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Parton model from bi-local solitonic picture of the baryon in two-dimensions

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Abstract

We study a previously introduced bi-local gauge invariant reformulation of two-dimensional QCD, called 2d hadron dynamics. The baryon arises as a topological soliton in hadron dynamics. We derive an interacting parton model from the soliton model, thus reconciling these two seemingly different points of view. The valence quark model is obtained as a variational approximation to hadron dynamics. A succession of better approximations to the soliton picture are obtained. The next simplest case corresponds to a system of interacting valence, ‘sea’ and anti-quarks. We also obtain this ‘embellished’ parton model directly from the valence quark system through a unitary transformation. Using the solitonic point of view, we estimate the quark and anti-quark distributions of 2d QCD. Possible applications to deep inelastic structure functions are pointed out. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

There are currently two distinct points of view on what a baryon is. One may be traced back to the quark model. In the other point of view, baryons arise as solitons of low energy effective local field theories of mesons, an idea that may be traced back to the Skyrme model [1]. In previous work by one of us, it was shown that in two dimensions, there is an exact description of QCD as a *bi-local* theory of mesons [2]. This description, called 2-dimensional hadron dynamics, is

not a low energy effective theory, but is equivalent to 2d QCD for all energies and numbers of colors. The exact description of the baryon is as a topological soliton of this bi-local theory. At the other extreme, we studied a 2-dimensional interacting quark model for the structure of the baryon [3]. The question then is whether the quark model picture can be derived from the exact bi-local solitonic picture in two dimensions. Here we derive the interacting valence quark model as a variational approximation to this bi-local soliton theory. Moreover, we find a succession of increasingly accurate variational approximations to the soliton model. The next simplest case turns out to be a system of interacting valence, ‘sea’ and anti-quarks. We show that this ‘embellished’ parton model could also have been obtained directly via a unitary transformation

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applied to the valence quark model, recovering the ‘Bogoliubov’ transformation introduced in [3]. Thus we have a reconciliation of the exact bi-local soliton model with the simpler relativistic parton picture of the baryon, in 2 dimensions.

The main advantage of this new point of view is that the semi-classical approximation of hadron dynamics corresponds to the large N_c limit of 2d QCD, and so is capable of describing non-perturbative phenomena such as the structure of hadrons. Thus this reconciliation between the soliton and parton pictures is more than just a mathematical correspondence. We illustrate its usefulness by calculating approximately, the quark and anti-quark distributions of the baryon in 2d QCD. In fact, due to the correspondence we establish here, the approximate solutions of the quark models presented in [3] are actually also approximate solutions of 2d QCD. Thus, in this paper, we will focus more on the passage from the bi-local soliton theory to quark models, rather than their actual solution. As mentioned in [3], our results agree well with the direct numerical solutions of Hornbostel et al. [4]. To summarize, we find that the valence quark approximation is accurate not only in the non-relativistic limit, but also in the ultra-relativistic chiral limit. In particular, we find that in the chiral limit, the valence quark approximation is exact, for $N_c \rightarrow \infty$.

As an aside, we speculate on the possible phenomenological implications of the above model. In deep inelastic scattering [5], the transverse momenta of the partons is small compared to their longitudinal momenta. Moreover, the observables of interest, the baryon structure functions, depend only on the proton momentum P and photon momentum q , which lie in a two-dimensional time like hypersurface, spanned by time and the beam direction. Thus, there must be an effective 2-dimensional theory that describes the structure of the proton as measured in deep inelastic scattering. Two-dimensional hadron dynamics has the correct symmetries to be a candidate for an approximate description of the ‘relevant’ interactions of the quarks, in such an effective action. This model can be thought of as a representative example of such an approximate effective action. The variational and many-body techniques developed here should be useful in understanding any such 2-dimensional effective action.

2. Two-dimensional quantum hadron dynamics

Let us begin with a summary of two-dimensional quantum hadron dynamics [2]. 2d QCD is quantized in the null gauge $A_- = 0$ in null coordinates. The elimination of longitudinal gluons leads to a linear potential between quark fields a, a^\dagger which satisfy canonical anti-commutation relations. The resulting hamiltonian is:

$$\begin{aligned} \frac{H}{N_c} = & \int dx a^\dagger a i \frac{1}{2} \left[\hat{p} + \frac{m^2}{\hat{p}} \right] a_{ai} \\ & - \frac{g^2}{2N_c} \int \frac{1}{2} |x - y| : a^\dagger a i (x) a_{aj}(x) : \\ & \times : a^\dagger b j (y) a_{bi}(y) : dx dy. \end{aligned}$$

‘ g ’ is a coupling constant with the dimensions of mass, i, j are color indices and a, b are flavour indices. Define the bi-local gauge invariant variable $\hat{M}_b^a(x, y) = \frac{2}{N_c} : a_{bi}(x) a^\dagger a i (y) :$. The points x, y are null separated. The operators $\hat{M}_b^a(x, y)$ form a complete set of observables in the color singlet sector of two-dimensional QCD. They provide a (projective) unitary irreducible representation of the infinite-dimensional unitary Lie algebra:

$$\begin{aligned} & [\tilde{M}_b^a(p, q), \tilde{M}_d^c(r, s)] \\ & = \frac{1}{N_c} (\delta_b^c 2\pi \delta(q - r) [\delta_d^a \text{sgn}(p - s) + \tilde{M}_d^a(p, s)]) \\ & \quad - \frac{1}{N_c} (\delta_d^a 2\pi \delta(s - p) [\delta_b^c \text{sgn}(r - q) + \tilde{M}_b^c(r, q)]). \end{aligned} \quad (1)$$

Note that the commutators are of order $1/N_c$ so that the large N_c limit is a sort of classical limit: $1/N_c$ plays the role that \hbar does in an ordinary field theory. In this classical limit the above commutators are replaced by the Poisson brackets of a set of classical dynamical variables $M_b^a(x, y)$. It was shown that the phase space of this system is an orbit of the unitary group, an infinite-dimensional grassmannian Gr_1 . It is the set of all hermitean operators M with integral kernel $M_b^a(x, y)$ satisfying the quadratic constraint $[\epsilon + M]^2 = 1$, with $\int |M(x, y)|^2 dx dy < \infty$. Gr_1 is a curved manifold with connected components labelled by an integer. The quadratic constraint is just the Pauli principle for quarks: the density matrix ($\rho = \frac{1}{2}(1 - M - \epsilon)$) must be a projection operator. Here ϵ is

the Hilbert transform operator, diagonal in momentum space, with $\tilde{\epsilon}(p, p) = \text{sgn}(p)$.

So in the large N_c limit, our problem reduces to solving the equations of motion obtained from the hamiltonian and Poisson brackets:

$$\begin{aligned} \frac{E[M]}{N_c} = & -\frac{1}{4} \int \left[p + \frac{\mu_a^2}{p} \right] \tilde{M}_a^a(p, p) \frac{dp}{2\pi} \\ & + \frac{\tilde{g}^2}{8} \int M_b^a(x, y) M_a^b(y, x) |x - y| dx dy, \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{1}{2i} \{ M_b^a(x, y), M_d^c(z, u) \} \\ = \delta_b^c \delta(y - z) [\epsilon_d^a(x, u) + M_d^a(x, u)] \\ - \delta_d^a \delta(x - u) [\epsilon_b^c(z, y) + M_b^c(z, y)]. \end{aligned} \quad (3)$$

The parameter μ_a^2 is related to the current quark masses m_a by a finite renormalization: $\mu_a^2 = m_a^2 - \tilde{g}^2/\pi$ and $\tilde{g}^2 = g^2 N_c$. Though the hamiltonian is quadratic, this is a non-linear interacting theory since the phase space is a curved manifold due to the constraints on $M(x, y)$. The linearization of the equation of motion around the vacuum $M_b^a = 0$ describes an infinite number of free mesons, and 't Hooft's integral equation [6] for the meson masses was recovered.

What kind of solution to this theory represents the baryon? The quantity $B = -\frac{1}{2} \int M_a^a(x, x) dx$ was shown to be an integer, a topological invariant and hence conserved under time evolution. We see that B is in fact baryon number. Thus, for $p \geq 0$, $-\frac{1}{2} \tilde{M}_a^a(p, p)$ and $-\frac{1}{2} \tilde{M}_a^a(-p, -p)$ represent the quark and anti-quark probability densities in the baryon. Thus the baryon is a topological soliton in this picture. It corresponds to a static solution of the equations of motion (minimum of energy subject to constraints), that has baryon number one. A Lorentz invariant formulation is to minimize the $(mass)^2$ of the baryon:

$$\begin{aligned} \frac{\mathcal{M}^2}{N_c^2} = & \left[-\frac{1}{2} \int p \tilde{M}(p, p) \frac{dp}{2\pi} \right] \\ & \times \left[-\frac{1}{2} \int \tilde{M}(p, p) \frac{\mu^2 dp}{2p 2\pi} \right. \\ & \left. + \frac{\tilde{g}^2}{8} \int dx dy |M(x, y)|^2 \frac{1}{2} |x - y| \right]. \end{aligned}$$

3. Separable or rank one ansatz and valence quark model

We have developed a method [2] to find the minimum of energy on the the phase space: a variant of the steepest descent method that takes into account the non-linear constraint. Here, we describe another method based on variational approximations, which brings out the connection to the quark model. The main difficulty in minimizing the energy is the non-linear constraint satisfied by $M(x, y)$. We find a succession of variational ansatzes for $M(x, y)$ that replace this constraint with simpler ones. These define an ascending family of sub-manifolds, which form a dense subset of the phase space. Minimizing the energy on these sub-manifolds will give us successively better approximations. These variational ansatzes turn out to correspond to interacting quark models.

To start with, consider an ansatz of the separable form

$$\tilde{M}_b^a(p, q) = -2\tilde{\psi}^a(p)\tilde{\psi}_b^{*}(q). \quad (4)$$

This satisfies the constraint $(\epsilon + M)^2 = 1$ if $\tilde{\psi}$ is of norm one and of positive momentum. The Poisson brackets of the $M_b^a(x, y)$ imply the relations

$$\{\tilde{\psi}_a(p), \tilde{\psi}_b(q)\} = 0 = \{\tilde{\psi}^{a*}(p), \tilde{\psi}^{b*}(q)\}, \quad (5)$$

$$\{\tilde{\psi}_a(p), \tilde{\psi}^{*b}(q)\} = -i2\pi\delta_a^b\delta(p - q). \quad (6)$$

The $\tilde{\psi}_a$ by themselves define a classical dynamical system with hamiltonian

$$\begin{aligned} \frac{E_1(\psi)}{N_c} = & \int_0^P \frac{1}{2} \left[p + \frac{\mu^2}{p} \right] |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} \\ & + \frac{\tilde{g}^2}{2} \int |\psi(x)|^2 |\psi(y)|^2 \frac{|x - y|}{2} dx dy. \end{aligned} \quad (7)$$

We can quantize this 'mini' theory by looking for operators satisfying canonical commutation relations. Let us denote the parameter that measures the quantum correction, analogous to \hbar , by $1/N_c$. The constraint on the norm can be implemented by restricting attention to those states $|V\rangle$ satisfying

$$\int_0^\infty \hat{\psi}^{*a}(p)\hat{\psi}_a(p) \frac{dp}{2\pi} |V\rangle = 1. \quad (8)$$

A representation for our commutation relations is provided by bosonic creation–annihilation operators:

$$\begin{aligned} [\hat{b}_a(p), \hat{b}_b(p')] &= 0 = [\hat{b}^{\dagger a}(p), \hat{b}^{\dagger b}(p')], \\ [\hat{b}_a(p), \hat{b}^{\dagger b}(p')] &= 2\pi\delta(p-q)\delta_a^b, \end{aligned} \quad (9)$$

with

$$\hat{\psi}_a(x) = \frac{1}{\sqrt{N_c}}\hat{b}_a(x), \quad \hat{\psi}^{\dagger a}(x) = \frac{1}{\sqrt{N_c}}\hat{b}^{\dagger a}(x). \quad (10)$$

Then the constraint becomes the condition that we restrict to states containing N_c particles: $\int_0^\infty \tilde{b}^{\dagger a}(p) \times \tilde{b}_a(p) \frac{dp}{2\pi} = N_c$. Therefore, N_c must be a positive integer! Thus we are dealing with a system of N_c bosons interacting through a linear potential.

What are these bosons? They are the valence quarks of the parton model. They appear like bosons in the momentum and spin-flavour quantum numbers since their wave function is totally anti-symmetric in color. N_c is interpreted as the number of colors. In the *mean field approximation* [3], their wave function is $\epsilon^{i_1 \dots i_{N_c}} \tilde{\psi}(p_1) \dots \tilde{\psi}(p_{N_c})$, which corresponds to

$$|V\rangle = a_{\tilde{\psi}}^{1\dagger} \dots a_{\tilde{\psi}}^{N_c\dagger} |0\rangle. \quad (11)$$

If $\hat{\rho}_b^a(p, q) = \frac{1}{N_c} a^{\dagger ai}(p) a_{bi}(q)$ is the quark density operator, then the expectation value of $\hat{\rho}(p, q)$ in the mean field state state $|V\rangle$ is equal to the ‘classical’ density matrix $\tilde{\rho}_1(p, q) = \frac{1}{2}(\tilde{\delta}(p, q) - \tilde{M}_1(p, q) - \tilde{\epsilon}(p, q))$. Thus the classical (or large N_c) limit we have been discussing is just the mean field approximation to this many-body problem, an idea that goes back to Witten [7]. The semi-classical approximation will give us the leading corrections in the case of finite N_c .

Since the quark null momenta are positive, their sum must equal the total baryon momentum P . In particular, the parton momenta cannot exceed P . This ensures that the quark distributions vanish beyond $p = P$. However, the total baryon momentum is extensive $P \sim N_c$. So in the limit as $N_c \rightarrow \infty$, $0 \leq p < \infty$. The valence quark wavefunction is determined by minimizing the $(mass)^2$ subject to the normalization and momentum sum rule conditions:

$$\frac{\mathcal{M}^2}{N_c^2} = \left[\int_0^P \frac{p}{2} |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} \right] \left[\int_0^P \frac{\mu^2}{2p} |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} \right]$$

$$+ \frac{\tilde{g}^2}{2} \int_{-\infty}^{\infty} |\psi(x)|^2 |\psi(y)|^2 \frac{|x-y|}{2} dx dy \Big],$$

$$\int_0^P |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} = 1, \quad N_c \int_0^P p |\tilde{\psi}(p)|^2 \frac{dp}{2\pi} = P.$$

Here $\psi(x) = \int_0^P \tilde{\psi}(p) e^{ipx} \frac{dp}{2\pi}$. Since we have ignored them, we will get the spin and flavor averaged wavefunction. The probability density of valence quarks is $V(x_B) = \frac{P}{2\pi} |\tilde{\psi}(x_B P)|^2$, where $x_B = p/P$.

The ground state of this many body problem was found in [3], in the guise of a valence quark model. Let us just summarize the result. In the limit $N_c \rightarrow \infty$ and $m = 0$, the *absolute minimum* of the variational principle is $\tilde{\psi}(p) = \sqrt{2\pi/\bar{P}} e^{-p/(2\bar{P})}$. Here \bar{P} is the mean baryon momentum per color, P/N_c . It turns out that $\mathcal{M} = 0$ for this configuration, so that it is not just a minimum on the separable submanifold, but on the entire phase space, in the chiral and large N_c limits. Thus we find that the ground state baryon is massless in this limit, in agreement with Hornbostel et al. [4].

A variational approximation to the ground state, after including the leading effects of finite N_c was also given in [3]. In the chiral limit, the valence quark probability distribution is $V(x_B) = (N_c - 1) \times [1 - x_B]^{N_c - 2}$. This variational approximation agrees well with our numerical solution [3] and is identical to the numerical solution of Hornbostel et al. (see Ref. [4] Eq. (22)).

4. Rank three ansatz: valence, sea and anti-quarks

We can get a better approximation to the exact soliton model, by considering a larger submanifold of the phase space, compared to the separable ansatz, which corresponded to the valence quark approximation.

The departure from the valence quark picture is determined by the dimensionless ratio m^2/\tilde{g}^2 , a measure of chiral symmetry breaking. Thus we should expect the anti-quark content to be small for small current quark masses. The leading effect of finite N_c is to constrain the range of momenta of the partons, as we have seen.

The mathematical advantage of the separable ansatz is that it ‘solves’ the nonlinear constraint on M :

more precisely, it replaces it with the condition that ψ is of norm one. In the same spirit, consider the configuration

$$M_r = \sum_{a,b=1}^r \xi_b^a \psi_a \otimes \psi^{\dagger b}. \quad (12)$$

Here we choose ψ_a to be a set of r orthonormal eigenvectors of the operator ϵ ; i.e., $\epsilon\psi_a = \epsilon_a\psi_a$, $\epsilon_a = \pm 1$. This implies that the operator M_r is of rank r : the special case of rank one is just the separable ansatz above. This ansatz will satisfy the constraint on M if the $r \times r$ matrix ξ is hermitean and satisfies the constraint

$$\xi_a^b \xi_b^c + [\epsilon_a + \epsilon_c] \xi_a^c = 0, \quad (13)$$

a ‘mini’ version of the constraint on M . Moreover, the baryon number is $B = -\frac{1}{2} \text{tr} M = -\frac{1}{2} \text{tr} \xi$. In the special case of rank one, we have simply $\xi = -2$. By choosing a large enough value of r this ansatz can produce as general a configuration in the phase space as needed. The simplest configuration of baryon number one that departs from the separable ansatz is of rank three. We will find that for small current quark masses, even this departure is very small, so we do not need to consider configurations of higher rank.

By a choice of basis among the ψ_a , we can always bring a rank three configuration of baryon number one to the form

$$M_3 = -2\psi \otimes \psi^\dagger + 2\zeta_-^2 [\psi_- \otimes \psi_-^\dagger - \psi_+ \otimes \psi_+^\dagger] + 2\zeta_- \zeta_+ [\psi_- \otimes \psi_+^\dagger + \psi_+ \otimes \psi_-^\dagger], \quad (14)$$

where ψ_-, ψ, ψ_+ are three vectors in $L^2(R)$ satisfying $\epsilon\psi_- = -\psi_-$, $\epsilon\psi = \psi$, $\epsilon\psi_+ = \psi_+$, $\|\psi_-\|^2 = \|\psi\|^2 = \|\psi_+\|^2 = 1$, $\langle \psi, \psi_+ \rangle = 0$. The conditions $\langle \psi_-, \psi \rangle = \langle \psi_-, \psi_+ \rangle = 0$ are then automatic. The parameter $0 \leq \zeta_- \leq 1$ measures the deviation from the rank one ansatz and hence, the anti-quark content of the baryon. $\zeta_+ = \sqrt{1 - \zeta_-^2}$. For example, baryon number is given by

$$B = \int_0^\infty \{ |\tilde{\psi}(p)|^2 + \zeta_-^2 [|\tilde{\psi}_+(p)|^2 - |\tilde{\psi}_-(-p)|^2] \} \frac{dp}{2\pi}. \quad (15)$$

ψ, ψ_+ vanish for $p < 0$ and describe valence and ‘sea’ quarks. Their orthogonality is a consequence of the

Pauli principle. $\tilde{\psi}_-$ is the anti-quark wavefunction. From our previous result we expect ζ_- to vanish as $m^2/\tilde{g}^2 \rightarrow 0$.

This rank 3 ansatz can also be understood as arising from a unitary transformation applied to the valence quark ansatz. The phase space of hadron dynamics carries a transitive action of the infinite-dimensional restricted unitary group [2]. Thus the configuration M_3 can be obtained from M_1 by a unitary transformation

$$U^\dagger (\epsilon - 2\psi \otimes \psi^\dagger) U = \epsilon + \xi_b^a \psi_a \otimes \psi^{\dagger b}. \quad (16)$$

Since both M_1 and M_3 have the same baryon number, U lies in the connected component of the identity and is of the form $U = e^{iA}$ for A hermitean. From the above expressions for M_1, M_3 , we see that U is the identity except on the span of ψ_+ and ψ_- . $\epsilon = -\sigma_3$, in this subspace. Thus $e^{-iA}(-\sigma_3)e^{iA} = s\sigma_1 + (r-1)\sigma_3$, where $r = 2\zeta_-^2$ and $s = 2\zeta_- \zeta_+$ and σ_i are the Pauli matrices. Therefore, on this subspace, A is a 2×2 traceless hermitean matrix $\sigma \cdot \mathbf{w}$. \mathbf{w} is the vector in \mathbf{R}^3 about which $(0, 0, -1)$ must be rotated by an angle $2|\mathbf{w}|$ to reach $(s, 0, r-1)$. Thus $A = i \arcsin(\zeta_-)(\psi_- \otimes \psi_+^\dagger - \psi_+ \otimes \psi_-^\dagger)$.

We can use the infinite-dimensional analogue of the Plücker embedding [8,9] of the grassmannian in the fermionic Fock space to reexpress this unitary transformation on the phase space of hadron dynamics, as a Bogoliubov transformation on the second quantized states. The operator that corresponds to U and acts on the fermionic Fock space is

$$\hat{U} = e^{i\hat{A}} = \exp [- \arcsin(\zeta_-) (a_{i\psi_-} a_{\psi_+}^{i\dagger} - a_{j\psi_+} a_{\psi_-}^{j\dagger})], \quad (17)$$

the sum over colors produces a singlet. The angle θ of [3] can be identified as $\arcsin(\zeta_-)$. The second quantized state after the Bogoliubov transformation is thus $|VSA\rangle = e^{-i\hat{A}}|V\rangle$. Here $|V\rangle$ is the valence quark state, and $|VSA\rangle$ stands for a state containing valence, ‘sea’ and anti-quarks. The condition $\langle VSA | \hat{\rho}(p, q) | VSA \rangle = \tilde{\rho}_3(p, q)$ is then automatic, since the corresponding condition was satisfied in the rank 1 case and we have performed the same unitary transformation on both sides.

Thus, we have derived the ‘embellished’ quark model, which contains valence, ‘sea’ and anti-quarks, as a variational approximation to the bi-local soliton theory. The wave functions ψ, ψ_\pm and the probability

of finding an anti-quark in a baryon $\zeta_-^2/(1 + 2\zeta_-^2)$, were estimated in [3]. To summarize, we found that the probability of finding an anti-quark in the baryon is $\sim 0.1\%$ and $a \sim 0.035$ for $m^2/\bar{g}^2 \sim 10^{-3}$. Moreover, the anti-quarks carry less than 0.1% of the baryon momentum. Hornbostel et al. [4] also find a similar suppression of the anti-quark content in the chiral limit.

5. Conclusion

Thus, while neither the quark model, nor the solitons of low energy effective *local* field theories provides a complete description of baryons in two dimensions, a *bi-local* quantum field theory provides an exact description. We have shown how the quark model arises as a variational approximation to this bi-local soliton theory. Moreover, we have used this bi-local theory to estimate the quark and anti-quark distributions in two dimensions and find good agreement with direct numerical approaches. It is interesting to know what the analogous non-local theory is in four dimensions.

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