The Idea of a Lax Pair–Part I*

Conserved Quantities for a Dynamical System

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Conserved quantities can help to understand and solve the equations of motion of various dynamical systems. Lax pairs are a useful tool to find conserved quantities of some dynamical systems. We give a motivated introduction to the idea of a Lax pair using examples such as the linear harmonic oscillator, Toda chain and Eulerian rigid body. A key step is to write the equations in 'Lax form', which makes it easy to read off conserved quantities. In Part II, these ideas will be extended from systems of particles to continuum systems of fields and also given a geometric interpretation in terms of curvature.

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1. Introduction

A dynamical system is one whose *state* evolves with time: planets moving in the solar system, tumbling stones, growing populations or the economy. For a system of particles, the *state variables* can be taken as the Cartesian components of the instantaneous positions and velocities (or momenta) of the particles. The number of such position coordinates is called the number of degrees of freedom of the system: a point particle moving on a plane has two degrees of freedom. By a dynamical variable, we mean a property of the system (such as the kinetic energy), which depends on the state variables and can change with time. We will be concerned with dynamical systems whose time evolution is governed by differential equations. The solutions of these equations may be visualized as curves (trajectories) traced out by the state as it evolves in time in the space of states (also known as the phase space).

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By a 'level set' of a function of three variables $f(x, y, z)$ we mean the set of points (x, y, z), where f takes a common value. The level sets of $f = x^2 + y^2 + z^2$ are spheres centered at the origin. Except for the zero radius sphere, they are surfaces of one lower dimension than the surrounding three-dimensional (3D) Euclidean space.

Each conserved quantity imposes one relation among coordinates and momenta and constrains the motion to lie on a level set of dimension one less than that of the state space. Thus, the presence of multiple independent conserved quantities restricts the portion of the phase space that a trajectory can explore.

Newton's equation $m\ddot{x} = -dV/dx$ is a second order *nonlinear* ordinary differential equation for $x(t)$ if the potential $V(x)$ is not just a quadratic polynomial in x.

A conserved quantity is a special type of dynamical variable which is constant along trajectories; its value may, however, change continuously from one trajectory to another. Conserved quantities are also called constants of motion or integrals of motion and play an important role in helping us understand the evolution of a system (see *Boxes* 1 and 3 for their role in quantum mechanics). Conserved quantities impose relations among dynamical variables (coordinates and momenta) thereby preventing a trajectory from exploring the whole of the state space. For example, for a free particle moving on a line, the conservation of momentum forces trajectories to lie on horizontal lines of the planar position-momentum phase space. Similarly, in the case of the harmonic oscillator, the conservation of energy forces trajectories to lie on a family of ellipses in the phase plane (see §4.). Each conserved quantity imposes one relation among coordinates and momenta and constrains the motion to lie on a 'level set' of dimension one less than that of the state space. If there are two conserved quantities, the trajectory is restricted to lie on the intersection of their level sets. Thus in general, a system with n degrees of freedom and a $2n$-dimensional state space can have at most $2n - 1$ independent conserved quantities in order to admit continuous time evolution. However, more often than not, there are far fewer than $2n - 1$ such constants of motion.

Conserved quantities are often related to symmetries. For example, in the Kepler problem, angular momentum $\mathbf{l}$ is a constant vector due to rotation invariance. The motion of a planet is confined to the plane perpendicular to $\mathbf{l}$.

Conserved quantities can also help to solve the equations of motion (EOM) of a dynamical system. For example, Newton's equation $m\ddot{x} = -V'(x)$ for a particle moving in one dimension under the influence of a conservative force $f = -dV/dx$ derived from a potential $V(x)$ always admits one conserved quantity. Here, $x(t)$ is the particle's position and dots denote time derivatives. Indeed, multiplying by the 'integrating factor' $\dot{x}$ we get

$$m\,\ddot{x}\,\dot{x} = -V'(x)\,\dot{x} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{m\dot{x}^2}{2}\right) = -\frac{d}{dt}V(x), \qquad (1)$$

implying the conservation of energy $E = \frac{1}{2}m\dot{x}^2 + V(x)$.

Box 1. Utility of conserved quantities in quantum mechanics: The hydrogen spectrum

Conserved quantities also play an important role in understanding quantum systems. In favourable cases, they can even be used to find the spectrum of bound state energy levels. For instance, consider the gravitational Earth-Sun Kepler problem or its quantum version: the electron-proton Hydrogen atom with both bodies treated as point particles located at $\mathbf{r}_1$ and $\mathbf{r}_2$. In the center of mass frame, this 'two-body' problem reduces to that of a particle of 'reduced mass' $m = m_1 m_2/(m_1 + m_2)$ moving in an attractive central potential $V(r) = -\alpha/r$. Here, $r = |\mathbf{r}| \equiv |\mathbf{r}_1 - \mathbf{r}_2|$ is the distance between particles and $\alpha > 0$ is a measure of the interaction strength. The system has three degrees of freedom, say the Cartesian components of the relative coordinate $\mathbf{r} = (x, y, z)$. Interestingly, it admits *seven* conserved quantities: energy ($E = p^2/2m - \alpha/r$, where $\mathbf{p} = \mathbf{p_1} - \mathbf{p_2}$ is the relative momentum), the three components each of the angular momentum and the Laplace–Runge–Lenz vectors ($\mathbf{l} = \mathbf{r} \times \mathbf{p}$ and $\mathbf{A} = \mathbf{p} \times \mathbf{l} - m\,\alpha\,\hat{r}$). The latter lies in the plane of the orbit (the ecliptic plane perpendicular to $\mathbf{l}$) and points from the origin (focus) to perihelion. However, as discussed earlier, at most five $(2 \times 3 - 1)$ of these conserved quantities can be independent. It turns out that there are two relations among them: $\mathbf{A} \cdot \mathbf{l} = 0$ and $A^2 = 2mEl^2 + m^2\alpha^2$. Remarkably, with some heuristic reasoning, these conserved quantities can be used to obtain the bound state spectrum of the hydrogen atom, where the Newtonian gravitational potential is replaced by the Coulomb potential with $\alpha = e^2/4\pi\epsilon_0$, where ϵ_0 is the permittivity of free space and e the proton charge. Let us consider circular orbits (zero eccentricity) so that every point on the orbit qualifies as perihelion and $\mathbf{A} = 0$. Bohr's quantization condition ($l = n\hbar$, $n = 1, 2, 3\ldots$, see *Box* 3) along with $A^2 = 2mEl^2 + m^2\alpha^2$ then leads to the well-known Hydrogen spectrum $E = -m\alpha^2/2\hbar^2 n^2$. Here, n is known as the principal quantum number. It turns out that there are no other bound energy levels so that the spectrum is independent of the angular momentum (azimuthal), magnetic and spin projection quantum numbers (l, m_l and m_s) which can be used to label linearly independent states with the same energy. This is typical: conserved quantities lead to degeneracies in the quantum mechanical energy spectrum. The more the number of compatible conserved observables (simultaneously diagonalizable commuting operators like L^2, L_z and S_z in Hydrogen), the greater the degeneracy of energy levels.

In other words, the conservation of energy has allowed us to reduce Newton's second order equation to a first-order one. The latter may be integrated by separation of x and t variables

$$dt = \frac{dx'}{\sqrt{(2/m)(E - V(x'))}} \quad \text{so that}$$

$$t - t_0 = \int_{x_0}^{x} \frac{dx'}{\sqrt{(2/m)(E - V(x'))}}. \quad (2)$$

A particle moving in 1D under the influence of a conservative force always admits a conserved energy. The latter may be used to reduce the determination of trajectories to the evaluation of an integral followed by the inversion of a function.

In a few cases (such as a quadratic/cubic or quartic potential), this integral can be evaluated explicitly to give a formula for $t(x)$. The trajectory $x(t)$ is obtained by inverting this formula. Such a reduction to quadrature (evaluating integrals) is in general not possible for systems with more than one degree of freedom (e.g. particle moving in 3D in a general potential $V(\mathbf{r})$).

Lax Pairs as a Tool to Generate Conserved Quantities

Lax pairs, introduced by Peter Lax [1] are a tool for finding conserved quantities of some evolution equations. As we will explain (especially in Part II [3]), they are based on the idea of expressing (typically) nonlinear evolution equations as 'compatibility' conditions for a pair of 'auxiliary' linear equations to admit simultaneous solutions. Since linear equations are often easier to understand than nonlinear ones, this can be a significant simplification. The idea only works for certain special systems, which, however, play an important role in our understanding of more general dynamical systems. Unfortunately, there is no recipe to find a Lax pair for a system or to know in advance whether one exists. So some knowledge of the nature of the system and its solutions (from numerical, analytical or experimental investigations) coupled with educated guesswork is involved. But, as we will see, once a Lax pair is known, it can be very helpful in understanding the system.

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2. Lax Pair for the Linear Harmonic Oscillator

The linear (or simple) harmonic oscillator is one of the simplest of mechanical systems. It describes, for instance, small oscillations of a particle of mass m due to a restoring force $-kx$ proportional to its displacement x from equilibrium. Here $k > 0$ is the force constant. Newton's second law '$F = ma$' for such a particle leads to the differential equation

$$m\ddot{x} = -kx. \quad (3)$$

We will use this example to illustrate the idea of a Lax pair. Though the general solution $x(t) = A\cos(\omega t + \phi)$ for constants of integration A and ϕ with $\omega = \sqrt{k/m}$ is well-known, we will not need the explicit solution to discuss a Lax pair formulation. Introducing the momentum $p = m\dot{x}$, we may rewrite (3) as a pair of first-order equations $\dot{x} = p/m$ and $\dot{p} = -m\omega^2 x$. It is convenient to regard them as equations for the variables ωx and p/m which have the same dimension (of velocity):

$$\frac{d(\omega x)}{dt} = \omega\left(\frac{p}{m}\right) \quad \text{and} \quad \frac{d(p/m)}{dt} = -\omega(\omega x). \qquad (4)$$

A Lax pair for a given system of equations (if it exists) is not unique. For instance, we may add to A a matrix that commutes with L and add to L a time-independent matrix that commutes with A without altering the Lax equation $\dot{L} = [L, A]$. Systems can even possess Lax pairs of different dimensions.

These equations are equivalent to the *Lax equation* $\dot{L} = [L, A]$ for the pair of 2×2 matrices [2]

$$L = \begin{pmatrix} p/m & \omega x \\ \omega x & -p/m \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & \omega/2 \\ -\omega/2 & 0 \end{pmatrix}, \qquad (5)$$

whose entries depend on the dynamical variables ωx and p/m. Here, $[L, A] = LA - AL$ is the commutator. How did we arrive at this (L, A) pair? We notice that (4) are linear in $\omega\dot{x}$ and $\dot{p}/m$. So, for $\dot{L} = [L, A]$ to reproduce (4) we choose L to be linear in ωx and p/m. The simplest possibility is to take L to be a 2×2 real matrix. However, in general, this would lead to four EOM. To ensure that there are only 2 independent equations as in (4), we will suppose that L is a traceless symmetric matrix with entries linear in p/m and ωx as in (5). As a consequence, $\dot{L}$ is also symmetric. Since the commutator of symmetric and anti-symmetric matrices is symmetric, it is natural to take A to be anti-symmetric[1]. Since the RHS of (4) are linear in ωx and p/m, we take A to be independent of these variables so that $[L, A]$ would also be linear in them. In terms of these variables, the RHS of (4) is independent of m and linear in ω, so the entries of the antisymmetric matrix A can depend only on ω and must be linear in it. This essentially leads to the A appearing in (5). One then verifies that the four Lax equations following from (5) coincide with (4):

$$\dot{L} = \begin{pmatrix} \dot{p}/m & \omega\dot{x} \\ \omega\dot{x} & -\dot{p}/m \end{pmatrix} = [L, A] = \begin{pmatrix} -\omega^2 x & \omega p/m \\ \omega p/m & \omega^2 x \end{pmatrix}. \qquad (7)$$

The Lax equation $\dot{L} = [L, A]$ bears a resemblance to the Heisenberg equation of motion for an operator Q in the Heisenberg picture of quantum mechanics:

$$i\hbar\frac{dQ}{dt} = [Q, H], \qquad (6)$$

where H is the Hamiltonian. If H and Q are finite dimensional matrices, then $\mathrm{tr}\,[Q, H] = 0$ so that $\mathrm{tr}\,Q$ is conserved. But often, operators in quantum mechanics are infinite dimensional and unbounded. The trace of the commutator of such operators may not vanish (or even be finite). In such cases, $\mathrm{tr}\,Q$ may not be a (finite) conserved quantity.

[1]For $[L, A]$ to be symmetric, A can differ from an anti-symmetric matrix at most by a multiple of the identity, which would not affect the commutator.

We notice the following feature of the Lax matrix, $\text{tr } L^2 = 2(p^2/m^2 + \omega^2 x^2)$ is $(m/4)\times$ the conserved energy of the harmonic oscillator. It turns out that this is a general feature: one may use the Lax matrix to obtain conserved quantities.

3. Isospectral Evolution of the Lax Matrix

The Lax equation ensures that the eigenvalues (spectrum) of L are independent of time. This property is known as isospectrality. To understand this, let us consider the Lax equation

Isospectral evolution means the spectrum of the Lax matrix is independent of time.

$$L_t \equiv \dot{L} = [L, A], \tag{8}$$

where L and A are matrices with entries depending on the dynamical variables. We have used subscripts to denote derivatives. Since the trace of the commutator of a pair of finite dimensional matrices vanishes, $\text{tr } L$ is independent of time . More generally, one may show that the eigenvalues of L are conserved. To see this, we begin with the eigenvalue problem $L\psi = \lambda\psi$. Differentiating in time,

The trace of a commutator of finite dimensional matrices $\text{tr } [A, B] = \text{tr } (AB - BA)$ vanishes because $\text{tr } AB = \text{tr } BA = \sum_{i,j} A_{ij} B_{ji}$.

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t. \tag{9}$$

Upon using the Lax equation (8) this becomes

$$(LA - AL)\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t. \tag{10}$$

Utilizing $L\psi = \lambda\psi$ and rearranging, we get

$$(L-\lambda)A\psi + (L-\lambda)\psi_t = \lambda_t\psi \quad \text{or} \quad (L-\lambda)(\psi_t + A\psi) = \lambda_t\psi. \tag{11}$$

For the eigenvalue λ to be time-independent ($\lambda_t = 0$), the LHS must vanish. For this to happen, $\psi_t + A\psi$ must be an eigenvector of L with eigenvalue λ. Recall that ψ too is an eigenstate of L with the same eigenvalue. Now, for simplicity, we will assume that the λ-eigenspace of L is one-dimensional, which implies that $\psi_t + A\psi$ must be a multiple of ψ (they are linearly dependent):

See *Box* 2 for other methods of showing the isospectrality of a Lax matrix without assuming that its eigenspaces are one-dimensional.

$$\psi_t + A\psi = \beta\psi \tag{12}$$

Box 2. Two more ways to show that the eigenvalues of L are time-independent

(a) In this approach, we assume that L is Hermitian so that λ is real. We take an inner product of $(L-\lambda)(\psi_t + A\psi) = \lambda_t\psi$ (see (11)) with the eigenfunction ψ and use Hermiticity to get

$$\langle (L-\lambda)(\psi_t + A\psi), \psi\rangle = \langle \lambda_t\psi, \psi\rangle \quad \text{or} \quad \langle (\psi_t + A\psi), (L-\lambda)\psi\rangle = \lambda_t\|\psi\|^2. \tag{14}$$

The LHS vanishes as $L\psi = \lambda\psi$. Moreover, being an eigenfunction, $\|\psi\| \neq 0$, so we must have $\lambda_t = 0$.
(b) The isospectrality of $L(t)$ may also be established by showing that $L(t)$ is similar (related by a similarity transformation) to $L(0)$. Indeed, suppose we define the invertible matrix $S(t)$ via the equation $\dot{S} = -AS$ with the initial condition $S(0) = \mathbf{1}$, then the solution of the Lax equation with initial value $L(0)$ is $L(t) = S(t)L(0)S^{-1}(t)$. This is easily verified:

$$\dot{L}(t) = \partial_t(S\,L(0)S^{-1}) = -AS\,L(0)S^{-1} - S\,L(0)S^{-1}\partial_t(S)S^{-1} = -AL(t) + L(t)A = [L(t), A]. \tag{15}$$

Here we used $\partial_t(SS^{-1}) = \partial_t\mathbf{1} = 0$, to write $\partial_t(S^{-1}) = -S^{-1}\partial_t(S)S^{-1}$. Finally, we observe that two matrices related by a similarity transformation have the same eigenvalues:

$$L(0)\psi = \lambda\psi \quad \Rightarrow \quad S\,L(0)S^{-1}(S\psi) = \lambda(S\psi) \quad \text{or} \quad L(t)(S\psi) = \lambda(S\psi). \tag{16}$$

Thus, the eigenvalues of L are conserved in time.

for some (possibly time-dependent) complex number β. This equation may be viewed as an evolution equation[2] for ψ:

$$\psi_t = (-A + \beta\mathbf{1})\psi. \tag{13}$$

Here, $\mathbf{1}$ is the identity matrix. Thus, the Lax equation $L_t = [L, A]$ and this evolution equation for ψ together imply that the eigenvalue λ is a conserved quantity. We say that L evolves isospectrally.

In §4. of Part II [3], we will revisit this problem from a different viewpoint where the Lax equation $L_t = [L, A]$ can be viewed as a compatibility condition among the two linear equations $L\psi = \lambda\psi$ and $\psi_t = -A\psi$ for constant λ.

[2] It is possible to absorb the $\beta\mathbf{1}$ term into A since it commutes with L and, therefore, does not affect the Lax equation.

4. Conserved Quantities From the Lax Equation

A mechanical system with p degrees of freedom can have at most $2p-1$ independent conserved quantities. So though tr L^n is conserved for any positive integer n, not all of them may be independent.

We have just seen that if the equations of motion of a system can be written in Lax form $L_t = [L, A]$, then the isospectrality of L gives us conserved quantities. These conserved quantities could be the eigenvalues of L or equivalently the basis independent quantities $\det L$ and $\operatorname{tr} L^n$ for $n = 1, 2, 3, \ldots$. For example, the familiar conserved energy of the harmonic oscillator may be expressed in terms of the Lax matrix of (5):

$$E = \frac{1}{2}\left(\frac{p^2}{m} + m\omega^2 x^2\right) = -\frac{m}{2}\det L = \frac{m}{4}\operatorname{tr} L^2. \tag{17}$$

We also notice that for any $E > 0$ this Lax matrix has two distinct eigenvalues ($\pm\sqrt{2E/m}$) leading to 1D eigenspaces (one linearly independent eigenvector for each eigenvalue), as was assumed in (12). Furthermore, for $n = 1, 2, 3, \ldots$,

$$L^{2n} = \left(\frac{p^2}{m^2} + x^2\omega^2\right)^n \mathbf{1} = \left(\frac{2E}{m}\right)^n \mathbf{1} \quad \text{and} \quad L^{2n+1} = \left(\frac{2E}{m}\right)^n L. \tag{22}$$

Thus, $\operatorname{tr} L^{2n} = 2\,(2E/m)^n$ while $\operatorname{tr} L^{2n+1} = 0$ so that the traces of higher powers of L do not furnish any new conserved quantities. Indeed, a system with one degree of freedom cannot have more than one independent conserved quantity. In fact, the conservation of energy restricts the trajectories of the harmonic oscillator to lie on a family of ellipses in the x-p phase plane. If there was an additional conserved quantity, trajectories would reduce to points which cannot describe nontrivial time evolution. Box 3 explores how conserved quantities can help find the quantum energy spectrum in a semiclassical ($\hbar \to 0$) approximation.

In the case of the simple harmonic oscillator, the traces of powers of L higher than two do not furnish any new independent conserved quantities.

We now discuss some more examples of Lax representations. Our first example is the Toda chain which admits a simple and elegant Lax pair. We then consider the Euler equations for a rigid body. They admit a simple Lax pair, which however does not allow us to obtain its conserved energy. This problem is solved by introducing a new Lax pair with a 'spectral parameter'.

Box 3. Bohr–Sommerfeld quantization condition and the simple harmonic oscillator

The harmonic oscillator allows us to illustrate how conserved quantities can help to find the quantum mechanical energy spectrum in a semiclassical approximation. The solutions of the EOM (4) can be expressed as

$$x(t) = \sqrt{\frac{2E}{m\omega^2}} \sin\theta(t) \quad \text{and} \quad p(t) = \sqrt{2mE}\cos\theta(t) \quad \text{where} \quad \theta = \omega \times (t - t_0). \tag{18}$$

Here, $E = p^2/2m + (1/2)m\omega^2 x^2$ is the conserved energy (§4., (17)). The trajectory $(x(t), p(t))$ on the phase plane is an ellipse with semiaxes $A = \sqrt{2E/m\omega^2}$ and $B = \sqrt{2mE}$. What is more, these ellipses are the level curves of the energy function $E(x, p)$. Interestingly, $I = E/\omega$ admits an elegant geometric interpretation: it is $(1/2\pi)\times$ the area (πAB) enclosed by the trajectory during one oscillation. The conserved quantity $I(x, p)$ is called an 'action variable' (it has dimensions of action (length × momentum)); and being an area, it may be expressed as a line integral clockwise around the closed trajectory:

$$I = \frac{1}{2\pi}\oint p\,dx = \frac{1}{2\pi}\int_0^{\frac{2\pi}{\omega}} 2E\cos^2(\theta(t))\,dt = \frac{2E}{2\pi}\frac{1}{2} \times \frac{2\pi}{\omega} = \frac{E}{\omega}. \tag{19}$$

The Bohr–Sommerfeld quantization condition postulates that in the semiclassical regime of highly excited states, the classically conserved action variable is quantized in units of Planck's constant:

$$I = \frac{1}{2\pi}\oint p\,dx = n\hbar \quad \text{for large positive integer } n. \tag{20}$$

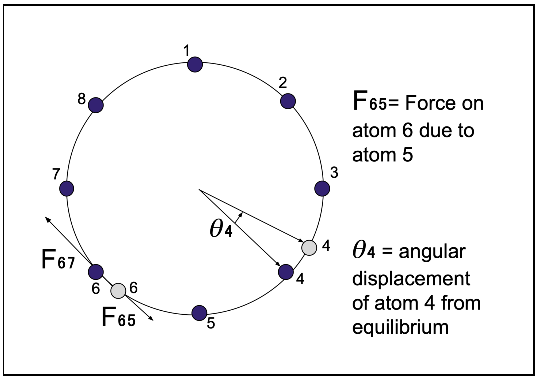
Thus, for the harmonic oscillator we get the semiclassical energy spectrum $E_n \approx n\hbar\omega$ for $n \gg 1$. The same method can be applied to obtain the highly excited energy levels of other systems such as the anharmonic oscillator where the potential $V(x) = ax^2 + bx^4$ is a quartic polynomial.

It turns out that the Bohr–Sommerfeld quantization rule can be derived from the Schrödinger equation of quantum mechanics in a semiclassical approximation. When one does this, one finds in the first approximation, a sub-leading correction to (20) for large n [4, 5]:

$$\oint p\,dx = 2\pi\hbar\left(n + \frac{\mu}{4}\right) \quad \text{for large positive integer } n. \tag{21}$$

Here, $\mu = N_{\text{soft}} + 2N_{\text{hard}}$ is called the Maslov index. N_{soft} is the number of classical turning points (x such that $V(x) = E$) where the potential may be approximated by a linear function and N_{hard} is the number of 'turning points' where the particle encounters an infinite potential barrier (since we are concerned with highly excited states). For the harmonic oscillator, $\mu = N_{\text{soft}} = 2$ and $E_n \approx (n + 1/2)\hbar\omega$. This happens to agree with the exact quantum mechanical spectrum including a zero point energy. In the Kepler/Hydrogen atom problem, there are no turning points (kinetic energy never vanishes along an orbit) and so $\mu = 0$ as in *Box* 1.

Figure 1. Toda chain of $N = 8$ particles with periodic boundary conditions.



5. Toda Chain: Lax Pair and Conserved Quantities

In 1967, Morikazu Toda introduced a model for a one-dimensional crystal in which a chain of identical atoms/particles of mass m interacts with their nearest neighbours via nonlinear springs with exponential forces. If x_i is the displacement of the i^{th} particle from its equilibrium position and p_i its momentum, then the EOM are

The Toda chain consists of particles interacting via nonlinear springs with exponential restoring forces. It can be used to model one-dimensional crystals.

$$m\dot{x}_i = p_i \quad \text{and} \quad \dot{p}_i = \kappa \left(e^{-(x_i - x_{i-1})} - e^{-(x_{i+1} - x_i)}\right). \tag{23}$$

Here, κ is a force constant and we will work in units where $\kappa = m = 1$. We will consider an N particle Toda chain subject to periodic boundary conditions: $x_{N+i} = x_i$ for all i. Thus, we may visualize the particles as lying on a circle and interpret x_i as the angular displacement θ_i from equilibrium (see *Figure* 1).

Falschka's variables convert the nonlinearities in the equations of motion (EOM) of the Toda chain from exponential to quadratic.

The exponential nonlinearity of the EOM (23) may be made quadratic by introducing Falschka's variables [6]

$$a_i = \frac{1}{2} e^{-(x_i - x_{i-1})/2} \quad \text{and} \quad b_i = -\frac{1}{2} p_{i-1}, \tag{24}$$

which evolve according to

$$\dot{a}_i = a_i(b_{i+1} - b_i) \quad \text{and} \quad \dot{b}_i = 2(a_i^2 - a_{i-1}^2). \tag{25}$$

These equations are equivalent to the Lax equation $\dot{L} = [L, A]$ if we define the essentially tridiagonal matrices L and A as below

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_N \\ a_1 & b_2 & a_2 & & \\ 0 & a_2 & b_3 & & \\ \vdots & & & \ddots & \\ a_N & & & & b_N \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & -a_1 & 0 & \cdots & a_N \\ a_1 & 0 & -a_2 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ -a_N & & & & 0 \end{pmatrix}. \quad (26)$$

The traces tr L^n for $n = 1, 2, \ldots, N$ then give us N conserved quantities [7]. The first two of these may be interpreted in terms of the total momentum and energy of the chain:

$$\text{tr } L = \sum_{i=1}^{N} b_i = -\frac{1}{2}\sum_{i=1}^{N} p_{i-1} = -\frac{P}{2} \quad \text{and}$$
$$\text{tr } L^2 = \sum_{i=1}^{N}\left(2a_i^2 + b_i^2\right) = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{1}{2}p_i^2 + e^{-(x_i - x_{i-1})}\right) = \frac{E}{2}. \quad (27)$$

In the absence of external forces, a top displays two types of motion: spinning about an instantaneous axis of rotation and precession of this axis about the fixed direction of angular momentum in the lab frame.

6. Euler Top: Lax Pair and Conserved Quantities

Next, we consider a rigid body (e.g. a top) free to rotate about its center of mass (which is held fixed) in the absence of external forces like gravity. In a frame that rotates with the body, its EOM may be written as a system of three first-order 'Euler' equations [8, 9] for the components of angular momentum about its center of mass:

$$\vec{S}_t = \vec{S} \times \vec{\Omega} \quad \text{or} \quad \dot{S}_1 = S_2\Omega_3 - S_3\Omega_2,$$
$$\dot{S}_2 = S_3\Omega_1 - S_1\Omega_3 \quad \text{and} \quad \dot{S}_3 = S_1\Omega_2 - S_2\Omega_1. \quad (28)$$

Here, $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is the angular velocity vector which is related to $\vec{S} = (S_1, S_2, S_3)$ via $\vec{S} = I\vec{\Omega}$. The inertia tensor[3] I is a 3×3 real symmetric matrix which encodes the distribution of mass in the body. The eigenvalues I_1, I_2 and I_3 of I are called the *principal moments of inertia*. In what follows, we will choose the axes of the co-rotating frame to be the principal axes of inertia (eigenvectors of I) so that the inertia tensor becomes diagonal: $I = \text{diag}(I_1, I_2, I_3)$.

[3]For a body with mass density $\rho(\mathbf{x})$, the components I_{ij} of the inertia tensor are given by an integral over the body $\int(\mathbf{x}^2\delta_{ij} - x_i x_j)\rho(\mathbf{x})\, d\mathbf{x}$.

There is a straightforward way of expressing the Euler equations

Box 4. From vectors to 3×3 anti-symmetric matrices using the Levi-Civita symbol

The Levi-Civita symbol ϵ_{ijk} (for i, j, k taking any values among $1, 2, 3$), is a tensor anti-symmetric under exchange of any pair of indices with $\epsilon_{123} = 1$. For instance, $\epsilon_{112} = 0, \epsilon_{312} = 1$ and $\epsilon_{132} = -1$. Now, given a vector $\vec{S}$, we may construct an anti-symmetric matrix $S_{ij} = \sum_k \epsilon_{ijk} S_k$ by 'contracting' the vector with the Levi-Civita symbol. Conversely, contracting the anti-symmetric matrix with the Levi-Civita symbol brings us back to the original vector $S_k = \sum_{i,j} \epsilon_{ijk} S_{ij}/2$. Under this transformation, the matrix corresponding to the cross product of vectors $\vec{S} \times \vec{\Omega}$ is the commutator $[\Omega, S]$.

in Lax form if we introduce the anti-symmetric matrices

$$S = \begin{pmatrix} 0 & S_3 & -S_2 \\ -S_3 & 0 & S_1 \\ S_2 & -S_1 & 0 \end{pmatrix} \quad \text{and} \quad \Omega = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix} \tag{29}$$

corresponding to the vectors $\vec{S}$ and $\vec{\Omega}$ (see *Box* 4 for how they are related).

The cross product

$$\vec{S} \times \vec{\Omega} = (S_2\Omega_3 - S_3\Omega_2, S_3\Omega_1 - S_1\Omega_3, S_1\Omega_2 - S_2\Omega_1), \tag{30}$$

then corresponds to (the negative of) the matrix commutator:

$$[S, \Omega] = \begin{pmatrix} 0 & S_2\Omega_1 - S_1\Omega_2 & S_3\Omega_1 - S_1\Omega_3 \\ S_1\Omega_2 - S_2\Omega_1 & 0 & S_3\Omega_2 - S_2\Omega_3 \\ S_1\Omega_3 - S_3\Omega_1 & S_2\Omega_3 - S_3\Omega_2 & 0 \end{pmatrix}. \tag{31}$$

Thus, the Euler equations (28) take the Lax form:

$$S_t = [\Omega, S]. \tag{32}$$

The Euler top admits two conserved quantities, the square of angular momentum ($\vec{S}^2$) and the energy (E).

Comparing with (8) we see that $(S, -\Omega)$ furnish a Lax pair. What is more, the Lax equation then implies that $-(1/2)$ tr $S^2 = S_1^2 + S_2^2 + S_3^2$ (square of angular momentum) is a conserved quantity. Indeed, it is straightforward to check using (28) that $S_1\dot{S}_1 + S_2\dot{S}_2 + S_3\dot{S}_3 = 0$. In addition to $\vec{S}^2$, the Euler top is known to possess another conserved quantity, its energy:

$$E = \frac{1}{2}\left(\frac{S_1^2}{I_1} + \frac{S_2^2}{I_2} + \frac{S_3^2}{I_3}\right). \tag{33}$$

However, E depends on the principal moments of inertia and cannot be obtained from the Lax matrix S by combining the traces of any of its powers. This is because S is independent of $I_{1,2,3}$.

Thus, we seek a new Lax pair (L, A) such that both $\vec{S}^2$ and E can be obtained from traces of L and its powers. We, therefore, introduce a new Lax matrix which is a combination of the angular momentum and inertia matrices, weighted by a parameter λ. However, in place of I, it turns out to be convenient to work with the diagonal matrix: $\mathcal{I} = \mathrm{diag}(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ with $\mathcal{I}_k = (1/2)(I_i + I_j - I_k)$ where (i, j, k) is any cyclic permutation of $(1, 2, 3)$. For example, $\mathcal{I}_1 = (1/2)(I_2 + I_3 - I_1)$. Now, we postulate the new Lax pair

The parameter λ that appears in the Lax matrix L of (34) is (somewhat confusingly) known as a spectral parameter. It is not to be confused with the symbol for an eigenvalue of the Lax matrix! The reason for this terminology will be clarified in §4 of Part II.

$$\begin{aligned} L(\lambda) &= \mathcal{I}^2 + \frac{S}{\lambda} = \begin{pmatrix} \mathcal{I}_1^2 & S_3/\lambda & -S_2/\lambda \\ -S_3/\lambda & \mathcal{I}_2^2 & S_1/\lambda \\ S_2/\lambda & -S_1/\lambda & \mathcal{I}_3^2 \end{pmatrix} \quad \text{and} \\ A(\lambda) &= -(\lambda\mathcal{I} + \Omega) = -\begin{pmatrix} \lambda\mathcal{I}_1 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & \lambda\mathcal{I}_2 & \Omega_1 \\ \Omega_2 & -\Omega_1 & \lambda\mathcal{I}_3 \end{pmatrix}. \end{aligned} \qquad (34)$$

To motivate this Lax pair we first note that putting $L = S/\lambda$ and $A = -\Omega$ in $\dot{L} = [L, A]$ gives the desired EOM (32). For the energy to emerge as a conserved quantity from tr L^2, we will augment this Lax pair by matrices involving the principal moments of inertia (or the matrix $\mathcal{I}$) while ensuring that the EOM are not affected. Since $\mathcal{I}_{1,2,3}$ are constant in time we can add any matrix function $f(\mathcal{I})$ to L without affecting $\dot{L}$. However, this will affect the commutator $[L, A]$. To cancel this contribution we will add another matrix function $g(\mathcal{I})$ to A. Thus, $L = S/\lambda + f$ and $A = -(\Omega + g)$. For the unwanted terms $[S/\lambda, g]$ and $[f, \Omega]$ in $[L, A]$ to have a chance of cancelling, we use the relation $\vec{S} = I\vec{\Omega}$ and dimensional analysis to pick $f = \mathcal{I}^2$ and $g = \lambda\mathcal{I}$ as in (34). Some algebra now shows that the Lax equation $\dot{L} = [L, A]$ is equivalent to (32). Indeed,

For the Euler top, the Lax equation $\dot{L} = [L, A]$ is equivalent to Euler's equations of motion for the angular momentum vector $\vec{S}$.

$$\dot{L} - [L, A] = \frac{1}{\lambda}(\dot{S} + [S, \Omega]) + [S, \mathcal{I}] + [\mathcal{I}^2, \Omega]. \qquad (35)$$

Using $\vec{S} = I\vec{\Omega}$, one finds that the sum $[S, \mathcal{I}] + [\mathcal{I}^2, \Omega]$ vanishes. Thus, requiring the Lax equation to hold for any value of λ leads to the Euler equations as in (32).

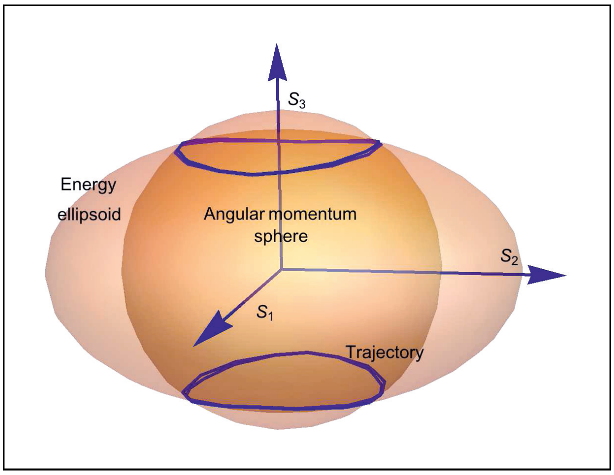


Figure 2. The intersection of the energy ellipsoid and angular momentum sphere is the orbit of the angular momentum vector $\vec{S}$ in the co-rotating frame of the Euler top.

The trace of this new Lax matrix L is conserved, but it is not a dynamical variable as it is simply a quadratic polynomial in the material constants $I_{1,2,3}$. Pleasantly, the traces of the second and third powers of L involve the square of angular momentum $\vec{S}^2$ and energy E, allowing us to deduce that both of them are conserved:

Traces of higher powers of L also lead to conserved quantities but they are simply functions of $\vec{S}^2$ and E.

$$\mathrm{tr}\, L^2 = \mathrm{tr}\, \mathcal{I}^4 - \frac{2}{\lambda^2}\vec{S}^2 \qquad \text{and}$$
$$\mathrm{tr}\, L^3 = \mathrm{tr}\left[\mathcal{I}^6 + \frac{3}{\lambda^2}\mathcal{I}^2 S^2\right] = \mathrm{tr}\, \mathcal{I}^6 - \frac{3}{\lambda^2}\left(\frac{(\mathrm{tr}\, I)^2\vec{S}^2}{4} - I_1 I_2 I_3 E\right). \quad (36)$$

These conservation laws may be used to determine how $\vec{S}$ evolves in the corotating frame. Indeed, since both E and $\vec{S}^2$ are conserved, trajectories must lie along the intersection of the energy ellipsoid and angular momentum sphere:

The constancy of energy E and square of angular momentum $\vec{S}^2$ confines the motion to a pair of surfaces in the space of angular momenta. The trajectories lie along the intersection of these two surfaces.

$$E = \frac{1}{2}\left(\frac{S_1^2}{I_1} + \frac{S_2^2}{I_2} + \frac{S_3^2}{I_3}\right) \quad \text{and} \quad \vec{S}\cdot\vec{S} = S_1^2 + S_2^2 + S_3^2. \quad (37)$$

These two quadratic surfaces typically intersect along a closed curve which forms the periodic orbit of the tip of the angular momentum vector $\vec{S}$ as shown in *Figure* 2.

Having found the evolution of the angular momentum vector, one still needs to use $\vec{S}(t)$ to solve three first order equations for the 'Euler angles' (θ, ϕ and ψ) to find the instantaneous orientation of the rigid body in space. For more on this, see the discussion in [8].

7. Discussion

In this article, we have explained what a Lax pair is and how it can be used to find conserved quantities for dynamical systems such as the simple harmonic oscillator, Toda chain and the Euler top. So far, we considered systems of particles with finitely many degrees of freedom. In Part II, we will extend the idea of a Lax pair to certain continuum mechanical systems with infinitely many degrees of freedom (systems of fields rather than finitely many particles). We will do this in the context of the linear wave equation for vibrations of a stretched string and the nonlinear Korteweg de-Vries (KdV) equation for water waves. The Lax pair framework will also be given a geometric reformulation in terms of the vanishing of a certain curvature, allowing us in principle to find infinitely many conserved quantities for certain field equations.

In Part II, we will introduce the notion of a field and extend the idea of a Lax pair to certain continuum mechanical systems of fields.

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Suggested Reading

[1] **P D Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.*, Vol.21, 467, 1968.**

[2] **O Babelon, D Bernard and M Talon, *Introduction to classical integrable systems*, Cambridge University Press, Cambridge, 2003.**

[3] **G S Krishnaswami and T R Vishnu, *The idea of a Lax pair–Part II : Continuum wave equations*, To appear in *Resonance* (2021).**

[4] **D J Griffiths, *Introduction to Quantum Mechanics*, Second edition, Pearson Education, Dorling Kindersley Indian Ed., New Delhi, 2005.**

[5] **A D Stone, Einstein's unknown insight and the problem of quantizing chaos, *Physics Today*, Vol.58, No.8, 37, 2005.**

[6] **H Flaschka, The Toda lattice. II. Existence of integrals, *Phys. Rev. B*, Vol.9, 1924, 1974.**

[7] **M Henon, Integrals of the Toda lattice, *Phys. Rev. B*, Vol.9, 1921, 1974.**

[8] L D Landau and E M Lifshitz, *Mechanics, Third edition, Volume 1 of course in theoretical physics*, Pergamon Press, Oxford, 1976.

[9] L N Hand and J D Finch, *Analytical Mechanics*, Cambridge University Press, Cambridge, 1998.

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