

Schwinger–Dyson operators as invariant vector fields on a matrix model analog of the group of loops

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For a class of large- N multimatrix models, we identify a group \mathbf{G} that plays the same role as the group of loops on space-time does for Yang–Mills theory. \mathbf{G} is the spectrum of a commutative shuffle-deconcatenation Hopf algebra that we associate with correlations. \mathbf{G} is the exponential of the free Lie algebra. The generating series of correlations is a function on \mathbf{G} and satisfies quadratic equations in convolution. These factorized Schwinger–Dyson or loop equations involve a collection of Schwinger–Dyson operators, which are shown to be right-invariant vector fields on \mathbf{G} , one for each linearly independent primitive of the Hopf algebra. A large class of formal matrix models satisfying these properties are identified, including as special cases, the zero momentum limits of the Gaussian, Chern–Simons, and Yang–Mills field theories. Moreover, the Schwinger–Dyson operators of the continuum Yang–Mills action are shown to be right-invariant derivations of the shuffle-deconcatenation Hopf algebra generated by sources labeled by position and polarization. © 2008 American Institute of Physics. [DOI: 10.1063/1.2940331]

I. INTRODUCTION

Quantum Yang–Mills (YM) theory is at the heart of the microscopic description of strongly interacting particles. The group of based loops on space-time plays an important role in the formulation of YM theory in terms of Wilson loops, which are gauge invariant variables containing much physical information.¹ Expectation values of Wilson loop observables are functions on this group. Expansion around the multicolor limit is a promising approximation method to solve YM theory.² However, in the absence of a full-fledged differential geometry and analysis on the space of loops, progress in understanding and approximately solving the multicolor limit of YM theory has been partly held up despite important early work.^{2–4}

On the other hand, Hermitian multimatrix models may be regarded as toy models for YM theory. The $N \times N$ matrices may be thought of as gauge fields at various space-time points, where N is the number of colors. It is natural to ask whether there is an analog of the group of loops such that multimatrix correlators are functions on this group. If so, can we interpret the factorized Schwinger–Dyson loop equations (fSDE)¹ of a matrix model in terms of the differential operators and products naturally associated with this group? Doing so may open up new perspectives and approximation methods for the large- N limit of multimatrix models and YM theory.

In this paper we show that there is indeed such a group \mathbf{G} associated with a large class of multimatrix models. We construct it indirectly as the group of characters of the commutative

¹The fSDE are quantum-corrected equations of motion for a matrix model in the multicolor (large- N) limit. They are analogous to the Makeenko–Migdal equations⁴ of large- N YM theory. $N=3$ in nature.

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shuffle-deconcatenation Hopf algebra.² The latter is defined using the shuffle and concatenation products and reversal of order of matrices in a correlator. These are analogous to pointwise products of functions of loops, concatenation of loops, and reversal of loop orientation. In the simplest case of a single matrix and real-valued characters, \mathbf{G} is the multiplicative group of nonzero real numbers. More generally, \mathbf{G} is identified with the exponential of the free Lie algebra (FLA). We develop the rudiments of differential calculus on \mathbf{G} using algebraic operations in the Hopf algebra. We find a large class of (formal) matrix models that can be formulated in terms of this group. We show that their fSDEs are quadratic equations (in the convolution product) for a function on \mathbf{G} . Moreover, there is one equation for each linearly independent primitive element of the Hopf algebra of functions on \mathbf{G} . The Schwinger–Dyson (SD) operators, one for each linearly independent primitive, are shown to be right-invariant vector fields on \mathbf{G} . Thus, given a prescription of which right-invariant vector field to associate with a given primitive, we can write down a system of fSDE for any group. For the group of relevance to matrix models, this prescription is encoded in the action. We find a large class of admissible actions and their SD operators, which include the Gaussian, Chern–Simons, and YM matrix models as special cases. Finally, the SD operators of the continuum YM action are obtained and shown to be right-invariant derivations of the shuffle-deconcatenation Hopf algebra on a continuously infinite number of generators labeled by space-time position and polarization. However, in the case of YM theory, we still need to pass to a quotient of this Hopf algebra to account for gauge invariance and recover the group of loops^{5,6} (or need to gauge fix and introduce additional generators for ghosts⁷), before we can look for physical solutions to the fSDE.

In Ref. 8 the fSDE were formulated as conditions for the extremum of a large- N “classical” action (the Legendre transform of the entropy of operator-valued random variables). This viewpoint applied to generic multimatrix models and also provided a variational approximation method. Here we develop a quite different group theoretic formulation, which only applies to a subclass of matrix models. The distinction is very roughly analogous to that between the generic classical mechanical system and one whose configuration space is a group. However, due to this restriction, we find additional structures which closely mimic those present in YM theory. We wanted to study these structures since they form the basis for an approximation scheme for multimatrix models proposed in Ref. 9. We hope our group theoretic formulation allows for a generalization to more familiar groups, where solutions to the fSDE may be more easily found. Our work continues the developments in the physics literature due to Migdal and Makeenko,^{4,10} Polyakov,¹¹ Cvitanovic *et al.*,¹² Gambini *et al.*,^{1,13} Tavares,⁶ Rajeev *et al.*,^{14,8} and others, and builds on our previous papers.^{9,7} There are of course other approaches to multimatrix models such as those related to integrable models and algebraic geometry, see, for instance, Refs. 15 and 16.

By a Λ -matrix model^{17,8,9,18} we mean a statistical system whose variables are a collection of random Hermitian $N \times N$ matrices A_i , $1 \leq i \leq \Lambda$ with partition function $Z = \int dA e^{-N \text{tr} S(A)}$. The integration is over all independent matrix elements. The action is the trace of a polynomial in the matrices $S(A) = \text{tr} S^I A_i$, where³ S^I are the coupling “tensors” (see Sec. V, for examples). A_i model gauge fields at a collection of space-time points labeled by i and are $N \times N$ matrices in color space. The action, measure of integration, and observables $\Phi_I = (1/N) \text{tr} A_i$ are invariant under the global $U(N)$ action $A_i \rightarrow U A_i U^\dagger$. We are interested in their expectation values in the large- N limit, the cyclic “gluon correlation” tensors,

²Technically, \mathbf{G} can be thought of as an analog of the group of generalized loops or extended loop group studied by Tavares⁶ and Bartolo *et al.*¹³ We also identify a subgroup of \mathbf{G} which can be regarded as an analog of the smaller group of loops on space-time, see Sec. III.

³Capitals denote multi-indices and repeated indices are summed, \emptyset is the empty word. If $I = i_1 i_2 \cdots i_n$ then $S^I = S^{i_1 \cdots i_n}$ and $A_I = A_{i_1} A_{i_2} \cdots A_{i_n}$ is a matrix product. δ_I^J equals one if $I = J$ and zero otherwise.

$$G_I = \left\langle \frac{\text{tr}}{N} A_I \right\rangle = \lim_{N \rightarrow \infty} \frac{1}{Z} \int dA e^{-N \text{tr} S(A)} \frac{\text{tr}}{N} A_I, \quad G_\emptyset = 1. \tag{1}$$

G_I satisfy a closed system of fSDE,⁴ conditions for the invariance of Z under infinitesimal non-linear changes of integration variable. The latter are infinitesimal automorphisms of the tensor algebra generated by A_i , $L_i^j A_j = \delta_i^j A_i$. The fSDEs, one for each letter i and word I , relate a change of action to a change in measure

$$S^{j_1 i j_2} G_{j_1 i j_2} = \delta_i^{j_1 j_2} G_{j_1} G_{j_2}. \tag{2}$$

If we define Λ noncommuting sources ξ^i , we can form the generating series of correlators $G(\xi) = \sum_I G_I \xi^I$. Then the fSDE can be written $S^i G(\xi) = G(\xi) \xi^i G(\xi)$. The SD operators

$$S^i = \sum_{n \geq 0} (n+1) S^{i j_1 \dots j_n} D_{j_n} \dots D_{j_1} \tag{3}$$

are expressed in terms of left annihilation operators D_j which satisfy $D_j \xi^{i_1 \dots i_n} = \delta_j^{i_1} \xi^{i_2 \dots i_n}$ or equivalently, $[D_j G]_I = G_{jI}$. The linear term on the left hand side of the fSDE is classical, while the quadratic term on the right hand side is a quantum correction, being suppressed by one power of $\hbar = 1$. However, both left hand and right hand sides are of the same order in $1/N$. Juxtaposition on the right hand side $G(\xi) \xi^i G(\xi)$, denotes the concatenation product ($\xi^{i_1} \xi^i \xi^{i_2} = \xi^{i_1 i_2}$). However, left annihilation does not satisfy the Leibnitz rule with respect to the concatenation product, so the fSDEs are not differential equations in the usual sense. On the other hand, D_j does satisfy the Leibnitz rule with respect to the shuffle product of correlations, a fact that we will exploit in our group theoretic reformulation. This property was also used to transform the fSDE into linear differential equations for the shuffle reciprocal of $G(\xi)$, at the zeroth order of an approximation method proposed in Ref. 9. For more details on the fSDE, we refer to Refs. 8, 9, and 18. In this paper we do not have anything to say about the convergence of matrix integrals. We only use them as a formal device to generate the fSDE, whose structure we wish to investigate.

II. HOPF ALGEBRA STRUCTURE ON CORRELATIONS

The space of based oriented loops γ on space-time (up to equivalence under backtracking or retracing) plays a basic role in the Wilson loop formulation of YM theory. This loop space forms an infinite dimensional non-Abelian group, with successive traversal of loops as product $\gamma_1 \gamma_2$ and reversal of orientation $\bar{\gamma}$ as inverse. The information in this group of loops can be encoded in the algebra of (complex-valued) functions defined on it. Wilson loop functions $W(\gamma) = \text{tr} P \exp \oint A_\mu(\gamma(t)) \dot{\gamma}^\mu(t) dt$ [trace of holonomy of the gauge connection $A_\mu(x)$ around the closed loop $\gamma^\mu(t)$] form an adequate class of functions for this purpose.¹ Since the underlying loop space is a non-Abelian group, the algebra of functions has the additional structure of a commutative but noncocommutative Hopf algebra (under suitable hypotheses, this is a general property of the algebra of functions on any group¹⁹). The pointwise product is $(W_1 W_2)(\gamma) = W_1(\gamma) W_2(\gamma)$, the coproduct $(\Delta W)(\gamma_1, \gamma_2) = W(\gamma_1 \gamma_2)$ encodes the concatenation of loops, and the antipode $(SW) \times (\gamma) = W(\bar{\gamma})$ encodes the inverse. The product and coproduct define compatible algebra and coalgebra structures, while the antipode turns this bialgebra into a Hopf algebra. Up to some technicalities, the underlying group of loops can be recovered as the spectrum (group of characters) of this Hopf algebra.^{19,25,6}

For a matrix model we did not know the analog of the group of loops, but did notice a bialgebra structure on correlators G_I , in connection with some approximation schemes for the loop equations.⁹ We recall this bialgebra and then define a compatible antipode to obtain the shuffle-

⁴Factorization¹⁰ is the property $\langle \Phi_{I_1} \Phi_{I_2} \dots \Phi_{I_n} \rangle = G_{I_1} \dots G_{I_n} + \mathcal{O}(1/N^2)$ as $N \rightarrow \infty$.

deconcatenation Hopf algebra, which is the analog of the Hopf algebra of Wilson loop functions. In Sec. III we extract the underlying group from this Hopf algebra. The shuffle-deconcatenation Hopf algebra has appeared previously in other contexts.^{20,19,6}

Let $G(\xi) = \sum_I G_I \xi^I$ denote the generating series of multimatrix correlators in the large- N limit. $G(\xi)$ is an element of the vector space $\mathbf{C}\langle\langle A \rangle\rangle$ of formal complex linear combinations of words ξ^I in generators chosen from the alphabet $A = \{\xi^i, 1 \leq i \leq \Lambda\}$ consisting of sources ξ^i , one for each matrix A_i . The commutative shuffle product sh of two such series is $(F \circ G)(\xi) = \sum_I (F \circ G)_I \xi^I$, where $(F \circ G)_I = \sum_{I=J \sqcup K} F_J G_K$. The sum is over all complementary order-preserving substrings J and K of I . For example,

$$(F \circ G)_{ijk} = F_\emptyset G_{ijk} + F_i G_{jk} + F_j G_{ik} + F_k G_{ij} + F_{ij} G_k + F_{ik} G_j + F_{jk} G_i + F_{ijk} G_\emptyset. \tag{4}$$

Physically, the shuffle product is the product induced on gluon correlations by the pointwise product of Wilson loop expectation values $\langle\langle (W_1 W_2)(\gamma) \rangle\rangle = \langle\langle W_1(\gamma) \rangle\rangle \langle\langle W_2(\gamma) \rangle\rangle$, when path ordered exponentials are expanded in iterated integrals of gluon correlations in the large- N limit.⁹ $\mathbf{C}\langle\langle A \rangle\rangle$ with the shuffle product is the shuffle algebra on Λ generators Sh_Λ . The empty word 1 is a unit element for sh , with $1 \circ F = F \circ 1 = F$ for all $F \in \text{Sh}_\Lambda$.

Concatenation is defined as $\xi^I \xi^J = \xi^{IJ}$, which extends linearly to $(FG)(\xi) = F_I G_J \xi^{IJ}$. Using the inner product on $\mathbf{C}\langle\langle A \rangle\rangle$ for which ξ^I form an orthonormal basis $(\xi^I, \xi^J) = \delta^{IJ}$, we can define the adjoint of concatenation or the deconcatenation coproduct $\Delta = \text{conc}^\dagger$ by $(F, GH) = (\Delta F, G \otimes H)$. On monomials, $\Delta \xi^I = \sum_{J,K} \delta_{JK}^I \xi^J \otimes \xi^K$. It is extended linearly to series $\Delta F = \sum_{J,K} F_{JK} \xi^J \otimes \xi^K$. Δ is not cocommutative, it mimics the coproduct on Wilson functions coming from concatenation of loops $(\Delta W)(\gamma_1, \gamma_2) = W(\gamma_1 \gamma_2)$. We showed in Ref. 9 that Δ is a homomorphism of sh . The homomorphism of sh , $\epsilon: \text{Sh}_\Lambda \rightarrow \mathbf{C}$ which picks out the constant term, $\epsilon(F) = F_\emptyset$, is a counit. Thus $(\text{sh}, \text{conc}^\dagger = \Delta, 1, \epsilon)$ is a bialgebra, the sh -deconc bialgebra.

To turn this bialgebra into a Hopf algebra, we define an antipode $S: \mathbf{C}\langle\langle A \rangle\rangle \rightarrow \mathbf{C}\langle\langle A \rangle\rangle$ that mimics the antipode $(SW)(\gamma) = W(\bar{\gamma})$ on functions on loop space. On basis elements, let

$$S(\xi^I) = (-1)^{|I|} \xi^{\bar{I}}, \tag{5}$$

where

$$\overline{i_1 i_2 \cdots i_n} = i_n \cdots i_2 i_1.$$

Extend it linearly to $S(F_I \xi^I) = \sum_I (-1)^{|I|} F_I \xi^{\bar{I}}$. For example, $S(\xi^{i_1 i_2 i_3}) = -\xi^{i_3 i_2 i_1}$. This comes from comparing the expansions of $W(\gamma)$ and $W(\bar{\gamma})$ in terms of gluon correlations. For S to be an antipode it must satisfy several conditions which are usually summarized in a commutative diagram (see Ref. 21).

(1) It must be a homomorphism of the commutative shuffle algebra. On basis elements, this is the requirement $S(\xi^I \circ \xi^J) = S(\xi^I) \circ S(\xi^J)$ or equivalently,

$$\sum_{I \sqcup J = K} (-1)^{|K|} \xi^{\bar{K}} = \sum_{\bar{I} \sqcup \bar{J} = L} (-1)^{|L|} \xi^L. \tag{6}$$

This is indeed true. Riffle shuffling two card packs (I, J) preserves the order of each. So reversing the order of the result of the shuffle (each summand on the left hand side) is the same as reversing the order of each card pack (\bar{I}, \bar{J}) and then shuffling them together (each summand on the right hand side). The minus signs just come along for the ride; S would be a homomorphism even without them.

(2) The next two conditions $S(1) = 1$ and $\epsilon S = \epsilon$ are obviously satisfied.

(3) The most interesting requirement for S to be an antipode is its compatibility with deconcatenation and shuffle (if $\mathbf{C}\langle\langle A \rangle\rangle$ were the algebra of functions on a group, these conditions would follow from the property that the product of a group element with its inverse in either order is the group identity),

$$\text{sh}(S \otimes 1)\Delta = \text{sh}(1 \otimes S)\Delta = 1 \epsilon$$

or equivalently,

$$\delta_{JK}^I (-1)^{|J|} \xi^{\bar{J}} \circ \xi^K = \delta_{JK}^I (-1)^{|K|} \xi^J \circ \xi^{\bar{K}} = \delta_{\emptyset}^I \quad \text{for all } I. \tag{7}$$

Putting $I=i_1 \cdots i_n$ these are the conditions

$$\sum_{p=0}^n (-1)^p \xi^{i_p \cdots i_1} \circ \xi^{i_{p+1} \cdots i_n} = \sum_{p=0}^n (-1)^{n-p} \xi^{i_1 \cdots i_p} \circ \xi^{i_{p+1} \cdots i_n} = \delta_0^n. \tag{8}$$

We have not found any nice proof of this, although we verified it explicitly for $n \leq 3$ and observed a pattern of cancelations for higher n which leads us to conjecture that it is an identity. Cartier¹⁹ mentioned that sh-deconc must form a Hopf algebra on general grounds, although we would still like an explicit proof of (7). The minus signs in the definition of the antipode are crucial for this compatibility condition to hold. In the sequel we will assume this condition is satisfied.

III. MATRIX MODEL ANALOG OF THE GROUP OF LOOPS

The sh-deconc Hopf algebra we described is a commutative but noncocommutative Hopf algebra, so it should be the algebra of functions on some non-Abelian group. Which group is it? In the case of YM theory, the corresponding group is that of based loops on space-time. Remarkably, there seems to be an analog of this group for Hermitian Λ -matrix models. One might speculate that it is a group built from $U(N)$ or a free group on Λ generators (since the concatenation algebra of correlations is the free associative algebra), but this is not the case. Rather, we will construct it as the group of complex valued characters (also known as the dual or spectrum) of the sh-deconc Hopf algebra. One might suspect that the analogs of loops are words in the generators of the shuffle algebra; but words do not form a group in a simple-minded way. Nevertheless, we will associate a family of group elements with each word and show that they form a subgroup of the spectrum. In another direction, using a result of Ree and Friedrichs,^{22,23} we will identify the spectrum with the exponential of the free Lie algebra (FLA).

Consider the set of real/complex-valued characters χ of the shuffle algebra, which are not identically zero. These are linear homomorphisms from the commutative shuffle algebra to the complex numbers. Suppose $F, G \in \text{Sh}$ then

$$\chi(F \circ G) = \chi(F)\chi(G)$$

and for $a, b \in \mathbb{C}$,

$$\chi(aF + bG) = a\chi(F) + b\chi(G). \tag{9}$$

It follows that $\chi(1)=1$ for all characters χ . We will define a group structure on this set and call it $\text{spec}(\text{Sh}_\Lambda)$ or spec_Λ . Suppose $\chi(\xi^I) = \chi^I$. Then the complex numbers χ^I , which we call the character coefficients, completely specify the character. For any $F \in \text{Sh}$, $\chi(F) = \chi(F_I \xi^I) = F_I \chi^I$. The χ form a dual space to the $F \in \text{Sh}$, which justifies the upper and lower indices. We can also think of a character as a formal power series $\chi = \chi^I \xi_I$. The identity is taken as the counit $\epsilon: \text{Sh} \rightarrow \mathbb{C}$ defined as $\epsilon(F) = F_\emptyset$, which is a rather trivial homomorphism. In terms of coefficients, $\epsilon^I = \delta_{\emptyset}^I$. The product is non-Abelian, in general, and is defined using $\Delta = \text{conc}^\dagger$. So it encodes the monoid structure of concatenation. More precisely, $\chi\psi = (\chi \otimes \psi)\Delta$, which is a map from $\text{Sh} \otimes \text{Sh} \rightarrow \mathbb{C} \otimes \mathbb{C}$. Then we identify $\mathbb{C} \otimes \mathbb{C}$ with \mathbb{C} by multiplying the two components, to get a map $\text{Sh} \otimes \text{Sh} \rightarrow \mathbb{C}$. On basis elements,

$$(\chi\psi)(\xi^I) = \delta_{JK}^I \chi(\xi^J) \psi(\xi^K) = \delta_{JK}^I \chi^J \psi^K. \tag{10}$$

It is extended linearly to the rest of Sh. So the product of characters is just the concatenation product of character power series. The formula for the product does not use the fact that characters

are homomorphisms of shuffle. But we need the latter property to show that products of characters are also characters. $\Delta = \text{conc}^\dagger$ is a homomorphism of the shuffle product as are χ and ψ . Therefore, $\chi\psi = (\chi \otimes \psi)\Delta$ is also a homomorphism of the shuffle product: $(\chi\psi)(F \circ G) = (\chi\psi)(F)(\chi\psi)(G)$. Indeed, each side is equal to (we identify $\mathbf{C} \otimes \mathbf{C}$ with \mathbf{C})

$$(\Delta F)_{I,J}(\Delta G)_{K,L} \delta_M^{I \sqcup K} \delta_N^{J \sqcup L} \chi^M \psi^N. \tag{11}$$

The inverse of a character is defined by composing with the antipode: $\chi^{-1} = \chi S$. S and χ are homomorphisms of shuffle and so χ^{-1} is also a homomorphism, and hence a character

$$\chi^{-1}(\xi^I) = \chi(S(\xi^I)) = (-1)^{|I|} \chi(\bar{\xi}^I) = (-1)^{|I|} \chi^{\bar{I}}. \tag{12}$$

The conditions $\chi\chi^{-1} = \chi^{-1}\chi = \epsilon$ are precisely the same as the compatibility conditions (7) of the antipode S with product sh and coproduct Δ . Indeed, using the homomorphism property of χ , the second equality in (12) and (7),

$$(\chi\chi^{-1})(\xi^I) = \delta_{JK}^I \chi(\xi^J) (-1)^{|K|} \chi(\bar{\xi}^K) = \chi(\delta_{JK}^I (-1)^{|K|} \xi^J \circ \bar{\xi}^K) = \chi(\delta_{\emptyset}^I) = \epsilon(\xi^I). \tag{13}$$

Similarly we verify that $\chi^{-1}\chi = \epsilon$ is equivalent to (7). Thus characters form a group. The shuffle algebra is the commutative algebra of functions with pointwise product on the group of characters. The value of a function F at the character χ is obtained by evaluating the character on F : $F(\chi) \equiv \chi(F) = \chi^I F_I$. Moreover, $(F \circ G)(\chi) = F(\chi)G(\chi)$ since $\chi(F)\chi(G) = \chi(F \circ G)$.

We still need to find nontrivial characters. If $\chi = \chi^I \xi_I$ wants to be a character, χ^I cannot be arbitrary. On the one hand, $\chi(F) = F_I \chi^I$ may not converge, but we can consider polynomial F so that the series terminates. On the other hand, χ must be a homomorphism of sh , and this imposes relations on the χ^I . For polynomials F and G , $\chi(F \circ G) = \chi(F)\chi(G)$ is satisfied if and only if $\chi(\xi^I \circ \xi^J) = \chi(\xi^I)\chi(\xi^J)$ for all I, J , or equivalently

$$\sum_{I \sqcup J = K} \chi^K = \chi^I \chi^J \quad \text{for all } I, J. \tag{14}$$

These conditions were called shuffle relations in another context.²² They are the complete set of conditions for χ to be a character. In detail, the first few shuffle relations are

$$\begin{aligned} \chi^\emptyset &= 1, & \chi^{ij} + \chi^{ji} &= \chi^i \chi^j, & \chi^{ijk} + \chi^{jik} + \chi^{kji} &= \chi^i \chi^j \chi^k, \\ \chi^{ijkl} + \chi^{ikjl} + \chi^{iklj} + \chi^{kijl} + \chi^{kilj} + \chi^{klij} &= \chi^i \chi^j \chi^k \chi^l, \\ \chi^{ijkl} + \chi^{jikl} + \chi^{jkil} + \chi^{kjli} &= \chi^i \chi^j \chi^k \chi^l, \text{ etc.} \end{aligned} \tag{15}$$

For rank $n \geq 1$ character coefficient tensors $\chi^{i_1 \dots i_n}$, there are $[n/2]$ systems of linear shuffle relations [i.e., either $\frac{1}{2}(n-1)$ or $n/2$ according as n is odd or even]. The shuffle relations are hierarchical, in the sense that the rank of the tensors on the left hand side ($|I| + |J|$) always exceeds the rank of the tensors on the right hand side ($|I|$ and $|J|$). So we can think of these as linear equations constraining the higher rank χ^K in terms of the lower rank ones which appear quadratically as sources on the right. This structure is reminiscent of the matrix model fSDE: $S^{J_1 i J_2} G_{J_1 J_2} = \delta_I^{i J_1 J_2} G_{J_1} G_{J_2}$.^{9,18} Naively, we expect a large space of solutions to these constraints, since there seem to be a lot more degrees of freedom in the χ^I than there are shuffle relations. In particular, the χ^i are unconstrained. Regard χ^{ij} as a matrix. Then its symmetric part is completely determined by the χ^i , but its antisymmetric part $\frac{1}{2}(\chi^{ij} - \chi^{ji})$ is not.

In the case of loop space, we can find examples of characters easily. For example, a loop $\gamma(t)_{0 \leq t \leq 1}$ on space-time M defines a character of $\text{Sh}(M)$, its value on an element of $F \in \text{Sh}(M)$ is $\gamma(F) = \int \gamma F$, where the right hand side is the iterated ‘‘Chen’’ integral^{5,6} of the linear combination of tensor products of one forms. For example if $F = \alpha \otimes \beta$ for a pair of 1-forms α and β , then

$$\gamma(F) = \int_0^1 dt_1 \int_0^{t_1} dt_2 \alpha_i(\gamma(t_1)) \beta_j(\gamma(t_2)) \dot{\gamma}^i(t_1) \dot{\gamma}^j(t_2). \tag{16}$$

For the shuffle algebra on a finite number of generators, we might imagine that the analog of a loop is a word ξ_I and define a linear functional on Sh by $\xi_I(F) = F_I$. However, this is not a character since $\xi_I(1) = \delta_I^\emptyset$, whereas for a character we must have $\chi(1) = 1$. Although single words are in the dual of Sh regarded as a vector space, they are not (with the exception of the empty word) in the dual of Sh regarded as an algebra.

Regarding a character as a formal series $\chi(\xi) = \sum_I \chi^I \xi_I$, we ask whether there are any characters aside from the identity ϵ . First, we show using the shuffle relations that there are no nontrivial polynomial characters. Suppose χ is one, of degree $n-1$. What this means is that $\chi^K = 0$ for all words K of length $|K| \geq n \geq 2$, but with $\chi^I \neq 0$ for some I of rank $|I| = n-1$. Then consider the shuffle relation $\chi(\xi^I \circ \xi^J) = \chi^I \chi^J$ which is the same as

$$\sum_{I, J \sqcup I = K} \chi^K = \chi^I \chi^J \quad (\text{no sum on } I). \tag{17}$$

The left hand side is equal to 0, since it is a linear combination of character coefficients of rank $2n-2 \geq n$. But the right hand side is not equal to 0 by assumption, a contradiction. So the only polynomial character is the identity $\chi = \epsilon$. To find nontrivial characters, let us specialize first to the case of a single generator.

A. Characters of the shuffle algebra on one generator Sh₁

For $\Lambda = 1$, a character is a formal series $\chi = \sum_{n=0}^\infty \chi_n \xi^n$ in one generator ξ . The condition that it be a homomorphism of sh is $\chi_0 = 1$, and the following shuffle relations for each $\chi_n, n \geq 1$:

$$\binom{n}{r} \chi_n = \chi_r \chi_{n-r} \quad \text{for } r = 1, 2, 3, \dots, [n/2]. \tag{18}$$

In more detail, $\chi_0 = 1, 2\chi_2 = \chi_1^2, 6\chi_4 = \chi_2^2, 4\chi_4 = \chi_1\chi_3, 5\chi_5 = \chi_1\chi_4, 10\chi_5 = \chi_2\chi_3$, etc. The general solution is a one-parameter family $\chi_n = (1/n!) \chi_1^n$ for $n \geq 0$. We write $\chi = e^{\chi_1 \xi}$. In particular, there are no polynomial characters. Moreover, if $\chi_1 \neq \psi_1$ then $e^{\chi_1 \xi}$ and $e^{\psi_1 \xi}$ are distinct characters as they have different coefficients in their power series expansions. The identity character is got by choosing $\chi_1 = 0$, in which case $\chi = 1$. The product $\chi\psi$ is the character whose value on monomials is

$$(\chi\psi)(\xi^n) = \sum_{r=0}^n \chi_{n-r} \psi_r. \tag{19}$$

$(\chi\psi)_1 = \chi_1 + \psi_1$ so $\chi\psi$ is the character $e^{(\chi_1 + \psi_1)\xi}$, which agrees with the usual rule for multiplying $\chi = e^{\chi_1 \xi}$ and $\psi = e^{\psi_1 \xi}$. The product is Abelian, since we have a single generator. The inverse of $\chi = e^{\chi_1 \xi}$ is $\chi^{-1} = e^{-\chi_1 \xi}$. We call the group of characters of Sh₁ as spec(Sh₁) or spec₁ for short. Although space-time has been reduced, in a sense, to a single point, spec₁ is a continuous Abelian group parametrized by one real/complex number χ_1 . Indeed we can even define a one dimensional Abelian Lie algebra on the vector space $\{\chi_1 \xi \mid \chi_1 \in \mathbf{C} \text{ or } \mathbf{R}\}$ with Lie bracket $[\chi_1 \xi, \psi_1 \xi] = 0$. $\chi = e^{\chi_1 \xi}$ is an exponential map from the Lie algebra to the group.

If we consider real-valued characters, then $f: e^{\chi_1 \xi} \mapsto e^{\chi_1}$ is an isomorphism from spec₁ to the multiplicative group of nonzero reals \mathbf{R}^* . For complex-valued characters, f is a homomorphism from spec₁ onto \mathbf{C}^* , the multiplicative group of nonzero complex numbers. Its kernel is the subgroup generated by the character $e^{2\pi i \xi}$, i.e., the subgroup $\{e^{2\pi i m \xi} \mid m \in \mathbf{Z}\}$.

We think of the shuffle algebra Sh₁ as the commutative algebra of functions on the group spec₁. The value of $F = F_n \xi^n$ at χ is

$$F(\chi) \equiv \chi\left(\sum_n F_n \xi^n\right) = \sum_n \chi_n F_n \tag{20}$$

or equivalently

$$\xi^n(\chi) \equiv \chi(\xi^n) = \chi_n.$$

The shuffle product is the same as the pointwise product of functions on spec_1 , since characters are homomorphisms of the shuffle algebra,

$$(\xi^n \circ \xi^m)(\chi) = \chi(\xi^n \circ \xi^m) = \binom{n+m}{n} \chi(\xi^{n+m}) = \binom{n+m}{n} \chi_{n+m} = \chi_n \chi_m = \xi^n(\chi) \xi^m(\chi). \tag{21}$$

B. Group of characters of Sh_Λ : Pure characters

We now discuss the group of characters of the sh-deconc Hopf algebra on $\Lambda > 1$ generators. Corresponding to the inclusions $\xi_i \mapsto \{\xi_1, \dots, \xi_\Lambda\}$ we get Λ Abelian one-parameter subgroups of $\text{spec}(\text{Sh}_\Lambda)$, namely, $e^{\chi^1 \xi_1}, e^{\chi^2 \xi_2}, \dots, e^{\chi^\Lambda \xi_\Lambda}$. For example, the value of $e^{\chi^3 \xi_3}$ on a basis element is

$$(e^{\chi^3 \xi_3})(\xi^I) = \begin{cases} \frac{(\chi^3)^n}{n!} & \text{if } I = 333 \dots 33 \text{ (} n \text{ times)} \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Call a character χ pure if it does not mix the letters $\{\xi_1, \dots, \xi_\Lambda\}$ i.e., $\chi^I = 0$ if ξ^I contains at least two distinct letters. The only pure characters are listed above. Equivalently, a letter ξ_{i_1} from the alphabet $\{\xi_1, \dots, \xi_\Lambda\}$ determines a 1-parameter family of pure characters $e^{\chi^1 \xi_{i_1}}$.

C. Mixed characters

A natural question is whether given a word ξ_I , it is possible to obtain a (family of) character(s) associated to it? This would in a sense be analogous to associating the character of the shuffle algebra $\text{Sh}(M)$, $\int_\gamma F$ with each based loop γ on space-time where $F \in \text{Sh}(M)$.

Call a character χ mixed if it is not pure, i.e., if there is a word ξ_I containing at least two distinct letters such that $\chi^I \neq 0$. Many mixed characters are obtained by multiplying pure ones. Given a word $\xi_I = \xi_{i_1} \xi_{i_2} \dots \xi_{i_n}$ and a sequence of complex numbers $\{\chi^1, \dots, \chi^n\}$, we define a character via the product of pure characters [products of characters are characters (Sec. III)],

$$\psi = e^{\chi^1 \xi_{i_1}} e^{\chi^2 \xi_{i_2}} \dots e^{\chi^n \xi_{i_n}}. \tag{23}$$

A word-sequence pair $(\xi_I, \vec{\chi}) = (\xi_{i_1} \dots \xi_{i_n}, \{\chi^1 \dots \chi^n\})$ determines a character. The inverse of χ is also of the same form, $\psi^{-1} = e^{-\chi^1 \xi_{i_1}} e^{-\chi^2 \xi_{i_2}} \dots e^{-\chi^n \xi_{i_n}}$ and satisfies $\psi \psi^{-1} = \psi^{-1} \psi = \epsilon$.

Reduced form of word-sequence pairs: Given a word $\xi_{i_1} \dots \xi_{i_{j-1}} \xi_{i_j} \xi_{i_{j+1}} \dots \xi_{i_n}$ and a sequence $(\chi^1, \dots, \chi^{j-1}, \chi^j, \chi^{j+1}, \dots, \chi^n)$, we define the reduced form of the pair. If a pair of adjacent letters coincide, $\xi_{i_j} = \xi_{i_{j+1}}$, then delete ξ_{i_j} and χ^j and replace χ^{j+1} with $\chi^j + \chi^{j+1}$ to get a new word $\xi_{i_1} \dots \xi_{i_{j-1}} \xi_{i_{j+1}} \dots \xi_{i_n}$ and a new sequence $\chi^1, \dots, \chi^{j-1}, \chi^j + \chi^{j+1}, \dots, \chi^n$. The resulting character is the same as the original one. Moreover, if any of the numbers χ_k vanishes, just delete it and the letter ξ_{i_k} . Proceeding this way, we get a word whose adjacent letters are always distinct and a sequence of nonzero complex numbers (the one exception is if the word is empty). Such a word along with its sequence of complex numbers is in reduced form. Of course, the length of the word is the same as the length of the sequence. Two pairs $(\xi_I, \vec{\chi})$ and $(\xi_J, \vec{\psi})$ are equivalent if they have the same reduced forms. Equivalent pairs correspond to the same character. For example, the reduced form of the pair $(\xi_3 \xi_1 \xi_1 \xi_2, \{i, -3, \pi, 0\})$ is $(\xi_3 \xi_1, \{i, \pi - 3\})$ and corresponds to the character $e^{i \xi_3} e^{(\pi - 3) \xi_1}$ whose inverse is $e^{(3 - \pi) \xi_1} e^{-i \xi_3}$. A pure character $e^{\chi^1 \xi_{i_1}}$ has reduced form $(\xi_{i_1}, \{\chi^1\})$ and the identity character the reduced form $(\emptyset, \{\})$.

We can multiply two characters corresponding to the words $\xi_{i_1} \cdots \xi_{i_n}$ and $\xi_{i_{n+1}} \cdots \xi_{i_{n+m}}$ to get a character corresponding to the concatenated word $\xi_{i_1} \cdots \xi_{i_{n+m}}$ and the concatenated sequence of complex numbers $\chi^1, \dots, \chi^{n+m}$,

$$e^{\chi^1 \xi_{i_1}} e^{\chi^2 \xi_{i_2}} \cdots e^{\chi^n \xi_{i_n}} e^{\chi^{n+1} \xi_{i_{n+1}}} e^{\chi^{n+2} \xi_{i_{n+2}}} \cdots e^{\chi^{n+m} \xi_{i_{n+m}}}. \tag{24}$$

This product is maximally non-Abelian. Products of pure characters form a subgroup of the group of characters of Sh_Λ . This subgroup is the free product of Λ copies of spec_1 ,

$$\text{Group of prod. of pure charac.} \cong \text{spec}_1 \times \text{spec}_1 \times \cdots \times \text{spec}_1 \quad (\Lambda \text{ factors}). \tag{25}$$

We know that the free group on n generators F_n is the same as the free product of n copies of the integers F_1 . By contrast, we will see that $\text{spec}_1 \times \text{spec}_1 \times \cdots \times \text{spec}_1$ is a proper subgroup of spec_Λ , at least if we restrict to finite products. It is interesting to know the appropriate topology for such free products of continuous groups ($\text{spec}_1 \cong \mathbf{R}^*$, for instance).

The exponential of any finite linear combination of generators $\chi = e^{\chi^j \xi_j}$ is a character. To show this, consider

$$\chi = e^{\chi^j \xi_j} = \sum_{n=0}^{\infty} \frac{1}{n!} \chi^{j_1} \xi_{j_1} \cdots \chi^{j_n} \xi_{j_n} = \sum_{n=0}^{\infty} \frac{1}{n!} \chi^{j_1} \cdots \chi^{j_n} \xi_{j_1} \cdots \xi_{j_n}. \tag{26}$$

The coefficients are symmetric tensors $\chi^{j_1 \cdots j_n} = (1/n!) \chi^{j_1} \cdots \chi^{j_n}$ which must satisfy the shuffle relations $\sum \chi^{I \cup J} = \chi^I \chi^J$. Taking $I = i_1 \cdots i_n$ and $J = j_1 \cdots j_m$, on the left hand side there are $\binom{n+m}{n}$ terms all of which are equal, so

$$\text{LHS} = \binom{n+m}{n} \chi^{i_1 \cdots i_n j_1 \cdots j_m} = \frac{1}{n!m!} \chi^{i_1} \cdots \chi^{i_n} \chi^{j_1} \cdots \chi^{j_m} = \text{RHS}. \tag{27}$$

$e^{\chi^j \xi_j}$ is thus a character, but since ξ_j do not commute it cannot be written as a finite product of the pure characters. The inverse of χ is $\chi^{-1} = e^{-\chi^j \xi_j}$, which is also of the same form. Since products of characters satisfy the shuffle relations, finite products of the form $\chi = e^{\chi^1 \xi_1} e^{\chi^2 \xi_2} \cdots e^{\chi^n \xi_n}$ form a group which properly contains $\text{spec}_1 \times \cdots \times \text{spec}_\Lambda$ and is a proper subgroup of spec_Λ . As before, we can put any such product in a reduced form.

Exponentials of arbitrary nonlinear polynomials in the generators are *not* characters, in general. For instance, it is easy to see that $\chi = e^{\xi_1 + \xi_1 \xi_2}$ does not satisfy the shuffle relation $\chi^{12} + \chi^{21} = \chi^1 \chi^2$. On the other hand, the Baker–Campbell–Hausdorff (BCH) formula tells us that $e^{\chi^i \xi_i} e^{\chi^j \xi_j} = e^{\chi^i \xi_i + \chi^j \xi_j + (1/2) \chi^i \chi^j [\xi_i, \xi_j] + (1/12) \chi^i \chi^j \chi^k [\xi_i, \xi_j, \xi_k] + \cdots}$. Using BCH we can re-express the products of exponentials occurring in the above subgroup of spec_Λ as exponentials of linear combinations of nested commutators of the generators ξ_i . Aside from $\chi^i \xi_i$, these will be certain infinite linear combinations since the ξ_i do not commute. This suggested to us that exponentials of finite (or *other* infinite) linear combinations of nested commutators of ξ_i may also be characters. While we verified this in some simple cases, the calculations rapidly get laborious. So we were pleasantly surprised to find this proven in the work of Ree²² using a theorem of Friedrichs.²³ More precisely, linear combinations of iterated commutators of the generators ξ_i are called Lie elements. They are obtained using the operations of taking Lie brackets $[.,.]$ and linear combinations but not products such as $\xi_i \xi_j$. For example, $C^i \xi_i + C^{ijk} [\xi_i, [\xi_j, \xi_k]] + C^{ijkl} [[\xi_i, \xi_j], [\xi_k, \xi_l]]$ are Lie elements for any tensors C^i, C^{ijk}, C^{ijkl} . Ree proved that exponentials of Lie elements are the only formal series satisfying the shuffle relations. In other words, the group spec_Λ consists precisely of exponentials of Lie elements. This characterization will be useful since the SD operators of an interesting class of matrix models related to YM theory turn out to be Lie elements. Now that we have identified the group $\text{spec}(\text{Sh}_\Lambda)$ which plays the role of the group of loops, we can formulate differential calculus on it.

Technically, the group of loops on space-time is a proper subgroup of a larger group of generalized loops (in the language of Ref. 6) or extended loop group (in the language of Refs. 1 and 13). spec_Λ is the analog of this larger group, while the group generated by pure characters is

the analog of the smaller group of loops. Both in the space-time and matrix model settings, the larger group behaves akin to a classical Lie group and appears to be the correct physical setting for the fSDE. For instance, the SD operators of several matrix models can be interpreted as right-invariant vector fields on the larger group, but not on the smaller one (Sec. V). Moreover, just as the group generated by pure characters is a free product of Λ copies of spec_1 , the group of loops is also a free product (free group generated by the based loops).

IV. DIFFERENTIAL CALCULUS ON THE GROUP $\mathbf{G}=\text{spec}_\Lambda$

A. Functions

By a function on the group $\mathbf{G}=\text{spec}_\Lambda$ we will mean an element of the shuffle algebra $F(\xi) = F_I \xi^I$. Its value at the character $\chi = \chi^I \xi_I \in \text{spec}_\Lambda$ is given by $F(\chi) \equiv \chi(F) = F_I \chi^I$. The ring of such functions is the commutative shuffle algebra Sh_Λ with pointwise product of F and G at χ given by the shuffle product $\sum_I (F \circ G)_I \chi^I$.

B. Vector fields and the Lie algebra of spec_Λ

By a vector field V on spec_Λ we will mean a derivation of the shuffle algebra, i.e., a map that takes functions to functions $V: \text{Sh} \rightarrow \text{Sh}$, that is, linear over the complex numbers and satisfies the Leibnitz rule $V(F \circ G) = VF \circ G + F \circ VG$. This extends the concept of vector fields to settings more general than differentiable manifolds.²⁴ The derivations must form a left module over the shuffle algebra as well as a Lie algebra over the ring of functions (i.e. FV must be a vector field and the structure functions of the Lie algebra of vector fields must be in Sh).

From Sec. III C, a necessary and sufficient condition for χ to be in $\mathbf{G}=\text{spec}_\Lambda$ is that $\log \chi(\xi)$ must be a Lie element, i.e., a linear combination of iterated commutators of $\xi_1, \dots, \xi_\Lambda$. The set of Lie elements is closed under commutators and forms the free Lie algebra of rank Λ (FLA_Λ). It is a FLA since there are no relations besides linearity, antisymmetry, and Jacobi identity that are satisfied by the commutator brackets. This is reminiscent of the exponential map from the Lie algebra to a Lie group. So we expect the Lie algebra of right (or left) invariant derivations of Sh_Λ (which should play the role of Lie algebra of the group spec_Λ) to be isomorphic to FLA_Λ . We will see that this is indeed the case.

In Ref. 9 we showed that linear combinations of iterated commutators of left annihilation D_i are derivations of the shuffle algebra, so they may be considered vector fields on spec_Λ . We recall why D_i satisfies the Leibnitz rule: $[D_i(F \circ G)]_I = [F \circ G]_{iI} = \sum_{I_1 \sqcup I_2 = iI} F_{I_1} G_{I_2}$. Now either $i \in I_1$ or $i \in I_2$, so

$$\begin{aligned}
 [D_i(F \circ G)]_I &= \sum_{I_1 \sqcup I_2 = I} (F_{iI_1} G_{I_2} + F_{I_1} G_{iI_2}) = \sum_{I_1 \sqcup I_2 = I} ([D_i F]_{I_1} G_{I_2} + F_{I_1} [D_i G]_{I_2}) \\
 &= [(D_i F) \circ G]_I + [F \circ (D_i G)]_I.
 \end{aligned}
 \tag{28}$$

Moreover, commutators of left annihilation do not satisfy any relations besides linearity, antisymmetry, and the Jacobi identity. So iterated commutators of D_i span a FLA of rank Λ . There is a standard basis $D_{(L)}$ for the FLA, labeled by Lyndon words L . $D_{(L)}$ is a particular iterated commutator of D_i 's, where $L = l_1 \cdots l_n$ is a Lyndon word (see Appendix A 1). So an element of the above FLA of vector fields is written $V = V_{\varnothing}^L D_{(L)}$, where V_{\varnothing}^L are (real or complex) constants and the sum is over all Lyndon words L . The value of such a vector field at the point $\chi = \chi^I \xi_I$ is the ‘‘tangent vector’’ $V_{\varnothing}^L \chi^{\varnothing} D_{(L)} = V_{\varnothing}^L D_{(L)}$, since $\chi^{\varnothing} = 1$ for a character. These ‘‘constant coefficient’’ vector fields can be regarded as forming a subalgebra of the Lie algebra of spec_Λ . For, by evaluating at the identity $\chi = \epsilon$, they span a space of tangent vectors at the identity. We believe these constant coefficient vector fields should be regarded as the whole of the Lie algebra of \mathbf{G} . The structure constants of the FLA of basis vector fields $D_{(L)}$ are denoted $[D_{(L)}, D_{(M)}] = c_{L,M}^N D_{(N)}$, see Appendix A 1, for examples.

A vector field V on a group is distinguished if it commutes with the action of the group on itself by multiplication (encoded in the coproduct Δ on the Hopf algebra of functions). Roughly,

this means ΔV must equal $V\Delta$. But that cannot be quite right since $\Delta: \text{Sh} \rightarrow \text{Sh} \otimes \text{Sh}$, while $V: \text{Sh} \rightarrow \text{Sh}$, so we must specify whether V acts on the first or second slot. In fact, we must distinguish the right action from the left action, which leads to the definitions $\Delta V = (V \otimes 1)\Delta$ and $\Delta V = (1 \otimes V)\Delta$.²⁵ Alternatively, let R_g be the right translation by g on a group \mathbf{G} and R_g^* and R_{g*} the pull-back and push-forward maps. Then the push forward of a vector field V acts on a function f according to $R_{g*} Vf = R_g^{*-1} V R_g^* f$. Now a vector field is right invariant if $R_{g*} V = V$ or, in other words, $R_g^* Vf = V R_g^* f$ for all f . But R_g^* is the pull back induced by right multiplication in the group, and multiplication in the group is encoded in the coproduct Δ in the Hopf algebra of functions on \mathbf{G} . This justifies the definitions of right and left invariant derivations by $\Delta V = (V \otimes 1)\Delta$ and $\Delta V = (1 \otimes V)\Delta$.

If vector fields V and W are right invariant, then so is their product VW (although not a vector field) and hence also their commutator $[V, W]$ (which is a vector field, since commutators of derivations are also derivations). To see this,

$$(V \otimes 1)\Delta = \Delta V \Rightarrow (V \otimes 1)\Delta W = \Delta VW. \tag{29}$$

Using the right invariance of W to re-express ΔW , this becomes

$$(V \otimes 1)(W \otimes 1)\Delta = \Delta VW \Leftrightarrow (VW \otimes 1)\Delta = \Delta VW. \tag{30}$$

Thus right-invariant derivations form a Lie algebra, which will serve as a substitute for the Lie algebra of right-invariant vector fields on the group \mathbf{G} .

It remains to identify the right-invariant derivations in the case $\mathbf{G} = \text{spec}(\text{Sh}_\Lambda)$. We will show that the constant coefficient vector fields $V = V_{\mathcal{O}}^L D_{(L)}$ are, in fact, right invariant. It is straightforward to check that D_i is a right-invariant derivation,

$$\begin{aligned} (D_i \otimes 1)\Delta G &= (D_i \otimes 1)G_{JK} \xi^J \otimes \xi^K = G_{iJK} \xi^J \otimes \xi^K, \\ \Delta(D_i G) &= \Delta(G_{iJ} \xi^J) = G_{iJ} \delta_{JK}^J \xi^J \otimes \xi^K = G_{iJK} \xi^J \otimes \xi^K. \end{aligned} \tag{31}$$

By the previous result, we deduce that iterated commutators of D_i are also right invariant. For example, $[[D_i, D_j], [D_k, D_l]]$ and $[D_i, [D_j, D_l]]$ are right-invariant derivations. Moreover, the condition $\Delta V = (V \otimes 1)\Delta$ is linear in V , so real/complex linear combinations of iterated commutators of D_i are also right invariant. In other words, constant coefficient vector fields $V = V_{\mathcal{O}}^L D_{(L)}$ are right invariant on \mathbf{G} . In particular, the SD operators $S^i = 4g^{ik} g^{jl} [D_j, [D_k, D_l]]$ of YM matrix models with action $S = \text{tr} g^{ik} g^{jl} [A_i, A_j][A_k, A_l]$ are right-invariant vector fields (see Sec. V).

A more general derivation of the shuffle algebra is obtained by allowing nonconstant coefficients (elements of Sh), $V = V^L(\xi) D_{(L)} = V_I^L \xi^I D_{(L)}$. V acts on a function as $VF = V^L(\xi) \circ (D_{(L)} F(\xi))$. The derivation property follows from that of $D_{(L)}$ and commutativity sh),

$$V^L(\xi) D_{(L)}(F \circ G) = V^L(\xi) (D_{(L)} F \circ G + F \circ D_{(L)} G) = VF \circ G + F \circ VG. \tag{32}$$

Thus, we can think of $V = V^L(\xi) D_{(L)}$ as a vector field on the group $\mathbf{G} = \text{spec}_\Lambda$. These derivations of Sh form a left module over the ring of functions on spec_Λ : shuffle multiplying a derivation $V^L(\xi) D_{(L)}$ on the left by a function $F(\xi)$ gives another derivation $\{F(\xi) \circ V^L(\xi)\} D_{(L)}$.

It is interesting to know whether there are any more derivations of the shuffle algebra; these seem to be adequate for us. We call the space of variable coefficient derivations $\text{Vect}(\mathbf{G})$. In a sense, the $D_{(L)}$ form a globally defined moving frame so that the tangent bundle of \mathbf{G} is trivial, i.e., spec_Λ is parallelizable just like any Lie group. $\text{Vect}(\mathbf{G})$ forms a Lie algebra with Lie bracket (all products are shuffle products),

$$[V, W] = (VW^L) D_{(L)} - (WV^L) D_{(L)} + c_{L,M}^N V^L \circ W^M D_{(N)}. \tag{33}$$

Here VW^L is the action of the vector field V on the function $W^L(\xi)$, $VW^L = V^M(\xi) \circ D_{(M)} W^L(\xi)$. We used the fact that $D_{(L)}$ satisfies the Leibnitz rule with respect to sh. This is the analog of the

formula for Lie brackets of vector fields on a manifold $[v, w] = v^i \partial_i w^j \partial_j - w^i \partial_i v^j \partial_j$. $D_{(L)}$ play the role of ∂_i , except they do not commute.

The value of vector field $V = V^L(\xi)D_{(L)}$ at $\chi \in \mathbf{G}$ is obtained by evaluating the coefficient functions on the character χ , to get a tangent vector $V^L \chi^J D_{(L)}$. In particular, the value of a vector field at the group identity $\chi^J = \delta^J_{\emptyset}$ is the tangent vector $V^L_{\emptyset} D_{(L)}$. Thus the space of tangent vectors at the identity is the same as the space of constant coefficient vector fields, which we have also observed to be right-invariant vector fields. Their Lie algebra is isomorphic to FLA_{Λ} ; we are now justified to think of it as the Lie algebra of the group $\mathbf{G} = \text{spec}_{\Lambda}$.

We argue that the nonconstant coefficient vector fields V cannot be right invariant. Suppose there were a right-invariant V , with $V^L_I \neq 0$ for some nonempty I . V as well as the distinct right-invariant vector field with constant coefficients $V^L_{\emptyset} D_{(L)}$, both evaluate to the same tangent vector at the identity: $V^L_{\emptyset} D_{(L)}$. However, there should be a unique right-invariant vector field on \mathbf{G} that is obtained by right translating the tangent vector $V^L_{\emptyset} D_{(L)}$, and the constant coefficient vector field $V^L_{\emptyset} D_{(L)}$ serves that purpose. So V could not be right invariant. We would still like a combinatorial proof that nonconstant coefficient vector fields cannot satisfy $\Delta V = (V \otimes 1)\Delta$.

C. One-forms

A one-form is a linear function from $\text{Vect}(\mathbf{G})$ to Sh . The dual basis of one-forms θ^L is also labeled by Lyndon words L . On basis vector fields, $\theta^L(D_{(M)}) = \delta^L_M$. Extended linearly to $\text{Vect}(\mathbf{G})$, $\theta^L(V^M(\xi)D_{(M)}) = V^L(\xi)$. A general one-form $\omega = \omega_L(\xi)\theta^L$ is a linear combination of the basis θ^L with coefficients coming from Sh . Although $D_{(L)}$ was defined (Appendix A) through iterated commutators of left annihilation D_i , we have not built θ^L from θ^j . The exterior derivative of an element of the shuffle algebra is a one-form defined by its action on vector fields,

$$dF(V^L(\xi)D_{(L)}) = V^L(\xi) \circ D_{(L)}F. \tag{34}$$

From this we can read off that if $dF = (dF)_L \theta^L$, then the components $(dF)_L = D_{(L)}F$.

D. Differential calculus on spec_1

Let us illustrate the above formalism in the simplest case of one generator. The spectrum of the shuffle algebra on one generator consists of the exponential series $\chi(\xi) = \sum_{n \geq 0} \chi_n \xi^n$ with $\chi_n = (1/n!) \chi_1^n$. Functions on spec_1 are elements of the shuffle algebra $F = \sum_n F_n \xi^n$ with the value $F(\chi) = \sum_{n \geq 0} F_n \chi_n$. $D\xi^n = \xi^{n-1}$ is the only left annihilation operator and its commutators vanish. So the Lyndon basis of the FLA on one generator is just D and it is an Abelian algebra. Moreover D is right invariant since

$$\Delta D \xi^n = \Delta \xi^{n-1} = \sum_{p+q=n-1} \xi^p \otimes \xi^q$$

and

$$(D \otimes 1)\Delta \xi^n = (D \otimes 1) \sum_{p+q=n} \xi^p \otimes \xi^q = \sum_{p+q=n} \xi^{p-1} \otimes \xi^q \tag{35}$$

are equal. The general vector field is $V = V(\xi)D$, where $V(\xi) = \sum_{n \geq 0} V_n \xi^n$ and it is right invariant if and only if $V(\xi)$ is a constant. V restricts to the tangent vector $\sum_{n \geq 0} V_n \chi_n D$ at the point χ on the group spec_1 and to the tangent vector $V_0 D$ at the identity.

The one-form dual to the vector field D is θ with $\theta(D) = 1$. The general one-form is $\omega = \omega(\xi)\theta$. The value of ω on the vector field $V(\xi)D$ is the shuffle product $\omega(V) = \omega(\xi) \circ V(\xi)$. The exterior derivative of a function is $dF = (DF)\theta$ and for a monomial, $d\xi^n = \xi^{n-1}\theta$. The one-form ω restricts to the covector $\sum_{n \geq 0} \omega_n \chi_n \theta$ at the point χ and to the covector $\omega_0 \theta$ at the identity.

V. fSDE on $\mathbf{G} = \text{spec}_\Lambda$

We found the group spec_Λ that plays the role of the group of based loops and identified the rudiments of differential calculus on it. Now we would like to formulate the fSDE of matrix models in terms of spec_Λ . This may indicate how to generalize the fSDE to groups we are more familiar with, and thereby provide more insight into their solutions. The fSDE $\mathcal{S}^i G(\xi) = G(\xi) \xi^i G(\xi)$ are a system of equations for the moment generating series $G(\xi) = G_I \xi^I$. $G(\xi)$ is an element of the shuffle algebra or, equivalently, a function on the group spec_Λ . \mathcal{S}^i are called the SD operators. The only *a priori* conditions are that G must evaluate to 1 at the group identity $\chi = \epsilon$ (i.e., $G_\emptyset = 1$), the coefficients G_I must be cyclically symmetric and satisfy the reality condition $G_I^* = G_{\bar{I}}$. These follow from the physical requirements of normalization of expectation values $\langle (1/N) \text{tr} 1 \rangle = 1$, cyclicity of the trace $G_I = \lim_{N \rightarrow \infty} \langle (1/N) \text{tr} A_I \rangle$ and reality of the matrix model action $S(A)$. It will be useful to keep the Gaussian (G), Chern–Simons (CS), and Yang–Mills (YM) ($g^{ij} = g^{ji}$) examples in mind,

$$S_G = \frac{1}{2} \text{tr } C^{ij} A_i A_j, \quad S_{\text{CS}} = \frac{2\sqrt{-1}\kappa}{3} \text{tr } \epsilon^{ijk} A_i A_j A_k, \quad S_{\text{YM}} = g^{ik} g^{jl} [A_i, A_j] [A_k, A_l]. \quad (36)$$

They are the zero-momentum limits of corresponding field theories. Their SD operators are⁹

$$\mathcal{S}_G^i = C^{ij} D_j, \quad \mathcal{S}_{\text{CS}}^i = \sqrt{-1} \kappa \epsilon^{ijk} [D_k, D_j], \quad \mathcal{S}_{\text{YM}}^i = 4g^{ik} g^{jl} [D_j, [D_k, D_l]]. \quad (37)$$

There is one fSDE for each letter ξ^i . But what does a *letter* mean in terms of the group? We noticed that letters are primitive elements of the commutative Hopf algebra of functions on \mathbf{G} . Primitive elements P are those that satisfy $\Delta P = 1 \otimes P + P \otimes 1$, where $\Delta = \text{conc}^\dagger$ is deconcatenation, which is not cocommutative. [In other words $\tau \Delta \neq \Delta$, where $\tau(F \otimes G) = G \otimes F$ reverses the order of factors. Note that this is distinct from the Poincaré–Birkhoff–Witt construction where the primitives of a *cocommutative* Hopf algebra form a Lie algebra whose universal envelope is the Hopf algebra.] On monomials, $\Delta \xi^I = \delta_{JK}^I \xi^J \otimes \xi^K$. Let us show that the only primitive elements are linear combinations of letters ξ^i , $1 \leq i \leq \Lambda$. For an element $P(\xi) = P_I \xi^I$ of the shuffle algebra to be primitive, we need $P_{JK} \xi^J \otimes \xi^K = P_I \xi^I \otimes 1 + 1 \otimes P_I \xi^I$. This is equivalent to the requirements $P_\emptyset = 0$ and $\sum_{J,K \neq \emptyset} P_{JK} \xi^J \otimes \xi^K = 0$. Since $\xi^J \otimes \xi^K$ span a basis for $\text{Sh} \otimes \text{Sh}$, these conditions are satisfied if and only if $P_\emptyset = 0$ and $P_I = 0$ for $|I| \geq 2$. In other words, linear combinations of letters are the only primitives. So if we pick any basis for the vector space of primitives (such as the letters themselves), we will have one fSDE for each basis element. This is again consistent with the fact that we could rewrite the fSDE as $w_i^j \mathcal{S}^i G(\xi) = G(\xi) w_i^j \xi^i G(\xi)$ for any nonsingular $\Lambda \times \Lambda$ matrix w_i^j . This characterization in terms of the primitives of the Hopf algebra of functions on \mathbf{G} applies to any group.

The right hand side of the fSDE $G(\xi) \xi^i G(\xi) = G_H G_J \xi^{HiJ}$ involves concatenation. It can be understood in terms of the convolution product of functions on the group \mathbf{G} . Given a non-Abelian group \mathbf{G} , there are two natural dual Hopf algebras associated with it,¹⁹ the commutative algebra $\mathbf{C}^{\mathbf{G}}$ of complex functions $F(g)$ on it with pointwise product $(FG)(g) = F(g)G(g)$ and the noncommutative group algebra $\mathbf{C}\mathbf{G} = \{\sum_{g \in \mathbf{G}} F(g)g\}$ with convolution product $(FG)(g) = \sum_{h \in \mathbf{G}} F(h)G(h^{-1}g)$. The coproduct of the first becomes the product in the second and vice versa using the duality $\langle \sum_{h \in \mathbf{G}} F(h)h, G \rangle = \sum_{g \in \mathbf{G}} F(g)G(g)$. In our case $\mathbf{G} = \text{spec}_\Lambda$, sh-deconc is the commutative Hopf algebra whose coproduct Δ is deconcatenation. The dual Hopf algebra conc-desshuffle is the convolution algebra of functions on \mathbf{G} , whose product is concatenation $(FG)_I \xi^I = F_J G_K \xi^{JK}$ and coproduct is desshuffle $\Delta' F = F_{I \sqcup J} \xi^I \otimes \xi^J$.⁹ Thus $G(\xi) \xi^i G(\xi) = G_H G_J \xi^{HiJ}$ is the convolution of $G(\xi)$ with the primitive element ξ^i convolved again with $G(\xi)$. This formulation again applies to any group.

For each primitive element ξ^i , the left hand side of the fSDE is the SD operator \mathcal{S}^i acting on the function $G(\xi)$ on the group. In the Gaussian, CS, and YM examples (37), \mathcal{S}^i is a complex-linear combination of iterated commutators of left annihilation. Let us restrict attention to models

where this is the case. Then from our discussion in Sec. IV B, we conclude that \mathcal{S}^i are right-invariant vector fields on \mathbf{G} . For this we think of the right-invariant vector fields (or Lie algebra of \mathbf{G}) as represented linearly on the space of functions on the group.

So far, we formulated practically everything in the fSDE in terms of concepts that generalize to any group, without reference to matrix integrals. It only remains to specify which right-invariant vector field \mathcal{S}^i to associate with a given primitive element ξ^i . For this we need additional data beyond the mere specification of a group \mathbf{G} . At present we do not have such a prescription that applies to an arbitrary group, but have a rough idea. The additional datum is the specification of an action, which should be an appropriate real-valued function on the universal envelope of the Lie algebra of \mathbf{G} . The prescription is the passage from action to SD operators, which involves differentiating the action along some vector fields. However, even for the group spec_Λ of relevance to matrix models, we only gave three examples (36) of actions leading to SD operators which are right-invariant vector fields for each primitive. Moreover, we know that many actions do not lead to SD operators that are right-invariant vector fields on \mathbf{G} . Among one-matrix models, the Gaussian $S(A)=c \text{tr} A^2$ is the only one that has a right-invariant vector field $\mathcal{S}=2cD$ for its SD operator. The quartic action $S(A)=\text{tr} A^4$ has $\mathcal{S}=4D^3$, which is not even a vector field on spec_1 (see Sec. IV D). So we postpone a general characterization of admissible actions and their passage to SD operators, which would apply to any group. Instead, we seek more examples of matrix model actions whose SD operators are right-invariant vector fields on spec_Λ .

The matrices A_i of a matrix model do not satisfy any relations in the large- N limit. So they live in the full tensor algebra or free associative algebra on Λ generators, which is the universal envelope of the FLA. A matrix model action $S(A)=\text{tr} S^I A_I$ specifies a real function on this universal envelope, and we define $S(G)=S^J G_J$ with cyclic S^J and G_J . The SD operators \mathcal{S}^i are obtained from the variation of the action under the infinitesimal automorphisms of the tensor algebra $\delta A_j = L_i^j A_j = \delta_j^i A_i$. L_i^j are vector fields on the space upon which the free associative algebra is the algebra of functions.^{8,9} Their action on G_J is $L_i^j G_J = \delta_j^{i_1 i_2} G_{J_1 i_2}$. Applying L_i^j to the action $[|J|]$ is the length of the word J and \bar{J} is the reversed word; cyclicity of S^J and G_I are used in the fourth equality; D_j is left annihilation, $(D_j G)_I = G_{jI} \Rightarrow (D_j D_k G)_I = G_{kjI}$, etc.],

$$\mathcal{S}^i G(\xi) = \xi^l L_i^j (S^l G_J) = \xi^l S^j \delta_j^{i_1 i_2} G_{J_1 i_2} = \xi^l S^{j_1 i_2} G_{J_1 i_2} = \sum_{J, I} |J_i| S^{iJ} G_{JI} \xi^l = \sum_{J, I} |J_i| S^{iJ} (D_{\bar{J}} G)_I \xi^l. \tag{38}$$

From this we read off \mathcal{S}^i (which are not right-invariant vector fields, in general),

$$\mathcal{S}^i = \sum_J |J_i| S^{iJ} D_{\bar{J}} = \sum_{n \geq 0} (n+1) S^{ij_1 \dots j_n} D_{j_n} \dots D_{j_1}. \tag{39}$$

We would like to know which actions lead to \mathcal{S}^i that are linear combinations of iterated commutators of D_j 's, i.e., Lie elements. Linearity of the passage from $S(A)$ to \mathcal{S}^i implies that it is sufficient to work with actions that are homogeneous polynomials of degree n for each $n = 2, 3, 4, \dots$. One difficulty is that actions are usually presented, for example, in the form $S(A) = \text{tr} C^{ijk} A_i [A_j, A_k]$, where C^{ijk} are not cyclic. It takes some relabeling to transform to $S(A) = \text{tr} S^{ijk} A_{ijk}$ with cyclic coupling tensors S^{ijk} , in terms of which the SD operators are expressed in (39). The other difficulty is to identify those actions for which (39) can be rewritten as a linear combination of iterated commutators. In what follows we carry out this program in part and identify a class of actions that lead to \mathcal{S}^i which are Lie elements.

Quadratic. The most general quadratic action is $S_2(A) = \text{tr} C^{ij} A_i A_j$. The cyclic coupling tensor is $S_2^{ij} = \frac{1}{2}(C^{ij} + C^{ji})$. By (39), $\mathcal{S}^i = 2S^{ij} D_j = (C^{ij} + C^{ji}) D_j$. In other words $\mathcal{S}^i = (C^{ij} + \text{cyclic}) D_j$ are Lie elements for all i , i.e., right-invariant vector fields on the group $\text{spec}(\text{Sh}_\Lambda)$. Formally, this holds for arbitrary tensors C^{ij} , although we must impose the reality condition $S^{ij} = S^{ji*}$ and ask that S^{ij} be a positive matrix to ensure that the matrix integrals converge and lead to correlators satisfying $G_I^* = G_{\bar{I}}$. In the sequel, we will largely work formally and suppress these reality and positivity conditions.

Cubic. The CS action (36) can also be written as $S(A) = (\sqrt{-1}\kappa/3) \text{tr} \epsilon^{ijk} A_i [A_j, A_k]$. Motivated by this, we consider the cubic actions $S_3(A) = \text{tr} C^{ijk} A_i [A_j, A_k]$ for arbitrary C^{ijk} . Writing this in the form $S = \text{tr} (C^{ijk} - C^{ikj}) A_{ijk} \equiv \text{tr} S^{ijk} A_{ijk}$ ensures that the coupling tensor $S^{ijk} = \frac{1}{3} (C^{ijk} + \text{cyclic})$ is cyclically symmetric. (The notation $C^{i[jk]} = C^{ijk} - C^{ikj}$.) We get $S^i = 3 S^{ijk} D_k D_j = (C^{ijk} + \text{cyclic}) [D_k, D_j]$ which are Lie elements. S^i is more easily expressed in terms of C^{ijk} , but the reality conditions are simpler in terms of the coupling tensors $(S^{ijk})^* = S^{kji}$.

Quartic. What is the appropriate generalization to higher degree polynomial actions, such that S^i remains a Lie element? In Ref. 9 we showed that the Yang–Mills type of quartic action $S_{\text{YM}} = \text{tr} g^{ik} g^{jl} [A_i, A_j] [A_k, A_l]$ leads to the SD operators $S_{\text{YM}}^i = 4 g^{ik} g^{jl} [D_j, [D_k, D_l]]$ which are Lie elements. However, explicit calculation indicates that the more general $S(A) = \text{tr} B^{ijkl} [A_i, A_j] [A_k, A_l]$ leads to SD operators which are not Lie elements for some B^{ijkl} . On the other hand, a quartic generalization of the CS action is $S_4 = \text{tr} C^{ijkl} A_i [A_j, [A_k, A_l]]$. Furthermore, the YM action is a special case of this. Using cyclicity of the trace,

$$\text{tr} g^{ik} g^{jl} [A_i, A_j] [A_k, A_l] = \text{tr} g^{ik} g^{jl} (A_{i[jkl]} - A_{i[klj]}) = \text{tr} g^{ik} g^{jl} A_i [A_j, [A_k, A_l]]. \tag{40}$$

Thus S_4 reduces to S_{YM} if $C^{ijkl} = g^{ik} g^{jl}$. This motivates us to check whether the SD operators corresponding to S_4 are Lie elements for arbitrary C^{ijkl} . Write S_4 as

$$S_4(A) = \text{tr} C^{ijkl} (A_{ijkl} - A_{ijlk} - A_{iklj} + A_{ilkj}) = \text{tr} (C^{ij[kl]} - C^{il[jk]}) A_{ijkl} \tag{41}$$

and define $\tilde{S}^{ijkl} = C^{ij[kl]} - C^{il[jk]}$. Then $S_4 = \text{tr} S^{ijkl} A_{ijkl}$ where the coupling tensor $S^{ijkl} = \frac{1}{4} (\tilde{S}^{ijkl} + \text{cyclic})$ is cyclically symmetric. Using (39) we read off the SD operators $S^i = (\tilde{S}^{ijkl} + \text{cyclic}) D_{lkj}$. After a lot of relabeling and simplification they can be written as Lie elements $S^i = (C^{ijkl} + \text{cyclic}) [[D_l, D_k], D_j]$. Thus in a sense, $S_4 = \text{tr} C^{ijkl} A_i [A_j, [A_k, A_l]]$ is the proper generalization of S_{YM} while preserving the property that S^i be right-invariant vector fields.

Quintic. We begin to see a pattern to a class of actions that lead to SD operators that are Lie elements. $S_5 = \text{tr} C^{ijklm} A_i [A_j, [A_k, [A_l, A_m]]]$ is the obvious quintic candidate. After some relabeling, we find

$$S_5(A) = \text{tr} \tilde{S}^{ijklm} A_{ijklm}, \tag{42}$$

where

$$\tilde{S}^{ijklm} = C^{ijk[lm]} - C^{ijm[kl]} - C^{imj[kl]} + C^{iml[jk]},$$

so that the cyclic coupling tensor is $S^{ijklm} = \frac{1}{5} (\tilde{S}^{ijklm} + \text{cyclic})$ and $S_5 = \text{tr} S^{ijklm} A_{ijklm}$. $S_5^i = (\tilde{S}^{ijklm} + \text{cyclic}) D_{mlkj}$ after some simplification become

$$S_5^i = (C^{ijklm} + \text{cyclic}) [[[D_m, D_l], D_k], D_j], \tag{43}$$

which are Lie elements. While the relation of C^{ijklm} to cyclic coupling tensors S^{ijklm} is nontrivial, the SD operators are simply expressed in terms of C^{ijklm} .

Sixth degree. For the sixth degree action $S_6(A) = \text{tr} C^{ijklmn} A_i [A_j, [A_k, [A_l, [A_m, A_n]]]]$, the cyclic coupling tensor turns out to be $S^{ijklmn} = \frac{1}{6} (\tilde{S}^{ijklmn} + \text{cyclic})$, where

$$\tilde{S}^{ijklmn} = C^{ijkl[mn]} - C^{ijkn[lm]} - C^{ijnk[lm]} - C^{injk[lm]} + C^{ijnm[kl]} + C^{injm[kl]} + C^{inmj[kl]} - C^{inml[jk]}. \tag{44}$$

The corresponding $S^i = (C^{ijklmn} + \text{cyclic}) [[[[D_n, D_m], D_l], D_k], D_j]$ are again Lie elements.

Conjecture. Based on these examples, we conjecture that the n th degree polynomial action

$$S_n(A) = \text{tr} C^{i_1 \dots i_n} A_{i_1} [A_{i_2}, [A_{i_3}, [\dots [A_{i_{n-1}}, A_{i_n}} \dots]]] \tag{45}$$

has SD operators that are the Lie elements,

$$S_n^{i_1} = (C^{i_1 \dots i_n} + \text{cyclic})[[\dots[[[D_{i_n}, D_{i_{n-1}}], D_{i_{n-2}}], D_{i_{n-3}}], \dots], D_{i_2}]. \tag{46}$$

We have exhibited a large class of matrix model actions (generalizing the Gaussian, Chern–Simons, and Yang–Mills ones) whose SD operators are right-invariant vector fields on \mathbf{G} . But it may not be an exhaustive list. We have not found the most general Lie elements S^i that arise as SD operators of *some* action nor the class of all such actions. Finally, this process must be generalized to other groups; we hope to return to these questions later.

VI. SD OPERATORS OF YANG–MILLS THEORY

Now we illustrate the above framework with the example of Yang–Mills theory. Instead of Λ matrices A_i , we now have the gluon field, one matrix $A_\mu(x)$ for each space-time point x , and $\mu = 1, \dots, d$ where d is the space-time dimension. The sources ξ^i are replaced by $\xi^\mu(x)$. We obtain the fSDE in terms of gluon correlations and write the SD operators $S^\mu(x)$ (53) as constant linear combinations of iterated commutators of left annihilation $D_\mu(x)$. [Constant coefficients must be independent of the sources $\xi^\mu(x)$, but could depend on x and μ which now play the role of the indices i, j, k of matrix models. Coefficients will be differential operators.] It follows (from Sec. IV B) that the SD operators of Yang–Mills theory are right-invariant derivations of the shuffle-deconcatenation Hopf algebra generated by the sources $\xi^\mu(x)$. However, our formulation is far from complete. The fSDE obtained here must be supplemented by gauge fixing and ghost contributions for a proper treatment of gauge invariance before we can look for physical solutions. In Ref. 7 we have indicated how to incorporate gauge fixing and ghost terms in the context of matrix models. Mathematically, this means the shuffle-deconcatenation Hopf algebra generated by $\xi^\mu(x)$ needs to be modified. This could be done either by adding generators corresponding to ghosts or by passing to the quotient by an ideal as in Chen’s work⁵ (see also Tavares⁶). Only then can we recover the group of (generalized) loops via the spectrum of the Hopf algebra. We hope to return to these issues in future work, but restrict ourselves here to the pure Yang–Mills action,

$$S = \text{tr} \int d^d x \left\{ \frac{1}{2} \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) - ig \partial_\mu A_\nu [A^\mu, A^\nu] - \frac{g^2}{4} [A_\mu, A_\nu][A^\mu, A^\nu] \right\}. \tag{47}$$

To get the SD equations, make the change of integration variable,

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \int v_\mu^{\mu_1 \dots \mu_n}(x; x_1 \dots x_n) A_{\mu_1}(x_1) \dots A_{\mu_n}(x_n) dx_1 \dots dx_n \tag{48}$$

in the Euclidean functional integral $Z = \int dA e^{-NS}$. v are infinitesimal tensors and we work to linear order in them. This is not a local change of variable, but it is not disallowed by any law of physics. If δS is the change in action, the SD equations relate it to the change in the measure in the sense of expectation values

$$\left\langle \frac{\delta S}{N} \right\rangle = \left\langle \frac{1}{N^2} \left\{ \det \left(\frac{\partial A'}{\partial A} \right) - 1 \right\} \right\rangle. \tag{49}$$

As in Sec. I, in the limit $N \rightarrow \infty$, the factorized SD equations can be written as

$$S^\mu(x)G(\xi) = G(\xi)\xi^\mu(x)G(\xi),$$

where

$$G(\xi) = \sum_{\mu_1, \dots, \mu_n} \int [dx] G_{\mu_1 \dots \mu_n}(x_1, \dots, x_n) \xi^{\mu_1}(x_1) \dots \xi^{\mu_n}(x_n)$$

and

$$G_{\mu_1 \cdots \mu_n}(x_1, \cdots, x_n) = \lim_{N \rightarrow \infty} \left\langle \frac{\text{tr}}{N} A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n) \right\rangle. \quad (50)$$

The generating series of gluon correlations $G(\xi)$ is an element of the shuffle algebra generated by $\xi^\mu(x)$ ($[dx] = dx_1 \cdots dx_n$ where dx_i is the volume element of space-time). To get the right hand side of the fSDE in (50), we need the Jacobian [hatted variables (e.g., $d\hat{y}_k$) are not integrated] $J = \langle \det(\partial A' / \partial A) \rangle$,

$$\begin{aligned} J &= 1 + N^2 \int dx [dx] dy_1 \cdots d\hat{y}_k \cdots dy_n v_{\mu}^{\mu_1 \cdots \mu_n}(x; x_1 \cdots x_n) \delta_{\mu_1 \cdots \mu_n}^{\nu_1 \cdots \nu_{k-1} \mu \nu_{k+1} \cdots \nu_n} \\ &\quad \times \delta(x_1 - y_1) \cdots \delta(x_{k-1} - y_{k-1}) \delta(x_k - x) \delta(x_{k+1} - y_{k+1}) \cdots \delta(x_n - y_n) \\ &\quad \times \left\langle \frac{\text{tr}}{N} A_{\nu_1}(y_1) \cdots A_{\nu_{k-1}}(y_{k-1}) \right\rangle \left\langle \frac{\text{tr}}{N} A_{\nu_{k+1}}(y_{k+1}) \cdots A_{\nu_n}(y_n) \right\rangle, \\ \frac{J-1}{N^2} &= \int [dx] \sum_{k=1}^n v_{\mu_k}^{\mu_1 \cdots \mu_n}(x_k; x_1 \cdots x_n) G_{\mu_1 \cdots \mu_{k-1}}(x_1 \cdots x_{k-1}) G_{\mu_{k+1} \cdots \mu_n}(x_{k+1} \cdots x_n). \end{aligned} \quad (51)$$

The infinitesimal change in action $\langle \delta S / N \rangle$ is also linear in the arbitrary tensors v . Equating the coefficients of common tensors v leads to the fSDE $S^\mu(x)G(\xi) = G(\xi)\xi^\mu(x)G(\xi)$, where the product on the right hand side is concatenation. Let us define left annihilation $D_\mu(x)$ (distinct from the covariant derivative) by its action on correlations,

$$(D_\mu(x)G)_{\mu_1 \cdots \mu_n}(x_1, \cdots, x_n) = G_{\mu\mu_1 \cdots \mu_n}(x, x_1, \cdots, x_n). \quad (52)$$

Then the SD operators are (square brackets denote antisymmetrization)

$$S^\mu(x) = \partial_\nu \partial^{\nu\mu} D^{\nu]} + ig \{ \partial_\nu [D^\mu, D^\nu] + [\partial^{\nu\mu} D^\mu, D_\nu] \} - g^2 [D^\nu, [D^\mu, D_\nu]]. \quad (53)$$

For example, let us show how $S^\mu(x)$ for the three-gluon term $S_3 = -ig \text{tr} \int dx \partial_\mu A_\nu [A^\mu, A^\nu]$ is obtained. The change in S_3 is (below $[dx]$ stands for $dx dx_1 \cdots dx_n$)

$$\begin{aligned} \delta S_3 &= -ig \text{tr} \int [dx] \partial_\mu (v_{\mu}^{\mu_1 \cdots \mu_n}(x; x_1 \cdots x_n) A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n)) [A^\mu(x), A^\nu(x)] - ig \text{tr} \int [dx] (\partial_\mu A_\nu(x)) \\ &\quad \times [v_{\mu_1 \cdots \mu_n}^\mu(x; x_1 \cdots x_n) A^{\mu_1}(x_1) \cdots A^{\mu_n}(x_n), A^\nu(x)] - ig \text{tr} \int [dx] (\partial_\mu A_\nu(x)) \\ &\quad \times [A^\mu(x), v_{\mu_1 \cdots \mu_n}^\nu(x; x_1 \cdots x_n) A^{\mu_1}(x_1) \cdots A^{\mu_n}(x_n)]. \end{aligned} \quad (54)$$

Integrating by parts in the first term we isolate the same v factor in all terms,

$$\begin{aligned} \delta S_3 &= ig \text{tr} \int [dx] v_{\mu_1 \cdots \mu_n}^\nu(x; x_1 \cdots x_n) A^{\mu_1}(x_1) \cdots A^{\mu_n}(x_n) \partial_\mu [A^\mu(x), A_\nu(x)] \\ &\quad - ig \text{tr} \int [dx] v_{\mu_1 \cdots \mu_n}^\nu(x; x_1 \cdots x_n) (\partial_\nu A_\mu(x)) [A^{\mu_1}(x_1) \cdots A^{\mu_n}(x_n), A^\mu(x)] \\ &\quad - ig \text{tr} \int [dx] v_{\mu_1 \cdots \mu_n}^\nu(x; x_1 \cdots x_n) (\partial_\mu A_\nu(x)) [A^\mu(x), A^{\mu_1}(x_1) \cdots A^{\mu_n}(x_n)]. \end{aligned} \quad (55)$$

We are interested in $\langle \delta S_3 / N \rangle$. Before taking expectation values, we should pull ∂_x and other coupling tensors to the left, while adding additional variables to ensure that derivatives act only on the appropriate fields. This ensures everything can be expressed in terms of gluon correlations,

$$\begin{aligned} \delta S_3 = & igg_{\rho\nu} \text{tr} \int [dx] v_{\mu_1 \dots \mu_n}^\nu(x; x_1 \dots x_n) \partial_\mu^x \{A^{\mu_1}(x_1) \dots A^{\mu_n}(x_n) [A^\mu(x), A^\rho(x)]\} \\ & - igg_{\rho\mu} \text{tr} \int [dxdy] v_{\mu_1 \dots \mu_n}^\nu(x; x_1 \dots x_n) \delta(x-y) \partial_\nu^x \{A^\rho(x) [A^{\mu_1}(x_1) \dots A^{\mu_n}(x_n), A^\mu(y)]\} \\ & - igg_{\rho\nu} \text{tr} \int [dxdy] v_{\mu_1 \dots \mu_n}^\nu(x; x_1 \dots x_n) \delta(x-y) \partial_\mu^x \{A^\rho(x) [A^\mu(y), A^{\mu_1}(x_1) \dots A^{\mu_n}(x_n)]\}. \end{aligned}$$

We take expectation values and use cyclicity to move $\mu_1 \dots \mu_n$ to the right. This facilitates re-expression in terms of left annihilation. Moreover, we omit the common factor v and the integration over $x, x_1 \dots x_n$, since this facilitates identifying the SD operators

$$\begin{aligned} \left\langle \frac{\delta S_3}{N} \right\rangle \propto & igg_{\rho\nu} \partial_\nu^x G^{[\mu\rho]\mu_1 \dots \mu_n}([x, x], x_1 \dots x_n) + igg_{\rho\mu} \int dy \delta(x-y) \partial_\nu^x G^{[\rho\mu]\mu_1 \dots \mu_n}([x, y], x_1 \dots x_n) \\ & + igg_{\rho\nu} \int dy \delta(x-y) \partial_\mu^x G^{[\mu\rho]\mu_1 \dots \mu_n}([y, x], x_1 \dots x_n) = igg_{\rho\nu} \partial_\mu^x ([D^\rho, D^\mu] G)^{\mu_1 \dots \mu_n}(x_1 \dots x_n) \\ & + igg_{\rho\mu} \int dy \delta(x-y) \partial_\nu^x ([D^\mu(y), D^\rho(x)] G)^{\mu_1 \dots \mu_n}(x_1 \dots x_n) \\ & + igg_{\rho\nu} \int dy \delta(x-y) \partial_\mu^x ([D^\rho(x), D^\mu(y)] G)^{\mu_1 \dots \mu_n}(x_1 \dots x_n). \end{aligned} \tag{56}$$

We read off the SD operators of the three-gluon terms in the Yang–Mills action,

$$S_3^\mu(x) = ig\{\partial_\nu [D^\mu, D^\nu] + [\partial^\nu D^\mu - \partial^\mu D^\nu, D_\nu]\} = ig\{\partial_\nu [D^\mu, D^\nu] + [\partial^\nu D^\mu, D_\nu]\}. \tag{57}$$

Similarly, the SD operators of the quadratic and four-gluon terms in the Yang–Mills action (47),

$$S_2 = \frac{1}{2} \text{tr} \int d^d x \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu), \quad S_4 = -\frac{g^2}{4} \text{tr} \int d^d x [A_\mu, A_\nu] [A^\mu, A^\nu],$$

are

$$S_2^\mu(x) = \partial_\nu (\partial^\mu D^\nu - \partial^\nu D^\mu) \equiv \partial_\nu \partial^{\mu\nu}$$

and

$$S_4^\mu(x) = -g^2 g^{\mu\rho} g^{\nu\sigma} [D_\nu, [D_\rho, D_\sigma]] = -g^2 [D^\nu, [D^\mu, D_\nu]]. \tag{58}$$

S_4 contains no derivatives that need to be treated with care when identifying coupling tensors. So $S_4^\mu(x)$ can be read off from the commutator-squared YM matrix model of Eq. (37). This completes the proof that SD operators of YM theory are right-invariant derivations of the sh-deconc Hopf algebra generated by $\xi^\mu(x)$. This holds independent of space-time dimension. However, due to local gauge invariance, YM theory is more than a matrix field theory. Gluon correlations are not gauge invariant. Physically, we are interested in certain (limits of possibly infinite) linear combinations of gluon correlations that are gauge invariant, such as Wilson loops or polynomials in the field strength and its covariant derivatives. Furthermore, we must gauge fix and introduce additional sources for ghosts, which will also contribute to the fSDE. Mathematically, we must pass to a quotient of this Hopf algebra by an ideal,^{5,6} before its spectrum is the group of (generalized) loops. We hope to address these issues in future work.

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APPENDIX A: LYNDON WORDS AND RIGHT STANDARD FACTORIZATION

The SD operators of many interesting matrix models are elements of the free Lie algebra (FLA). We identified the FLA with the Lie algebra of right-invariant vector fields on the group $\mathbf{G}=\text{spec}_\Lambda$. To work with these vector fields it is useful to have a basis. Lyndon words introduced by Lyndon²⁶⁻²⁹ are interesting since they label a basis for the FLA. There are other bases for the FLA, such as the Hall basis.³⁰ To be self-contained, we summarize some facts (without proofs) about the Lyndon word basis for the FLA.

Suppose we are given the alphabet $\xi^1, \dots, \xi^\Lambda$ with the order $\xi^1 < \xi^2 < \dots < \xi^\Lambda$. There does not seem to be any physically preferred choice for an ordering of the letters. We extend the order on letters to the alphabetical or lexicographic order on all words in the alphabet. For example, $\xi^1 \xi^2 < \xi^2 \xi^1$ and $\xi^2 \xi^1 \xi^1 < \xi^2 \xi^1 \xi^1 \xi^3$. If $\xi^j < \xi^l$ we say ξ^j precedes ξ^l . (If $I=i_1 i_2 \dots i_n$ we abbreviate $\xi^{i_1} \xi^{i_2} \dots \xi^{i_n}$ as ξ^I . For brevity we will sometimes talk of the word L when we mean the word ξ^L . For instance, $I < J$ really means $\xi^I < \xi^J$.) A Lyndon word is one which is strictly minimal among its conjugates. Conjugates are words related by cyclic permutations; the conjugates of the Lyndon word $\xi^1 \xi^2 \xi^3$ are $\xi^2 \xi^3 \xi^1$ and $\xi^3 \xi^1 \xi^2$. It follows that Lyndon words ξ^L must be primitive, i.e., cannot be written as $(\xi^M)^n$ for some word ξ^M and $n \geq 2$. In particular, a Lyndon word must be aperiodic, since otherwise it would equal one of its nontrivial conjugates. Equivalently, a word is Lyndon if and only if it precedes every nonempty proper right factor. That is, ξ^L is Lyndon if and only if for any factorization $\xi^L = \xi^M \xi^N$ with ξ^M and ξ^N nonempty, we have $\xi^L < \xi^N$. Letters are automatically Lyndon words. Lyndon words of length two are $\xi^i \xi^j$ with $\xi^i < \xi^j$. There is also a recursive characterization of Lyndon words: ξ^L is Lyndon if and only if there exist Lyndon words ξ^M and ξ^N such that $\xi^M < \xi^N$ and $\xi^L = \xi^M \xi^N$. Of course, there may be more than one choice of M, N that do the job. The number of Lyndon words of length n over an alphabet of cardinality Λ is given by Witt's formula,

$$l(n, \Lambda) = \frac{1}{n} \sum_{d|n} \mu(d) \Lambda^{n/d},$$

where the Mobius function

$$\mu(d) = \begin{cases} 0, & \text{if } d \text{ has a repeated prime factor} \\ 1, & \text{if } d = 1 \\ (-1)^k, & \text{if } d \text{ is a product of } k \text{ distinct primes.} \end{cases} \tag{A1}$$

For example, the numbers of Lyndon words of lengths 1, 2, and 3 are given by $l(1, \Lambda) = \Lambda, l(2, \Lambda) = \frac{1}{2}(\Lambda^2 - \Lambda)$, and $l(3, \Lambda) = \frac{1}{3}(\Lambda^3 - \Lambda)$.

The *right standard factorization* of a Lyndon word $L, |L| > 1$ is the unique factorization $L = MN$, where M and N are Lyndon words such that N is of maximal length. In particular, M must be of length at least one. It follows that $M < L = MN < N$. For clarity we will denote the right standard factorization by $L = M \cdot N$. For example, $\xi^1 \xi^2 = \xi^1 \cdot \xi^2$ and $\xi^1 \xi^3 \xi^3 = \xi^1 \xi^3 \cdot \xi^3$.

1. Lyndon word basis for FLA

We will specify a basis labeled by Lyndon words, for the FLA generated by left annihilation $D_i, 1 \leq i \leq \Lambda$. The FLA consists of linear combinations of iterated commutators of D_i . Here we are implicitly thinking of the FLA as embedded in the free associative algebra generated by D_i . Elements of the FLA are called Lie elements or Lie polynomials. A Lie element has a definite degree d if it is a homogeneous polynomial of degree d in the free associative algebra. Finding a basis for the FLA is complicated because antisymmetry and the Jacobi identity relate many different Lie elements. This problem was solved^{27,20} using the right standard factorization of Lyndon words. The basis elements will be called $D_{(L)}$, where ξ^L runs over all Lyndon words. $D_{(L)}$

is a certain iterated commutator of D_i 's, where ξ^i are the letters of the Lyndon word ξ^L . We use the notation $D_{(L)}$ to distinguish this iterated commutator from the word $D_L = D_{l_1 \dots l_n} = D_{l_1} \dots D_{l_n}$ which is not a Lie element for $n > 1$.

Letters ξ^i are Lyndon words, and the corresponding degree-one basis elements are D_i , which are independent by definition. Given a Lyndon word L with right standard factorization $L = M \cdot N$, we associate the recursively defined Lie element, $D_{(L)} = [D_{(M)}, D_{(N)}]$, with the Lyndon word L . To express $D_{(L)}$ as an iterated commutator of left annihilation operators, we need to apply this rule recursively until M and N are both single letters. The degree of the basis element $D_{(L)}$ is equal to the length $|L|$. $D_{(L)}$ is not defined when L is not a Lyndon word, although it is sometimes convenient to define $D_{(L)} = 0$ if L is not Lyndon.

The Lyndon words of length two are $\xi^i \xi^j, i < j$. Using the right standard factorization $\xi^i \xi^j = \xi^i \cdot \xi^j$ for $i < j$, we get the Lyndon basis $D_{(ij)} = [D_i, D_j]$ for $i < j$. There are clearly $\frac{1}{2}(\Lambda^2 - \Lambda)$ of these basis elements. The restriction $i < j$ is explained by the antisymmetry of the commutator. For an alphabet of two letters ($\Lambda = 2$), there are two Lyndon words of length of 3. Their right standard factorizations are

$$\xi^1 \xi^1 \xi^2 = \xi^1 \cdot \xi^1 \xi^2 \quad \text{with } \xi^1 < \xi^1 \xi^1 \xi^2 < \xi^1 \xi^2,$$

and

$$\xi^1 \xi^2 \xi^2 = \xi^1 \xi^2 \cdot \xi^2 \quad \text{with } \xi^1 \xi^2 < \xi^1 \xi^2 \xi^2 < \xi^2. \tag{A2}$$

The corresponding Lyndon basis elements are $D_{(112)} = [D_1, [D_1, D_2]]$ and $D_{(122)} = [[D_1, D_2], D_2]$. We see that after accounting for antisymmetry and the Jacobi identity, there are only two independent Lie elements of degree three for a two letter alphabet. For an alphabet of length $\Lambda = 3$, there are eight Lyndon words of length of 3. Six of them involve only two of the letters each and can be obtained from the previous example. We list the eight Lyndon basis elements,

$$\begin{aligned} & [D_1, [D_1, D_2]], \quad [[D_1, D_2], D_2], \quad [D_1, [D_1, D_3]], \quad [[D_1, D_3], D_3], \\ & [D_2, [D_2, D_3]], \quad [[D_2, D_3], D_3], \quad [D_1, [D_2, D_3]], \quad [[D_1, D_3], D_2]. \end{aligned} \tag{A3}$$

The right standard factorization of the corresponding Lyndon words may be read off, for example, $\xi^{132} = \xi^{13} \cdot \xi^2$ and $\xi^{123} = \xi^1 \cdot \xi^{23}$.

The structure constants of the FLA in the Lyndon basis are defined as $[D_{(L)}, D_{(M)}] = c_{L,M}^N D_{(N)}$, where L, M, N are Lyndon words and $|L| + |M| = |N|$. For example,

$$[D_i, D_j] = \begin{cases} D_{(ij)}, & \text{if } i < j \\ D_{(ji)}, & \text{if } j < i \\ 0, & \text{if } i = j. \end{cases}$$

Thus for $|K| = 2$,

$$c_{i,j}^K = \begin{cases} 1, & \text{if } K = ij \quad \text{with } i < j \\ -1, & \text{if } K = ji \quad \text{with } j < i \\ 0, & \text{otherwise.} \end{cases} \tag{A4}$$

Another example is (for $j < k$ so that jk is Lyndon)

$$[D_i, D_{(jk)}] = \begin{cases} D_{(ijk)}, & \text{if } i < jk \\ D_{(jki)} - D_{(kji)}, & \text{if } jk < i \text{ and } k < i \text{ and } k < i \\ D_{(jki)} + D_{(jik)}, & \text{if } jk < i \text{ and } k < i \text{ and } ji < k \\ -D_{(jki)}, & \text{if } jk < i \text{ and } i \leq k. \end{cases} \quad (\text{A5})$$

Another simple example, $[D_{(M)}, D_{(N)}] = D_{(MN)}$ if $MN = M \cdot N$, is the right standard factorization of the Lyndon word MN . For this to be the case, it is necessary that $M < MN < N$.

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