelastic Electron Scattering,* Ann. Rev. Nucl. Part. Sci. **22**, 203 (1972).

[39] K. Fujikawa, *Path Integral Measure For Gauge Invariant Fermion Theories,* Phys. Rev. Lett. **42**, 1195 (1979).

[40] M. Gell-Mann, *A Schematic Model Of Baryons And Mesons,* Phys. Lett. **8**, 214 (1964).

[41] H. Georgi and H. D. Politzer, *Freedom At Moderate Energies: Masses In Color Dynamics,* Phys. Rev. D **14**, 1829 (1976). R. Barbieri, J. R. Ellis, M. K. Gaillard and G. G. Ross, *Mass Corrections To Scaling In Deep Inelastic Processes,* Nucl. Phys. B **117**, 50 (1976).

[42] P. Ginsparg and G. W. Moore, *Lectures On 2-D Gravity And 2-D String Theory,* arXiv:hep-th/9304011.

[43] R. Gopakumar and D. J. Gross, *Mastering the master field,* Nucl. Phys. B **451**, 379 (1995) [arXiv:hep-th/9411021].

[44] M. B. Green, J. H. Schwarz and E. Witten,

[45] V. N. Gribov and L. N. Lipatov, *Deep Inelastic E P Scattering In Perturbation Theory,* Yad. Fiz. **15**, 781 (1972) [Sov. J. Nucl. Phys. **15**, 438 (1972)]. G. Altarelli

and G. Parisi, *Asymptotic Freedom In Parton Language,* Nucl. Phys. B **126**, 298 (1977). Y. L. Dokshitzer, *Calculation Of The Structure Functions For Deep Inelastic Scattering And E+E- Annihilation By Perturbation Theory In Quantum Chromodynamics. (In Russian),* Sov. Phys. JETP **46**, 641 (1977) [Zh. Eksp. Teor. Fiz. **73**, 1216 (1977)].

[46] D. J. Gross and C. H. Llewellyn Smith, *High-Energy Neutrino - Nucleon Scattering, Current Algebra And Partons,* Nucl. Phys. B **14**, 337 (1969). S. A. Larin and J. A. M. Vermaseren, *The Alpha-S**3 Corrections To The Bjorken Sum Rule For Polarized Electroproduction And To The Gross-Llewellyn Smith Sum Rule,* Phys. Lett. B **259**, 345 (1991).

[47] T. Guhr, A. Muller-Groeling and H. A. Weidenmuller, *Random matrix theories in quantum physics: Common concepts,* Phys. Rept. **299**, 189 (1998) [arXiv:cond-mat/9707301].

[48] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras,* Texts and monographs in physics, Springer, Berlin, Germany (1992).

[49] W. Hackbusch, *Integral Equations: Theory and Numerical Treatment* Birkhauser Verlag, Basel Switzerland (1995); P. J. Davis and P. Rabinowitz, *Methods of Numerical Integration* Academic Press, Orlando Florida (1984).

[50] D. R. Herschbach, Int. J. Quant. Chem. 57, 295, (1996); S. G. Rajeev, *New classical limits of quantum theories,* To appear in the proceedings of 70th Meeting between Theoretical Physicists and Mathematicians, Strasbourg, France, 23-25 May 2002. arXiv:hep-th/0210179.

[51] D. R. Herschbach, D. Z. Goodson, M. Lopez-Cabrera and J. D. Morgan, *Large-order dimensional perturbation theory for two-electron atoms,* J. Chem. Phys. 97 (11), 1 (1992).

[52] G. 't Hooft, *On The Convergence Of Planar Diagram Expansions,* Commun. Math. Phys. **86**, 449 (1982).

[53] G. 't Hooft, *A Two-Dimensional Model For Mesons,* Nucl. Phys. B **75**, 461 (1974).

[54] G. 't Hooft, *Under The Spell Of The Gauge Principle,* Adv. Ser. Math. Phys. **19**, 1 (1994).

[55] G. 't Hooft, *A Planar Diagram Theory For Strong Interactions,* Nucl. Phys. B **72**, 461 (1974).

[56] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **44**, 189 (1972).

[57] G. 't Hooft, unpublished. D. J. Gross and F. Wilczek, *Ultraviolet Behavior Of Non-Abelian Gauge Theories,* Phys. Rev. Lett. **30**, 1343 (1973). H. D. Politzer, *Reliable Perturbative Results For Strong Interactions?,* Phys. Rev. Lett. **30**, 1346 (1973).

[58] K. Hornbostel, S. J. Brodsky and H. C. Pauli, *Light Cone Quantized QCD In (1+1)-Dimensions,* Phys. Rev. D **41**, 3814 (1990).

[59] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring,* Nucl. Phys. B **498**, 467 (1997) [arXiv:hep-th/9612115].

[60] C. Itzykson and J. M. Drouffe, *Statistical Field Theory. 2 Vols.* Cambridge University Press (1998).

[61] C. Itzykson and J. B. Zuber, *Quantum Field Theory,* New York, USA: Mcgraw-hill (1980).

[62] A. Jevicki and B. Sakita, *The Quantum Collective Field Method And Its Application To The Planar Limit,* Nucl. Phys. B **165**, 511 (1980).

[63] V. John, G. S. Krishnaswami and S. G. Rajeev, *An Interacting Parton Model For Quark And Anti-Quark Distributions In The Baryon,* Phys. Lett. B **487**, 125 (2000).

[64] V. John, G. S. Krishnaswami and S. G. Rajeev, *Parton Model From Bi-Local Solitonic Picture Of The Baryon In Two-Dimensions,* Phys. Lett. B **492**, 63 (2000) [arXiv:hep-th/0310014].

[65] V. A. Kazakov, *Ising Model On A Dynamical Planar Random Lattice: Exact Solution,* Phys. Lett. A 119(1986) 140.

[66] V. A. Kazakov and P. Zinn-Justin, *Two-matrix model with ABAB interaction,* Nucl. Phys. B **546**, 647 (1999) [arXiv:hep-th/9808043].

[67] J. P. Keating, Comm. Math. Phys. 220 (2001) 429.

[68] G. S. Krishnaswami, *A model of interacting partons for hadronic structure functions,* Undergraduate Thesis, University of Rochester, arXiv:hep-ph/9911538.

[69] G. S. Krishnaswami, *Variational ansatz for gaussian + Yang-Mills two matrix model compared with Monte-Carlo simulations in 't Hooft limit,* arXiv:hep-th/0310110.

[70] G. S. Krishnaswami and S. G. Rajeev, *A model of interacting partons for hadronic structure functions,* Phys. Lett. B **441**, 429 (1998) [arXiv:hep-ph/9807345].

[71] I. K. Kostov, *Exact solution of the three-color problem on a random lattice,* Phys. Lett. B **549**, 245 (2002) [arXiv:hep-th/0005190].

[72] C. W. H. Lee and S. G. Rajeev, *Symmetry algebras of large-N matrix models for open strings,* Nucl. Phys. B **529**, 656 (1998) [arXiv:hep-th/9712090]. C. W. H. Lee and S. G. Rajeev, *A Lie algebra for closed strings, spin chains and gauge theories,* J. Math. Phys. **39**, 5199 (1998) [arXiv:hep-th/9806002].

[73] J. Madore, *An Introduction To Noncommutative Differential Geometry And its Physical Applications,* London Mathematical Society lecture note series. 257, Cambridge Univ. Press. (2000).

[74] Y. M. Makeenko and A. A. Migdal, *Exact Equation For The Loop Average In Multicolor QCD,* Phys. Lett. B **88**, 135 (1979) [Erratum-ibid. B **89**, 437 (1980)].

[75] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity,* Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[76] A. D. Martin, R. G. Roberts, W. J. Stirling and R. S. Thorne, *Parton distributions: A new global analysis,* Eur. Phys. J. C **4**, 463 (1998) [arXiv:hep-ph/9803445].

[77] M. L. Mehta *Random Matrices*, Academic Press, New York, 2nd ed.(1991).

[78] M. L. Mehta, *A Method Of Integration Over Matrix Variables,* Commun. Math. Phys. **79**, 327 (1981).

[79] J. Mickelsson, *Current Algebras And Groups,* Plenum (1989).

[80] J. A. Minahan and K. Zarembo, *The Bethe-ansatz for N = 4 super Yang-Mills,* JHEP **0303**, 013 (2003) [arXiv:hep-th/0212208].

[81] H. L. Montgomery, Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo. (1972) 181.

[82] J. Nishimura, T. Okubo and F. Sugino, *Convergent Gaussian expansion method: Demonstration in reduced Yang-Mills integrals,* JHEP **0210**, 043 (2002) [arXiv:hep-th/0205253].

[83] K. Hagiwara *et al.* [Particle Data Group Collaboration], *Review Of Particle Physics,* Phys. Rev. D **66**, 010001 (2002).

[84] A. Pressley and G. Segal, *Loop Groups,* Clarendon Press, Oxford (1986).

[85] R. Rajaraman, *Solitons And Instantons. An Introduction To Solitons And Instantons In Quantum Field Theory,* Amsterdam, Netherlands: North-holland ( 1982).

[86] S. G. Rajeev, *Derivation of the hadronic structure functions from quantum chromodynamics,* arXiv:hep-th/9905072.

[87] S. G. Rajeev, *Quantum Hadrodynamics In Two-Dimensions,* Int. J. Mod. Phys. A **9**, 5583 (1994) [arXiv:hep-th/9401115].

[88] S. G. Rajeev and O. T. Turgut, "Poisson brackets of Wilson loops and derivations of free algebras," J. Math. Phys. **37**, 637 (1996) [hep-th/9508103].

[89] C. Rubbia, *Experimental Observation Of The Intermediate Vector Bosons W+, W-, And Z0,* Rev. Mod. Phys. **57**, 699 (1985).

[90] B. Sakita, *Quantum Theory Of Many Variable Systems And Fields,* World Sci. Lect. Notes Phys. **1**, 1 (1985).

[91] S. Guruswamy, *Renormalizable Quantum Field Theories in the Large-N limit*, Ph.D. Thesis, University of Rochester 1995.

[92] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory,* Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [arXiv:hep-th/9407087]; *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD,* Nucl. Phys. B **431**, 484 (1994) [arXiv:hep-th/9408099].

[93] T. H. R. Skyrme, *A Nonlinear Field Theory,* Proc. Roy. Soc. Lond. A **260**, 127 (1961).

[94] T. H. R. Skyrme, *Kinks And The Dirac Equation,* J. Math. Phys. **12**, 1735 (1971).

[95] T. H. R. Skyrme, *A Unified Field Theory Of Mesons And Baryons,* Nucl. Phys. **31**, 556 (1962).

[96] H. E. Stanley, *Spherical Model As The Limit Of Infinite Spin Dimensionality,* Phys. Rev. **176**, 718 (1968).

[97] M. Staudacher, it Combinatorial solution of the two matrix model, Phys. Lett. B **305**, 332 (1993) [hep-th/9301038].

[98] M. J. Teper, *SU(N) gauge theories in 2+1 dimensions,* Phys. Rev. D **59**, 014512 (1999) [arXiv:hep-lat/9804008].

[99] D. V. Voiculescu, Invent. Math **118**, 411 (1994), **132**, 189 (1998); Indiana U Math. J **49**, 837 (2000); *Lect. Notes in Math 1738* (Springer, 1998); D. V. Voiculescu, K. J. Dykema and A. Nica *Free Random Variables*, CRM Monograph Series, AMS (1992).

[100] W. Taylor, *M(atrix) theory: Matrix quantum mechanics as a fundamental theory,* Rev. Mod. Phys. **73**, 419 (2001) [arXiv:hep-th/0101126].

[101] J. J. M. Verbaarschot and T. Wettig, *Random matrix theory and chiral symmetry in QCD,* Ann. Rev. Nucl. Part. Sci. **50**, 343 (2000) [arXiv:hep-ph/0003017].

[102] S. Weinberg, *A Model Of Leptons,* Phys. Rev. Lett. **19**, 1264 (1967). A. Salam, *Weak And Electromagnetic Interactions,* in Svartholm: Elementary Particle Theory, Proceedings Of The Nobel Symposium (1968).

[103] E. Witten, *Global Aspects Of Current Algebra,* Nucl. Phys. B **223**, 422 (1983).

[104] E. Witten, *Baryons In The 1/N Expansion,* Nucl. Phys. B **160**, 57 (1979).

[105] L. G. Yaffe, *Large N Limits As Classical Mechanics,* Rev. Mod. Phys. **54**, 407 (1982).

[106] C. N. Yang and R. L. Mills, *Conservation Of Isotopic Spin And Isotopic Gauge Invariance,* Phys. Rev. **96**, 191 (1954).