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ENTROPY OF OPERATOR-VALUED RANDOM VARIABLES: A VARIATIONAL PRINCIPLE FOR LARGE N MATRIX MODELS

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We show that, in 't Hooft's large N limit, matrix models can be formulated as a classical theory whose equations of motion are the factorized Schwinger–Dyson equations. We discover an action principle for this classical theory. This action contains a universal term describing the entropy of the noncommutative probability distributions. We show that this entropy is a nontrivial one-cocycle of the noncommutative analog of the diffeomorphism group and derive an explicit formula for it. The action principle allows us to solve matrix models using novel variational approximation methods; in the simple cases where comparisons with other methods are possible, we get reasonable agreement.

Keywords: Large N matrix models; QCD; noncommutative probability theory; free entropy; cohomology; variational principle.

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1. Introduction

There are many physical theories in which random variables which are operators — matrices of finite or infinite order — appear: For example, Yang–Mills theories, models for random surfaces and M-theory (an approach to a string theory of quantum gravity). In all these theories, the observables are functions of the matrices which are invariant under changes of basis; in many cases — as for Yang–Mills theories — the invariance group is quite large since it contains changes of basis that depend on position. We address the question of how to construct an effective action (probability distribution) for these gauge-invariant observables induced by the original probability distribution of the matrices.

Quantum chromodynamics (QCD) is the matrix model of greatest physical interest. QCD is the widely accepted theory of strong interactions. It is a Yang–Mills theory with a non-Abelian gauge group $SU(N)$. Thus, the microscopic degrees of freedom include a set of $N \times N$ Hermitian matrices at each point of space–time: The components of the one-form that represents the gauge field. In addition there are the quark fields that form an N-component complex vector at each point of space–time. The number of "colors," N, is equal to three in nature. Nevertheless, it will be useful to study the theory for an arbitrary value of N. Also, it will be convenient to regard $U(N)$ rather than $SU(N)$ as the gauge group.

The microscopic degrees of freedom — quarks and gluons — do not describe the particles that we can directly observe.[1–3] Only certain bound states called hadrons — those that are invariant under the gauge group — are observable. This phenomenon — called confinement — is one of the deepest mysteries of theoretical physics.

In earlier papers we have postulated that there is a self-contained theory of color invariant observables fully equivalent to QCD at all energies and all values of N. We have called this theory we seek "quantum hadron dynamics"[4] and fully constructed it in the case of two-dimensional space–time. Also we have shown that this theory is a good approximation to four-dimensional QCD applied to deep inelastic scattering: It predicts with good accuracy the parton distributions observed in experiments.[5]

Certain simplifications of the two-dimensional theory allowed us to eliminate all gluon (matrix-valued) degrees of freedom. This helped us to construct two-dimensional quantum hadron dynamics quite explicitly. To make further progress, it is necessary to understand theories in which the degrees of freedom are $N \times N$ matrices. Before studying a full-fledged matrix field theory we need to understand how to reformulate a theory of a finite number of matrices in terms of their invariants.[a] This is the problem we will solve in this paper.

It is well known that matrix models simplify enormously in the limit $N \to \infty$.[1,2] The quantum fluctuations in the gauge-invariant observables in gauge-invariant states can be shown to be of order $\frac{\hbar}{N}$. Thus, as long as we restrict to gauge-invariant observables, in the limit $N \to \infty$ QCD must tend to some classical theory. This classical theory cannot be Yang–Mills theory, however, since the fluctuations in all states (not just the gauge-invariant ones) would vanish in that limit. An important clue to discovering quantum hadron dynamics would be to study its classical limit first. This is the strategy that worked in the case of two dimensions.

The analog of the field equations of this "classical hadron dynamics" has been known for a long time — they are the factorized Schwinger–Dyson equations.[3] It is natural to ask if there is a variational principle from which this equation can be derived. Finding this action principle would be a major step forward in understanding hadronic physics: It would give a formulation of hadron dynamics in terms of hadronic variables, entirely independent of Yang–Mills theory. A quantization of the theory based on this action principle would recover the corrections of order $\frac{1}{N}$. Moreover, we would be able to derive approximate solutions of the large-N field equations by the variational method.

[a] Since we understand by now how to deal with the quark degrees of freedom in terms of invariants, it is sufficient to consider toy models for pure gauge theory, without vectorial degrees of freedom.

Even after the simplifications of the large N limit, generic matrix models have proved to be not exactly solvable: The factorized Schwinger–Dyson equations have proved to be generally intractable. Diagrammatic methods have been pushed to their limit.[1] To make further progress, new approximation methods are needed — based on algebraic, geometric and probabilistic ideas. Moreover, the entire theory has to be reformulated in terms of manifestly $U(N)$-invariant observables. Thus, the basic symmetry principle that determines the theory has to be something new — the gauge group acts trivially on these observables. In previous papers[6] we had suggested that the group $\mathcal{G}$ of automorphisms of a free algebra — the noncommutative analog of the diffeomorphism group — plays this crucial role in such a gauge-invariant reformulation of matrix models. In this paper we finally discover this manifestly gauge-invariant formulation of finite dimensional matrix models. We find that the configuration space of the theory is a coset space of $\mathcal{G}$ — justifying our earlier anticipation.

If we restrict to observables which are invariant under the action of $U(N)$, we should expect that the effective action should contain some kind of entropy. The situation is analogous to that in statistical mechanics, with the gauge-invariant observables playing the role of macroscopic variables. However, there are an infinite number of such observables in our case. Moreover, there is no reason to expect that the systems we are studying are in thermal equilibrium in any sense. The entropy should be the logarithm of the volume of the set of all Hermitian matrices that yield a given set of values for the $U(N)$-invariant observables. This physical idea, motivated by Boltzmann's notions in statistical mechanics, allows us to derive an explicit formula for entropy.

Our approach continues the point of view in the physics literature on large N limit of matrix models.[8,10,1–3,11,6,16–18] It should not be surprising that our work has close relations to the theory of von Neumann algebras-nowadays called noncommutative probability theory: After all operators are just matrices of large dimension. Voiculescu[12–15] has another, quite remarkable, approach to noncommutative probability distributions.[b] Our definition in terms of moments and the group of automorphisms is closer in spirit to the physics literature. Also, the connection of entropy to the cocycle of the automorphism group is not evident in that approach. A closer connection between the mathematical and physical literature should enrich both fields.

Although our primary motivation has been to study toy models of Yang–Mills theory, the matrix models we study also arise in some other physical problems. There are several recent reviews that establish these connections, so we make only some brief comments. See e.g. Ref. 19.

In the language of string theory, what we seek is the action of closed string field theory. We solve this problem in a "toy model" — for strings on a model of space–time with a finite number of points. Closed string theory turns out to be a kind

[b]We thank Prof. Dan Voiculescu for bringing Refs. 13–15 to our attention.

of Wess–Zumino–Witten model on the coset space[c] $\mathcal{G}/\mathcal{SG}$; we discover an explicit formula for the classical action, including a term which represents an anomaly — a nontrivial one-cocycle. Our work complements the other approaches to closed string field theory.[20]

Random matrices also appear in another approach to quantum geometry.[21] Our variational method could be useful to approximately solve these matrix models for Lorentzian geometry.

Quantum chaos are often modeled by matrix models.[19] In that context the focus is often on universal properties that are independent of the particular choice of the matrix model action (which we call S below).[27] These universal properties are thus completely determined by the entropy. Our discovery of an explicit formula for entropy should help in deriving such universal properties for multi-matrix models: So far results have been mainly about the one-matrix model. In the current paper our focus is on the joint probability distribution, which is definitely *not* universal.

A preliminary version of this paper was presented at the MRST 2001 conference.[7]

2. Operator-Valued Random Variables

Let ξ_i, $i = 1 \cdots M$ be a collection of operator valued random variables. We can assume without any loss of generality that they are Hermitian operators: We can always split any operator into its "real" and "imaginary" parts. If the ξ_i were real-valued, we could have described their joint probability distribution as a density (more generally a measure) on R^M. When ξ_i are operators that cannot be diagonalized simultaneously, this is not a meaningful notion; we must seek another definition for the notion of joint probability distribution (jpd).

The quantities of physical interest are the expectation values of functions of the basic variables (generators) ξ_i; the jpd simply provides a rule to calculate these expectation values. We will think of these functions as polynomials in the generators (more precisely formal power series). Thus, each random variable will be determined by a collection of tensors $u = \{u^\emptyset, u^i, u^{i_1 i_2}, \ldots\}$ which are the coefficients in its expansion in terms of the generators:

$$u(\xi) = \sum_{m=0}^{\infty} u^{i_1 \cdots i_m} \xi i_1 \cdots \xi i_m \,. \tag{1}$$

The constant term is just a complex number: The set of indices on it is empty.

If u is a polynomial, all except a finite number of the tensors will be zero. It is inconvenient to restrict to polynomials: We would not be able to find inverses of functions, for example, within the world of polynomials. The opposite extreme would be to impose no restriction at all on the tensors u: Then the random variable

[c] $\mathcal{G}$ is a noncommutative analog of the diffeomorphism group; $\mathcal{SG}$ is the subgroup that preserves a noncommutative analog of volume. See below for precise definitions.

is thought of as a formal power series. We pay a price for this: It is no longer possible to "evaluate" the above infinite series for any particular collection of operators ξ_i: The series may not converge. Nevertheless, it makes sense to take linear combinations and to multiply such formal power series:

$$[\alpha u + \beta v]^{i_1\cdots i_m} = \alpha u^{i_1\cdots i_m} + \beta v^{i_1\cdots i_m}, \quad [uv]^{i_1\cdots i_m} = \sum_{n=0}^{m} u^{i_1\cdots i_n} v^{i_{n+1}\cdots i_m}. \qquad (2)$$

Note that even if there are an infinite number of nonzero elements in the tensors u or v, the sum and product is always given by finite series: There are no issues of convergence in their definition. Thus the set of formal power series form an associative[d] algebra; this is the *free algebra* $\mathcal{T}_M$ on the generators ξ_i. Note that the multiplication is just the direct product of tensors.

As we noted above, the joint probability distribution of the ξ_i is just a rule to calculate the expectation value of an arbitrary function of these generators. If we restrict to functions of ξ_i that are polynomials,[e] such expectation values are determined by the moments

$$G_{i_1 i_2\cdots i_n} = \langle \xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}\rangle . \qquad (3)$$

If the variables commute among each other, these moments are symmetric tensors. The most general situation that can arise in physics is that the ξ_i satisfy no relations at all among each other, (in particular they do not commute) except the associativity of multiplication. In this case the moments form tensors with no particular symmetry property. All other associative algebras can be obtained from this "free algebra" by imposing relations (i.e. quotienting by some ideal of the free algebra.) Such relations can be expressed as conditions on the moments. For example, if $\xi_i\xi_j = R_{ij}^{kl}\xi_k\xi_l$, the moment tensors will satisfy conditions

$$G_{i_1\cdots i_a i_{a+1}\cdots i_m} = R_{i_a i_{a+1}}^{kl} G_{i_1\cdots i_{a-1}kli_{a+2}\cdots i_m} \qquad (4)$$

involving neighboring indices.

2.1. *The space of paths*

Thus, in our theory, a random variable is a tensor $u = (u^{\emptyset}, u^{i}, u^{i_1 i_2}, \ldots)$. We can regard each sequence of indices $I = i_1 i_2\cdots i_m$ as a path in the finite set $1, 2, \ldots, M$ in which the indices take their values; a tensor is then a function on this space of paths. Now, given two paths $I = i_1\cdots i_m$ and $J = j_1\cdots j_n$, we can concatenate them: Follow I after traversing $J : IJ = i_1\cdots i_m j_1\cdots j_n$. This concatenation operation is associative but not in general commutative; it has the empty path $\emptyset$ as an identity element. There is however no inverse operation, so the concatenation defines the structure of a semi-group on the space of paths.

[d] This algebra is commutative only if the number of generators M is one.
[e] Not all formal power series may have finite expectation values: The series might diverge. This does not need to worry us: There is a sufficiently large family of "well-behaved" random variables, the polynomials.

We will use upper case latin indices to denote sequences of indices or paths. Repeated indices are to be summed as usual; for example

$$u^I G_I = \sum_{m=0}^{\infty} u^{i_1 \cdots i_m} G_{i_1 \cdots i_m} \,. \tag{5}$$

Also, define $\delta_I^{I_1 I_2}$ to be one if the paths I_1 and I_2 concatenate to give the path I; and zero otherwise. $\bar{I} = i_m i_{m-1} \cdots i_1$ denotes the reverse path. Now we can see that the direct product on tensors is just the multiplication induced by concatenation on the space of paths:

$$[uv]^I = \delta^I_{I_1 I_2} u^{I_1} v^{I_2}. \tag{6}$$

A more refined notion of a path emerges if we regard the indices i as labelling the edges of a directed graph; a sequence $I = i_1 i_2 i_3 \cdots i_n$ is a path only when i_1 is incident to i_2, and i_2 incident with i_3 etc. The space of paths associated with a directed graph is still a semigroup; the associative algebra induced by concatentations is just the algebra of functions on this semi-group. Lattice gauge theory can be interpreted as a matrix model (of unitary matrices) on a directed graph[f] that approximates space–time; for example the cubic lattice.

The free algebra arises from the graph with one vertex, with every edge connecting that vertex to itself; that is why every edge is incident to every other edge. This is the case we will mostly consider in this paper. Other cases can be developed analogously.

2.2. *Noncommutative probability distributions*

We *define* the "noncommutative joint probability distribution" of the variables ξ_i to be a collection of tensors $G_\emptyset, G_i, G_{i_1 i_2}, \ldots$ satisfying the normalization condition

$$1 = \langle 1 \rangle \,, \tag{7}$$

the hermiticity condition

$$G^*_{i_1 i_2 \cdots i_m} = G_{i_m i_{m-1} \cdots i_1} \,, \quad \text{i.e.} \quad G^*_I = G_{\bar{I}} \tag{8}$$

as well as the positivity condition

$$\sum_{m,n=0}^{\infty} G_{i_1 i_2 \cdots i_m j_1 j_2 \cdots j_n} u^{i_m \cdots i_1\, *} u^{j_1 j_2 \cdots j_n} \geq 0 \,, \quad \text{i.e.} \quad G_{IJ} u^{\bar{I}\, *} u^J \geq 0 \,. \tag{9}$$

for any polynomial $u(\xi) = u^\emptyset + u^{i\xi_i} + u^{i_1 i_2} \xi_{i_1} \xi_i \xi_2 \cdots$. Denote by $\mathcal{P}_M$ the space of such noncommutative probability distributions. We define the *expectation value* of a polynomial $v(\xi) = \sum_m v^{i_1 i_2 \cdots i_m} \xi_{i_1} \cdots \xi_{i_m}$ to be

$$\langle v(\xi) \rangle = \sum_m v^{i_1 i_2 \cdots i_m} G_{i_1 i_2 \cdots i_m} \,; \quad \text{i.e.} \quad \langle v(\xi) \rangle = v^I G_I \,. \tag{10}$$

[f] Directed graphs approximating space–time are called "lattices" in physics terminology.

If the variables ξ_i are commutative with joint pdf $p(x_1 \cdots x_n) d^M x$,

$$G_{i_1 i_2 \cdots i_n} = \int x_{i_1} \cdots x_{i_n} p(x_1 \cdots x_M) d^M x \,; \tag{11}$$

it is clear then that the above conditions on G follow from the usual normalization, hermiticity and positivity conditions on $p(x)d^n x$. For example, the contravariant tensors $u^{\emptyset}, u^i, u^{i_1 i_2}$ define a polynomial

$$u(\xi) = \sum_m u^{i_1 i_2 \cdots i_m} \xi_{i_1} \cdots \xi_{i_m} \tag{12}$$

and the quantity on the lhs of the positivity condition above is just the expectation value of $u^{\dagger}(\xi)u(\xi)$. In this case, the moment tensors will be real and symmetric. Up to some technical assumptions, the pdf $p(x)d^M x$ is determined uniquely by the collection of moments $G_{i_1 \cdots i_n}$. (In the case of a single variable, the reconstruction of the pdf from the moments is the "moment problem" of classical analysis; it was solved at varying levels of generality by Markov, Chebycheff etc. See the excellent book by Akhiezer.[22])

In the noncommutative case, the pdf no longer makes sense — even then the moments allow us to calculate expectation values of arbitrary polynomials. This motivates our definition.

In the cases of interest to us in this paper, G_I is cyclically symmetric. This corresponds to closed string theory or to glueball states of QCD. Open strings, or mesons would require us to study moments which are not cyclically symmetric. The theory adapts with small changes; but we will not discuss these cases here.

3. Large N Matrix Models

The basic examples of such operator-valued random variables are random matrices of large size.

A *matrix model* is a theory of random variables which are $N \times N$ Hermitian matrices A_i, $i = 1, \ldots, M$. The matrix elements of each of the A_i are complex-valued random variables, with the joint probability density function on $R^{N^2 M}$

$$\frac{1}{Z(S)} e^{N \operatorname{tr} S(A)} d^{N^2 M} A \,, \tag{13}$$

where $S(A) = \sum_{n=1} S^{i_1 i_2 \cdots i_n} A_{i_1} \cdots A_{i_n}$ is a polynomial called the *action.* Also, $Z(S)$ is determined by the normalization condition:

$$Z(S) = \int e^{N \operatorname{tr} S(A)} d^{N^2 M} A \,. \tag{14}$$

The expectation value of any function of the random variables is defined to be

$$\langle f(A) \rangle = \int f(A) \frac{1}{Z(S)} e^{N \operatorname{tr} S(A)} d^{N^2 M} A \,. \tag{15}$$

The tensors $S^{i_1 \cdots i_n}$ may be chosen to be cyclically symmetric. We assume they are such that the integrals above converge: $S(A) \to -\infty$ as $|A| \to \infty$. The interesting observables are invariant under changes of bases. We can regard the indices $i = 1, \ldots, M$ as labelling the links in some graph; then a sequence $i_1 \cdots i_n$ is a path in this graph. Then $\Phi_{i_1 \cdots i_n}(A) = \frac{1}{N} \operatorname{tr}[A_{i_1} \cdots A_{i_n}]$ is a random variable depending on a loop in this graph. the moment we will consider every link in the graph to be incident with every other link, so that all sequences $i_1 \cdots i_n$ are allowed loops. In this case the loop variables are invariant under simultaneous changes of basis in the basic variables:

$$A_i \to g A_i g^\dagger , \quad g \in \mathrm{U}(N) . \tag{16}$$

If we choose some other graph, a sequence of indices is a closed loop only if the edge i_2 is adjacent to i_1, i_3 is adjacent to i_2 and so on. The invariance group will be larger; as a result, S^I is nonzero only for closed loops I.

Given S, the moments of the loop variables satisfy the Schwinger–Dyson equations

$$S^{J_1 i J_2} \langle \Phi_{J_1 I J_2} \rangle + \delta_I^{I_1 i I_2} \langle \Phi_{I_1} \Phi_{I_2} \rangle = 0 . \tag{17}$$

This equation can be derived by considering the infinitesimal change of variables

$$[\delta_v A]_i = v_i^I A_I \tag{18}$$

on the integral

$$Z(S) = \int e^{N \operatorname{tr} S(A)} dA . \tag{19}$$

The variation of a product of A's is easy to compute:

$$[\delta_v A]_J = v_i^I \delta_J^{J_1 i J_2} A_{J_1} A_I A_{J_2} . \tag{20}$$

The first term in the Schwinger–Dyson equation follows from the variation of the action under this change. The second term is more subtle as it is the change of the measure of integration — the divergence of the vector field $v_i(A)$:

$$\delta_v(dA) = v_i^I \frac{\partial A^a_{Ib}}{\partial A^a_{ib}} dA , \quad \frac{\partial A^a_{Ib}}{\partial A^a_{ib}} = \delta_I^{I_1 i I_2} \operatorname{tr} A_{I_1} \operatorname{tr} A_{I_2} . \tag{21}$$

Returning to the Schwinger–Dyson equations, we see that they are not a closed system of equations: The expectation value of the loop variables is related to that of the product of two loop variables. However, there is a remarkable simplification as $N \to \infty$.

In the *planar limit* $N \to \infty$ keeping $S^{i_1 \cdots i_n}$ fixed, the loop variables have no fluctuations:[g]

$$\langle f_1(\Phi) f_2(\Phi) \rangle = \langle f_1(\Phi) \rangle \langle f_2(\Phi) \rangle + O\left(\frac{1}{N^2} \right) , \tag{22}$$

[g]This is called the planar limit, since in perturbation theory, only Feynman diagrams of planar topology contribute.[1] In the matrix model of random surface theory, one is interested in another large N limit, the *double scaling limit*. Here the coupling constants S^I have to vary as $N \to \infty$ and tend to certain critical values at a specified rate. The fluctuations are not small in the double scaling limit.

where $f_1(\Phi)$, $f_2(\Phi)$ are polynomials of the loop variables. This means that the probability distribution of the loop variables is entirely determined by the expectation values (*moments*)

$$G_{i_1\cdots i_n} = \lim_{N\to\infty}\left\langle \frac{1}{N}\,\mathrm{tr}\, A_{i_1}\cdots A_{i_n}\right\rangle . \tag{23}$$

Thus we get the *factorized Schwinger–Dyson equations*:

$$S^{J_1 i J_2} G_{J_1 I J_2} + \delta_I^{I_1 i I_2} G_{I_1} G_{I_2} = 0\,. \tag{24}$$

Since the fluctuations in the loop variables vanishes in the planar limit, there must be some effective "classical theory" for these variables of which these are the equations of motion. We now seek a variational principle from which these equations follow.

Matrix models arise as toy models of Yang–Mills theory as well as string theory. The cyclically symmetric indices $I = i_1\cdots i_n$ should be interpreted as a closed curve in space–time. The observable Φ_I correspond to the Wilson loop in Yang–Mills theory and to the closed string field.

To summarize, the most important examples of noncommutative probability distributions are large N matrix models:

$$G_{i_1\cdots i_n} = \lim_{N\to\infty}\int \frac{1}{N}\,\mathrm{tr}[A_{i_1}\cdots A_{i_n}]e^{N\,\mathrm{tr}\, S^J A_J}\frac{dA}{Z(S)}\,. \tag{25}$$

3.1. *Example: The Wigner distribution*

The most ubiquituous of all classical probability distributions is the Gaussian; the noncommutative analog of this is the Wigner distribution[8] (also called the semicircular distribution).

We begin with the simplest where we have just one generator ξ for our algebra of random variables. The algebra of random variables is then necessarily commutative and can be identified with the algebra of formal power series in one variable. The simplest example of a matrix-valued random variable is this: ξ is an $N\times N$ Hermitian matrix whose entries are mutually independent random variables of zero mean and unit variance. More precisely, the average of ξ^n is

$$\langle \xi^n\rangle = \int \frac{1}{N}\,\mathrm{tr}\,\xi^n e^{-\frac{N}{2}\,\mathrm{tr}\,\xi^\dagger\xi}\frac{d^{N^2}\xi}{Z_N}\,. \tag{26}$$

The normalization constant Z_N is chosen such that $\langle 1\rangle = 1$.

The Wigner distribution with unit covariance is the limit as $N\to\infty$:

$$\Gamma_n = \lim_{N\to\infty}\int \frac{1}{N}\,\mathrm{tr}\,\xi^n e^{-\frac{N}{2}\,\mathrm{tr}\,\xi^\dagger\xi}\frac{d^{N^2}\xi}{Z_N}\,. \tag{27}$$

The factorized Schwinger–Dyson equations reduce to the following recursion relations for the moments:

$$\Gamma_{k+1} = \sum_{m+n=k-1}\Gamma_m\Gamma_n\,. \tag{28}$$

Clearly the odd moments vanish and $\Gamma_0 = 1$. Set $\Gamma_{2k} = C_k$. Then

$$C_{k+1} = \sum_{m+n=k} C_m C_n \,. \tag{29}$$

The solution of the recursion relations give the moments in terms of the *Catalan numbers*

$$\Gamma_{2k} = C_k = \frac{1}{k+1}\binom{2k}{k} \,. \tag{30}$$

3.2. *The multivariable Wigner distribution*

Let K^{ij} be a positive matrix; i.e. such that

$$K^{ij} u_i^* u_j \geq 0 \tag{31}$$

for all vectors u^i with zero occuring only for $u = 0$. Then the moments of the Wigner distribution on the generators ξ_i, $i = 1, 2, \ldots, M$ are given by

$$\Gamma_{i_1 \cdots i_n} = \lim_{N\to\infty} \int \frac{1}{N} \operatorname{tr}\,[\xi_{i_1} \cdots \xi_{i_n}]\, e^{-\frac{N}{2} K^{ij} \operatorname{tr} \xi_i \xi_j} \frac{d^{N^2 M}\xi}{Z_N} \tag{32}$$

(Again, Z_N is chosen such that $\langle 1 \rangle = 1$.) It is obvious that the moments of odd order vanish; also, that the second moment is

$$\Gamma_{ij} = K^{-1\; ij} \,. \tag{33}$$

The higher order moments are given by the recursion relations:

$$\Gamma_{iI} = \Gamma_{ij} \delta_I^{I_1 j I_2} \Gamma_{I_1} \Gamma_{I_2} \,. \tag{34}$$

Note that each term on the rhs corresponds to a partition of the original path I into subsequences that preserve the order. By repeated application of the recursion the rhs can be written in terms of a sum over all such "non-crossing partitions" into pairs. The Catalan number C_k is simply the number of such non-crossing partitions into pairs of a sequence of length $2k$.

Our strategy for studying more general probability distributions will be to transform them to the Wigner distribution by a nonlinear change of variables. Hence the group of such transformations is of importance to us. In the next section we will study this group.

4. Automorphisms of the Free Algebra

The free algebra generated by ξ_i remains unchanged (is isomorphic) if we change to a new set of generators,

$$\phi(\xi)_i = \sum_{m=0}^{\infty} \phi_i^{i_1 \cdots i_m} \xi_{i_1} \cdots \xi_{i_m} \tag{35}$$

provided that this transformation is invertible. We will often abbreviate $\xi_I = \xi_{i_1} \cdots \xi_{i_m}$ so that the above equation would be $\phi(\xi)_i = \phi_i^I \xi_I$. The composition of two transformations ψ and ϕ can be seen to be

$$[(\psi \circ \phi)]_i^K = \sum_{n \le |K|} \delta^K_{P_1 \cdots P_n} \psi_i^{j_1 \cdots j_n} \phi^{P_1}_{j_1} \cdots \phi^{P_n}_{j_n} . \tag{36}$$

Note that the composition involves only finite series even when each of the series ϕ and ψ is infinite.

The inverse, $(\phi^{-1})_i(\xi) \equiv \chi_i(\xi)$, is determined by the conditions

$$[(\chi \circ \phi)_i]^K = \delta^K_{P_1 \cdots P_n} \chi_i^{j_1 \cdots j_n} \phi^{P_1}_{j_1} \cdots \phi^{P_n}_{j_n} = \delta_i^K . \tag{37}$$

They can be solved recursively for χ^i_J:

$$\begin{aligned} \chi^i_j &= (\phi^{-1})^i_j , \\ \chi^i_{j_1 j_2} &= -\chi^{k_1}_{j_1} \chi^{k_2}_{j_2} \chi^i_{l_1} \phi^{l_1}_{k_1 k_2} , \\ &\cdots \\ \chi^i_{j_1 \cdots j_n} &= -\sum_{m<n} \delta^{P_1 \cdots P_m}_{k_1 \cdots k_n} \chi^{k_1}_{j_1} \cdots \chi^{k_n}_{j_n} \chi^i_{l_1 \cdots l_m} \phi^{l_1}_{P_1} \cdots \phi^{l_m}_{P_m} . \end{aligned} \tag{38}$$

Thus an automorphism ϕ has an inverse as a formal power series if and only if the linear term $\phi^i_j \xi_j$ has an inverse; i.e. if the determinant of the matrix ϕ^i_j is nonzero. The set of such automorphisms form a group $\mathcal{G}_M = \mathrm{Aut}\, \mathcal{T}_M$. This group plays a crucial role in our theory.

4.1. *Transformation of moments under $\mathcal{G}_M$*

Given an automorphism and a probability distribution with moments G_I

$$[\phi(\xi)]_i = \phi_i^I \xi_I \tag{39}$$

the expectation value of $[\phi(\xi)]_{i_1} \cdots [\phi(\xi)]_{i_n}$:

$$[\phi_* G]_I = \sum_{n=1}^{\infty} \phi^{J_1}_{i_1} \cdots \phi^{J_n}_{i_n} G_{J_1 \cdots J_n} . \tag{40}$$

We might regard these as the moments of some new probability distribution. There is a technical problem however: The sums may not converge, the group $\mathcal{G}_M$ of formal power series includes many transformations that may not map positive tensors to positive tensors: They may not preserve the "measure class" of the joint probability distribution G_I.

Given some fixed jpd (say the Wigner distribution with unit covariance), there is a subgroup $\tilde{\mathcal{G}}_M$ that maps it to probability distributions; this is the open subset of $\mathcal{G}_M$ defined by the inequalities:[h]

$$\tilde{\mathcal{G}}_M = \{\phi \in \mathcal{G}_M | [\phi_* G]_{I_1 I_2} u^{*\bar{I}_1} u^{I_2} \ge 0\} \tag{41}$$

[h] $\bar{I}$ denotes the reverse of the sequence: $I = i_1 i_2 \cdots i_n$, $\bar{I} = i_n i_{n-1} \cdots i_1$.

for all polynomials u. Thus in the neighborhood of the identity the two groups $\mathcal{G}_M$ and $\tilde{\mathcal{G}}_M$ are the same; in particular, they have the same Lie algebra. The point is that $\mathcal{G}_M$ and $\tilde{\mathcal{G}}_M$ are Lie groups under different topologies: The series $\phi \in \tilde{\mathcal{G}}_M$ have to satisfy convergence conditions implied by the above inequalities.

It is plausible that any probability distribution can be obtained from a fixed one by some automorphism; indeed there should be many such automorphisms. As a simple example, the Wigner distribution with covariance G_{ij} can be obtained from the one with covariance δ^{ij} by the linear automorphism $\phi^i_j \xi_j$ provided that $G_{ij} = \sum_k \phi^k_i \phi^k_j$. Thus the space of Wigner distributions (the space of positive covariance matrices) is the coset space GL_M/O_M.

In the same spirit, we will regard the space of all probability distributions as the coset space of the group of automorphisms $\tilde{\mathcal{G}}_M/\mathcal{SG}_M$, where $\mathcal{SG}_M$ is the subgroup of automorphisms that leave the Wigner distribution of unit covariance invariant. We can parametrize an arbitrary distribution with moments G_I by the transformation that relates it to the unit Wigner distribution with moments Γ_I:

$$G_I = \sum_{n=1}^{\infty} \phi^{J_1}_{i_1} \cdots \phi^{J_n}_{i_n} \Gamma_{J_1 \cdots J_n} \,. \tag{42}$$

Indeed, we will see below that all moments that differ infinitesimally from a given one are obtained by such infinitesimal transformations. To rigorously justify our point of view, we must prove that (in an appropriate topology) the Lie group $\tilde{\mathcal{G}}$ is the exponential of this Lie algebra. We will not address this somewhat technical issue in this paper. In the next section we describe the Lie algebra in some more details.

4.2. *The Lie algebra of derivations*

An automorphism that differs only infinitesimally from the identity is a derivation of the tensor algebra. Any derivation of the free algebra is determined by its effect on the generators ξ_i. They can be written as linear combinations $v^I_i L^i_I$, where the basis elements L^i_I are defined by

$$[L^i_I \xi]_j = \delta^i_j \xi_I \,. \tag{43}$$

The derivations form a Lie algebra with commutation relations

$$[L^i_I, L^j_J] = \delta^{J_1 i J_2}_J L^j_{J_1 I J_2} - \delta^{I_1 j I_2}_I L^i_{I_1 J I_2} \,. \tag{44}$$

The change of the moments under such a derivations is

$$[L^i_I G]_J = \delta^{J_1 i J_2}_J G_{J_1 I J_2} \,. \tag{45}$$

We already encountered these infinitesimal variations in the derivation of the Schwinger–Dyson equation.

Let us consider the infinitesimal neighborhood of some reference distribution Γ_I — for example the unit Wigner distribution. We will assume that Γ_I satisfies the strict positivity condition, $\Gamma_{IJ}u^{*I}u^J > 0$; i.e. that this quadratic form vanishes only when the polynomial u is identically zero. (This condition is satisfied by the unit Wigner distribution.) It is the analog of the condition in classical probability theory that the probability distribution does not vanish in some neighborhood of the origin. Then, the *Hankel matrix* $H_{I;J} = G_{IJ}$ is invertible on polynomials: It is an inner product.

The infinitesimal change of moments under a derivation $v = v_i^I L_I^i$ is

$$[L_v\Gamma]_{k_1\cdots k_n} = v_{k_1}^I \Gamma_{Ik_2\cdots k_n} + \text{ cyclic permutations in } (k_1 \cdots k_n)\,. \tag{46}$$

Now, it is clear that the addition of an arbitrary infinitesimal cyclically symmetric tensor g_I to Γ_I can be achieved by some derivation: We just find some tensor w_I of which g_I is the cyclically symmetric part and put $w_{kk_1\cdots k_n} = v_k^J \Gamma_{JK}$. Since the Hankel matrix is invertible, we can always find such a v. Thus an arbitrary infinitesimal change in Γ_I can be achieved by some v_i^I.

Indeed there will be many such derivations, differing by those that leave Γ_I invariant. The isotropy Lie algebra of Γ_I is defined by

$$v_{k_1}^I \Gamma_{Ik_2\cdots k_n} + \text{ cyclic permutations in } (k_1 \cdots k_n) = 0\,. \tag{47}$$

We can simplify this condition for the choice where Γ_I is the Wigner distribution. For $n = 1$ this is just

$$v_k^I \Gamma_I = 0\,; \tag{48}$$

for $n = 2$, we get, using the recursion relation for Wigner moments,

$$v_{k_1}^{IjL} \Gamma_{k_2j} \Gamma_I \Gamma_L + k_1 \leftrightarrow k_2 = 0\,. \tag{49}$$

In general, using the iterations of the Wigner recursion relation

$$\Gamma_{Ik_1k_2} = \Gamma_{k_1j_2}\Gamma_{k_2j_1}\delta_I^{I_1j_1I_2j_2I_3}\Gamma_{I_1}\Gamma_{I_2}\Gamma_{I_3} \tag{50}$$

etc. we get

$$\begin{aligned}\Gamma_{k_1j_{n-1}}\Gamma_{k_2j_{n-2}}\cdots\Gamma_{k_{n-1}j_1} v_{k_n}^{I_1j_1I_2j_2\cdots j_{n-1}I_n}\Gamma_{I_1}\cdots\Gamma_{I_n}\\ + \text{cyclic permutations in } (k_1\cdots k_n) = 0\,.\end{aligned} \tag{51}$$

In other words, we should lower a certain number of indices on v_i^I using the second moment and contract away the rest; the resulting tensor should not have a cyclically symmetric part. It would be interesting to find the solutions of these conditions more explicitly. We will not need this for our present work.

5. The Action Principle and Cohomology

We seek an action $\Omega(G)$ such that its variation under an infinitesimal automorphism $[L_I^i G]_J = \delta_J^{J_1iJ_2} G_{J_1IJ_2}$ is the factorized SD equation:

$$L_I^i \Omega(G) = S^{J_1iJ_2} G_{J_1IJ_2} + \delta_I^{I_1iI_2} G_{I_1} G_{I_2}\,. \tag{52}$$

It is easy to identify a quantity that will give the first term:

$$L^i_I[S^J G_J] = S^{J_1 i J_2} G_{J_1 I J_2}\,. \tag{53}$$

This term is simply the expectation value of the matrix model action. So

$$\Omega(G) = S^J G_J + \chi(G) \tag{54}$$

with

$$L^i_I \chi(G) = \eta^i_I \equiv \delta^{I_1 i I_2}_I G_{I_1} G_{I_2}\,. \tag{55}$$

This term arises from the change in the measure of integration over matrices; hence it is a kind of "anomaly."

Now, in order for such a function $\chi(G)$ to exist, the anomaly η^i_I must satisfy an integrability condition $L^i_I(L^i_J\chi) - L^j_J(L^i_I\chi) = [L^i_I, L^j_J]\chi$; i.e.

$$L^i_I \eta^j_J - L^j_J \eta^i_I - \delta^{J_1 i J_2}_J \eta^j_{J_1 I J_2} + \delta^{I_1 j I_2}_I \eta^i_{I_1 J I_2} = 0\,. \tag{56}$$

A straightforward but tedious calculation shows that this is indeed satisfied. We were not able to find a formal power series of moments satisfying this condition even after many attempts.

Then we realized that, even in the case of a single matrix (treated in the appendix) there is no solution of the above equation! The condition above is in fact the statement that $\eta^i_I(G)$ is a one-cocycle of the Lie-algebra cohomology of $\underline{\mathcal{G}}_M$ valued in the space of formal power series in G. (See the appendix.[24]) Although η itself is a quadratic polynomial in the G, there is no polynomial or even formal power series of which it is a variation: It represents a nontrivial element of the cohomology of $\underline{\mathcal{G}}$ twisted by its representation on the space of formal power series in the moments.

We need to look for χ in some larger class of functions on the space $\mathcal{P}_M$ of probability distributions. Now, $\mathcal{P}_M = \tilde{\mathcal{G}}_M/\mathcal{SG}_M$, a coset space of the group of automorphisms. We can parametrize the moments in terms of the automorphism that will bring them to a standard one: $G_I = [\phi_*\Gamma]_I$. So, another way of thinking of functions on $\mathcal{P}_M$ would be as functions on $\tilde{\mathcal{G}}_M$ invariant under the action[i] of $\mathcal{SG}_M$. Thus, instead of power series in G_I, we will have power series in the coefficients ϕ^I_i determining an automorphism. In order to stand in for a function on $\mathcal{P}_M$, such a power series would have to be invariant under the subgroup $\mathcal{SG}_M$.

Clearly, any power series of G_I can be expressed as a power series of the ϕ^I_i: Simply substitute $[\phi_*\Gamma]_I$ for G_I. But there could be a power series in ϕ that is invariant under $\mathcal{SG}$ and still is not expressible as a power series in G. This subtle distinction is the origin of the cohomology we are discussing.[j] We can now guess that the quantity we seek is a function of this type on $\mathcal{P}_M$.

A hint is also provided by the origin of the term $\eta^i_I(G)$ in the Schwinger–Dyson equations. It arises from the change of the measure of integration over matrices

[i]Such an idea was used successfully to solve a similar problem on cohomologies.[25]
[j]We give a simple example in the appendix.

under an infinitesimal, but nonlinear, change of variables. Thus, it should be useful to study this change of measure under a finite nonlinear change of variables — an automorphism.

More precisely, let $\phi(A)$ be a nonlinear transformation $A_i \mapsto \phi(A)_i = \sum_{n=1}^{\infty} \phi_i^I A_I$, on the space of Hermitian $N \times N$ matrices. Also, let $\sigma(\phi, A) = \frac{1}{N^2} \log\det J(\phi, A)$, where $J(\phi, A)$ is the Jacobian determinant of ϕ.

By the multiplicative property of Jacobians, we have

$$\sigma(\phi_1\phi_2, A) = \sigma(\phi_1, \phi_2(A)) + \sigma(\phi_2, A)\,. \tag{57}$$

For example, $\sigma(\phi, A) = \log\det\phi_0$ if $[\phi(x)]_i = \phi_{0i}^j \xi_j$ is a linear transformation: The Jacobian matrix is then a constant. It is convenient to factor out this linear transformation and write

$$[\phi(A)]_i = \phi_{0i}^j [\tilde{\phi}(A)_j]\,, \quad \tilde{\phi}(A)_i = A_i + \sum_{n=2}^{\infty} \tilde{\phi}_i^{i_1 \cdots i_n} A_{i_1} \cdots A_{i_n}\,. \tag{58}$$

We will show in the appendix that $\sigma(\phi, A)$ can be written in terms of the traces $\Phi_I = \frac{1}{N} \operatorname{tr} A_I$:

$$\sigma(\phi, A) = \log\det\phi_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \tilde{\phi}_{i_1}^{K_1 i_2 L_1} \tilde{\phi}_{i_2}^{K_2 i_3 L_2} \cdots \tilde{\phi}_{i_n}^{K_n i_1 L_n} \Phi_{K_1 \cdots K_n} \Phi_{L_n \cdots L_1}\,.$$

Thus, the expectation value of $\sigma(\phi, A)$ with respect to some distribution can be expressed in terms of its moments $G_I = \langle \Phi_I \rangle$, in the large N limit:

$$\langle \sigma(\phi, A) \rangle = c(\phi, G) = \log\det\phi_1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \tilde{\phi}_{i_1}^{J_1 i_2 K_1} \tilde{\phi}_{i_2}^{J_2 i_3 K_2} \cdots \tilde{\phi}_{i_n}^{J_n i_1 K_n} G_{J_1 \cdots J_n} G_{K_n \cdots K_1}\,.$$

The above equation for $\sigma(\phi_1\phi_2, A)$ then shows that the expectation value $c(\phi, G)$ satisfies the cocycle condition:

$$c(\phi_1\phi_2, G) = c(\phi_1, \phi_{2*}(G)) + c(\phi_2, G)\,. \tag{59}$$

Moreover, if we now restrict to the case of infinitesimal transformations, $\phi(\xi)_i = \xi_i + v_i^I \xi_I$, this $c(\phi, G)$ reduces to η:

$$c(\phi, G) = v_i^I \eta_I^i(G) + \mathcal{O}(v^2)\,. \tag{60}$$

Let us look at it another way: Let $G = \phi_* \Gamma$ for some reference probability distribution Γ. Then the cocycle condition gives

$$c(\phi_1, G) = c(\phi_1\phi, \Gamma) - c(\phi, \Gamma)\,. \tag{61}$$

Choosing ϕ_1 to be infinitesimal gives then,

$$\eta_i^I(G) = L_i^I c(\phi, \Gamma)\,. \tag{62}$$

Thus, we have solved our problem!: $\chi(\phi) = c(\phi, \Gamma)$ is a function on $\widetilde{\mathcal{G}}_M$ whose variation is η. But is it really a function on $\mathcal{P}_M$. In other words, is $\chi(\phi)$-invariant under the right action of $\mathcal{SG}$?

If $\phi_{2*}\Gamma = \Gamma$ the cocycle condition reduces to

$$c(\phi\phi_2, \Gamma) = c(\phi, \Gamma) + c(\phi_2, \Gamma)\,. \tag{63}$$

We need to show that the last term is zero.

We will only consider the case where the reference distribution Γ is the unit Wignerian. If $\phi_2(\xi)_i = \xi_i + v_i^I \xi_I$ is infinitesimal, $c(\phi_2, \Gamma)$ is just $v_i^I \eta_I^i(\Gamma) = v_i^{JiL}\Gamma_J\Gamma_L$. But, since v must leave the Wigner moments Γ_I unchanged, it must satisfy (49). If we contract that equation by $\Gamma^{k_1k_2}$ we will get $v_i^{JiL}\Gamma_J\Gamma_L = 0$. Thus $\chi(\phi)$ in invariant at least under an infinitesimal $\phi_2 \in \mathcal{SG}$. Within the ultrametric topology of formal power series, the group $\mathcal{SG}$ should be connected, so that any element can be reached by a succession of infinitesimal transformations.

To summarize, we now have an action principle for matrix models,

$$\begin{aligned}\Omega(\phi) = &\sum_{n=1}^{\infty} S^{i_1\cdots i_n}\phi_{i_1}^{J_1}\cdots\phi_{i_n}^{J_n}\Gamma_{J_1\cdots J_n} + \log\det\phi_{0j}^{i}\\ &+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\tilde{\phi}_{i_2}^{K_1i_1L_1}\tilde{\phi}_{i_3}^{K_2i_2L_2}\cdots\tilde{\phi}_{i_1}^{K_ni_nL_n}\Gamma_{K_n\cdots K_1}\Gamma_{L_1\cdots L_n}\,.\end{aligned}$$

The factorized Schwinger–Dyson equations follow from requiring that this action be extremal under infinitesimal variations of ϕ. By choosing an ansatz for ϕ that depends only on a few parameters and maximizing Ω with respect to them, we can get approximate solutions to the factorized SD equations.

6. Entropy of Noncommutative Probability Distributions

Whenever we restrict the set of allowed observables of a system, some entropy is created: It measures our ignorance of the variables we are not allowed to measure. Familiar examples arise from thermodynamics, where only a finite number of macroscopic parameters are measured. In black hole physics where only the charge, mass and angular momentum of a black hole can be measured by external particle scattering: The interior of a black hole is not observable to an outside observer.

There should be a similar entropy in the theory of strong interactions due to confinement: Only "macroscopic" observables associated to hadrons are measurable by a scattering of hadrons against each other. Quarks and gluons are observable only in this indirect way. More precisely, only color invariant observables are measurable.

In this paper, we have a toy model of this entropy due to confinement of gluons: We restrict to the gauge-invariant functions $\Phi_I = \frac{1}{N}\operatorname{tr} A_I$, of the matrices $A_1\cdots A_M$. It turns out that the term χ in the action principle above is just the entropy caused by this restriction.

Let Q be some space of "microscopic" variables with a probability measure μ, and $\Phi : Q \to \bar{Q}$ some map to a space of "macroscopic" variables. We can now define the volume of any subset of $\bar{Q}$ to be the volume of its preimage in Q: This is the induced measure $\bar{\mu}$ on $\bar{Q}$.

In particular we can consider the volume of the pre-image of a point $\bar{q} \in \bar{Q}$. It is a measure of our ignorance of the microscopic variables, when $\bar{q}$ is the result of measuring the macroscopic ones. Any monotonic function of this volume is just as good a measure of this ignorance. The best choice is the logarithm of this volume, since then it would be additive for statistically independent systems. Let us denote this function on $\Phi(Q)$ by

$$\sigma(\bar{q}) = \log(\mu[\Phi^{-1}(\bar{q})])\,. \tag{64}$$

The average of this quantity over $\bar{Q}$ is the *entropy* of the induced probability distribution $\bar{\mu}$.

Let us apply this idea to the case where the "microscopic" observable is a single Hermitian $N \times N$ matrix A; the "macroscopic" observable is the spectrum, the set of eigenvalues. We disregard the information in the basis in which A is presented. We do so even if this information is measurable in principle; e.g. by interference experiments in quantum mechanics. The natural measure on the space of matrices is the uniform (Lebesgue) measure dA on R^{MN^2}. Although the uniform measure is not normalizable, the volume of the space of matrices with a given spectrum $\{a_1 \cdots a_N\}$ is finite.[10] Up to a constant (i.e. depending only on N), it is $\prod_{1\leq i<j\leq N}(a_i - a_j)^2$. Thus the entropy is $2\int_{x<y} \rho(x)\rho(y) \log|x-y| dx\, dy$ where $\rho(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - a_i)$. This expression make sense even in the limit $N \to \infty$: We get a continuous distribution of eigenvalues $\rho(x)$.

What is the "joint spectrum" of a collection of M Hermitian matrices $A_1 \cdots A_M$? Clearly they cannot be simultaneously diagonalized, so a direct definition of this concept is impossible. Now, recall that the set of eigenvalues $\{a_1 \cdots a_N\}$ can be recovered from the elementary symmetric functions $G_n = \frac{1}{N}\sum_{i=1}^{N} a_i^n$ as the solutions of the algebraic equation

$$\frac{1}{N}x^N = G_1 x^{N-1} - G_2 x^{N-2} + \cdots + (-1)^{N-1} G_N\,. \tag{65}$$

The moments G_n for $n > N$ are not independent: They can be expressed as polynomials of the lower ones. Although the set $\{a_1 \cdots a_N\}$ is determined by the sequence $G_1 \cdots G_N$, there is no explicit algebraic formula: The Galois theory shows that this is impossible for $N > 4$. Galois theory also shows that any gauge-invariant polynomial of A can be expressed as a polynomial of the $G_1 \cdots G_N$. Thus we can regard this sequence $\frac{1}{N}\operatorname{tr} A, \frac{1}{N}\operatorname{tr} A^2 \cdots$ as the spectrum of the matrix A.

The volume $\prod_{i<j}(a_i - a_j)^2$ of the space of matrices with a given spectrum is a symmetric polynomial of order $\frac{N(N-1)}{2}$ in the eigenvalues. Hence, in principle, it can be expressed as a polynomial in $G_1 \cdots G_N$, athough there does not appear to be a simple universal formula.[k]

[k]It is possible to get a formula for the volume in terms of the first $2N$ moments. The complication is that only the first N moments can be freely specified. The remaining moments are determined by these, and yet, there is no algebraic formula that expresses $G_{N+1} \cdots G_{2N}$ in terms of $G_1 \cdots G_N$.

This point of view suggests a generalization to several matrices: We can define the *joint spectrum* of a collection of matrices to be the quantities $G_{i_1\cdots i_n} = \frac{1}{N}\operatorname{tr} A_{i_1}\cdots A_{i_n}$. Again, there are relations among these quantities when I is longer than N; but it is difficult to characterize these relations explicitly. Nevertheless, it is meaningful to ask for the volume (with respect to the uniform measure on R^{MN^2}) of the set of all matrices with a given value for the sequence $G_{i_1\cdots i_n}$. Again, we will not get any explicit formula for entropy by pursuing this point of view.

So we look for yet another way to think of the joint spectrum of a collection of matrices. We can ask how the entropy of a collection of matrices with joint spectrum G_I changes if we transform them by some power series:

$$A_i \mapsto \phi(A)_i = \phi_i^I A_I\,. \tag{66}$$

Let $c(\phi, G)$ be this change. Then, if we perform another transformation, we must have

$$c(\phi\phi_2, G) = c(\phi, \phi_{2*}G) + c(\phi_2, G)\,; \tag{67}$$

i.e. it must be a one-cocyle. Under infinitesimal variations, it reduces to η, since it is just the infinitesimal change in the uniform measure dA.

In the last section we obtained this $c(\phi, G)$ explicitly as a formal power series in G. It can be written as the variation

$$c(\phi, G) = \chi(\phi_*(G)) - \chi(G) \tag{68}$$

of some function χ of the joint spectrum G. However this χ is not a formal power series in G, so we cannot get an explicit formula for it. We can write it as an explicit formal power series in ϕ which is invariant under the action of $\mathcal{SG}$.

Thus we see the confluence of three apparently unrelated questions: An action principle for the planar limit of matrix models (our main interest), cohomology of the automorphism of formal power series and entropy of noncommutative variables.

Voiculsecu has a somewhat different approach[12] to defining the entropy of noncommuting random variables. Up to some additive constant his definition seems to agree with ours. However, the transformation property of entropy under change of variables is not discussed there. This transformation property is the cornerstone of our approach.

7. Example: Two-Matrix Models

Let us consider a quartic multi-matrix model with the action:

$$S(M) = -\operatorname{tr}\left[\frac{1}{2}K^{ij}A_{ij} + \frac{1}{4}g^{ijkl}A_{ijkl}\right]. \tag{69}$$

Our reference action is the Gaussian[l] $S_0(M) = -\operatorname{tr} \frac{1}{2}\delta^{ij} A_i A_j$. We are interested in estimating the Green functions and vacuum energy in the large N limit:

$$E^{\text{exact}} = -\lim_{N\to\infty} \frac{1}{N^2} \log \frac{Z}{Z_0}\,, \tag{70}$$

where Z and Z_0 are partition functions for S and S_0. Choose the linear change of variable $A_i \to \phi_i(A) = \phi_i^j A_j$. The variational matrix ϕ_i^j that maximizes Ω determines the multi-variable Wigner distribution that best approximates the quartic matrix model. For a linear change of variables,

$$\Omega[\phi] = \operatorname{tr}\log[\phi_i^j] - \frac{1}{2}K^{ij}G_{ij} - \frac{1}{4}g^{ijkl}G_{ijkl}\,. \tag{71}$$

Here $G_{ij} = \phi_i^k \phi_j^k$ and $G_{ijkl} = G_{ij}G_{kl} + G_{il}G_{jk}$ are the Green functions of $S_0(\phi^{-1}(A))$. Thus, the matrix elements of G may be regarded as the variational parameters and the condition for an extremum is

$$\frac{1}{2}K^{pq} + \frac{1}{4}[g^{pqkl}G_{kl} + g^{ijpq}G_{ij} + g^{pjqk}G_{jk} + g^{ipql}G_{il}] = \frac{1}{2}[G^{-1}]^{pq}\,. \tag{72}$$

This is a nonlinear equation for the variational matrix G, reminiscent of the self consistent equation for a mean field. To test our variational approach, we specialize to a two matrix model for which some exact results are known from the work of Mehta.[28]

7.1. *Mehta's quartic two-matrix model*

Consider the action

$$S(A,B) = -\operatorname{tr}\left[\frac{1}{2}(A^2 + B^2 - cAB - cBA) + \frac{g}{4}(A^4 + B^4)\right], \tag{73}$$

which corresponds to the choices $K^{ij} = \begin{pmatrix} 1 & -c \\ -c & 1 \end{pmatrix}$, $g^{1111} = g^{2222} = g$ and $g^{ijkl} = 0$ otherwise.[m] We restrict to $|c| < 1$, where K^{ij} is a positive matrix. Since $S(A,B) = S(B,A)$ and $G_{AB} = G^*_{BA}$ we may take

$$G_{ij} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \tag{74}$$

with α, β real. For $g > 0$, Ω is bounded above if G_{ij} is positive. Its maximum occurs at (α, β) determined by $\beta = \frac{c\alpha}{1+2g\alpha}$ and

$$4g^2\alpha^3 + 4g\alpha^2 + (1 - c^2 - 2g)\alpha - 1 = 0\,. \tag{75}$$

[l] In the language of noncommutative probability theory we used earlier, what we call the Gaussian in this section is really the multivariate Wignerian distribution. There should be no confusion, since the Wignerian moments are realized by a Gaussian distribution of matrices.
[m] Kazakov relates this model to the Ising model on the collection of all planar lattices with coordination number four.[29]

We must pick the real root $\alpha(g,c)$ that lies in the physical region $\alpha \geq 0$. Thus, the Gaussian ansatz determines the vacuum energy ($E(g,c) = -\frac{1}{2}\log(\alpha^2 - \beta^2)$) and *all* the Green functions (e.g. $G_{AA} = \alpha, G_{AB} = \beta, G_{A^4} = 2\alpha^2$ etc.) approximately.

By contrast, only a few observables of this model have been calculated exactly. Mehta[28,n] obtains the exact vacuum energy $E^{\rm ex}(g,c)$ implicitly, as the solution of a quintic equation. $G^{\rm ex}_{AB}$ and $G^{\rm ex}_{A^4}$ may be obtained by differentiation. As an illustration, we compare with Mehta's results in the weak and strong coupling regions:

$$\begin{aligned}
E^{\rm ex}\left(g,\frac{1}{2}\right) &= -0.144 + 1.78g - 8.74g^2 + \cdots\,,\\
G^{\rm ex}_{AB}\left(g,\frac{1}{2}\right) &= \frac{2}{3} - 4.74g + 53.33g^2 + \cdots\,,\\
G^{\rm ex}_{AAAA}\left(g,\frac{1}{2}\right) &= \frac{32}{9} - 34.96g + \cdots\,,\\
E^{\rm var}\left(g,\frac{1}{2}\right) &= -0.144 + 3.56g - 23.7g^2 + \cdots\,,\\
G^{\rm var}_{AB}\left(g,\frac{1}{2}\right) &= \frac{2}{3} - 4.74g + 48.46g^2 + \cdots\,,\\
G^{\rm var}_{AAAA}\left(g,\frac{1}{2}\right) &= \frac{32}{9} - 31.61g + 368.02g^2 + \cdots \text{etc.}]\,,\\
E^{\rm ex}(g,c) &= \frac{1}{2}\log g + \frac{1}{2}\log 3 - \frac{3}{4} + \cdots\,,\\
G^{\rm ex}_{AB}(g,c) &\to 0 \text{ as } g \to \infty\,,\\
G^{\rm ex}_{A^4}(g,c) &= \frac{1}{g} + \cdots\,,\\
E^{\rm var}(g,c) &= \frac{1}{2}\log g + \frac{1}{2}\log 2 + \frac{1}{\sqrt{8g}} + \mathcal{O}\left(\frac{1}{g}\right),\\
G^{\rm var}_{AB}(g,c) &= \frac{c}{2g} - \frac{c}{(2g)^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{g^2}\right),\\
G^{\rm var}_{A^4}(g,c) &= \frac{1}{g} - \frac{2}{(2g)^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{g^2}\right), \text{ etc.}
\end{aligned} \tag{76}$$

We see that the Gaussian variational ansatz provides a reasonable first approximation in both the weak and strong coupling regions. The Gaussian variational ansatz is not good near singularities of the free energy (phase transitions). As $|c| \to 1^-$, the energy $E^{\rm ex}$ diverges; this is not captured well by the Gaussian ansatz. This reinforces our view that the Gaussian variational ansatz is the analog of mean field theory.

[n]Some other special classes of Green functions are also accessible (see Refs. 30 and 31).

7.2. *Two-matrix model with* $\mathrm{tr}[A,B]^2$ *interaction*

The power of our variational methods is their generality. We present an approximate solution to a two-matrix model, for which we could find no exact results in the literature. The action we consider is

$$S(A,B) = -\,\mathrm{tr}\left[\frac{m^2}{2}(A^2+B^2) + \frac{c}{2}(AB+BA) - \frac{g}{4}[A,B]^2\right]. \tag{77}$$

This is a caricature of the Yang–Mills action. A supersymmetric version of this model is also of interest (see Ref. 32). Consider the regime where $K^{ij} = \begin{pmatrix} m^2 & c \\ c & m^2 \end{pmatrix}$ is positive, $k = (m^4 - c^2) \geq 0$. As before, we pick a Gaussian ansatz and maximize Ω. We get $\beta = -\frac{c}{m^2}\alpha$ and

$$\begin{aligned} G^{AA} = G^{BB} = \alpha &= \frac{m^2}{2g}\left[\sqrt{1+\frac{4g}{k}} - 1\right], \\ E &= -\frac{1}{2}\log\left[\frac{2g+k-\sqrt{k^2+4kg}}{2g^2}\right]. \end{aligned} \tag{78}$$

All other mean field Green functions can be expressed in terms of α.

It is possible to improve on this Gaussian variational ansatz by using nonlinear transformations. It is much easier to find first a Gaussian approximation and then expand around it in a sort of loop expansion. This is the analog of the usual Goldstone methods of many body theory. We have performed such calculations for these multimatrix models, but we will not report on them in this paper for the sake of brevity. The results are qualitatively the same. In the next section (an appendix) we will give the departures from the Gaussian ansatz in the case of the single-matrix models.

Appendix A. Group Cohomology

Given a group G and a G-module V (i.e. a representation of G on a vector space V), we can define a cohomology theory.[24] The r-cochains are functions

$$f : G^r \to V\,. \tag{A.1}$$

The coboundary is

$$\begin{aligned} df(g_1, g_2, \ldots, g_{r+1}) &= g_1 f(g_2,\ldots,g_{r+1}) + \sum_{s=1}^{r}(-1)^s f(g_1, g_2, \ldots, g_{s-1}, g_s g_{s+1}, g_{s+2}, \ldots, g_{r+1}) \\ &\quad + (-1)^{r+1} f(g_1,\ldots,g_r)\,. \end{aligned} \tag{A.2}$$

It is straightforward to check that $d^2 f = 0$ for all f. A chain c is a *cocycle* or is *closed* if $df = 0$; a cocycle is *exact* or is a *coboundary* if $b = df$ for some f; the rth cohomology of G twisted by the module V, $H^r(G,V)$ is the space of closed chains modulo exact chains. $H^0(G,V)$ is the space of invariant elements in V; i.e. the space of v satisfying $gv - v = 0$ for all $g \in G$. A one-cocycle is a function $c : G \to V$ satisfying

$$c(g_1 g_2) = g_1 c(g_2) + c(g_1)\,. \tag{A.3}$$

Solutions to this equation modulo one-coboundaries (which are of the form $b(g) = (g-1)v$ for some $v \in V$) is the first cohomology $H^1(G,V)$. If G acts trivially on V, a cocycle is just a homomorphism of G to the additive group of V: $c(g_1 g_2) = c(g_2) + c(g_1)$.

A one-cocycle gives a way of turning a representation on V into an affine action:

$$(g, v) \mapsto gv + c(g)\,. \tag{A.4}$$

If $c(g)$ is a coboundary (i.e. $b(g) = (g-1)u$ for some u), this affine action is really a linear representation in disguise: If the origin is shifted by u we can reduce it to a linear representation. Thus the elements of $H^1(G,V)$ describe "true" affine actions on V. For example, let G be the loop group of a Lie group G', the space of smooth functions from the circle to G': $G = S^1G' = \{g : S^1 \to G'\}$. Let $V = S^1\underline{G'}$ be the corresponding loop of the Lie algebra $\underline{G'}$ of G'. Then there is an obvious adjoint representation of G on V; a nontrivial one-cocycle is $c(g) = g\,dg^{-1}$, d being the exterior derivative on the circle:

$$c(g_1 g_2) = g_1[g_2 d(g_2^{-1})]g_1^{-1} + g_1\,dg_1^{-1} = \text{ad}\ g_1 c(g_2) + c(g_1)\,. \tag{A.5}$$

Appendix B. A Single Random Matrix

In the special case where there is only one matrix ($M = 1$), there is a probability distribution on the real line $\rho(x)dx$ such that

$$G_n = \int x^n \rho(x) dx\,. \tag{B.1}$$

This follows because the G_n satisfy the positivity condition

$$\sum_{m,n=0}^{\infty} G_{n+m} u_m^* u_n \geq 0\,; \tag{B.2}$$

up to technical conditions, any sequence of real numbers satisfying this condition determine a probability distribution on the real line. (This is the *classical moment problem* solved in the nineteenth century.[22]) There is an advantage to transforming the factorized SD equations into an equation for $\rho(x)$ — it becomes a linear integral equation, the *Mehta–Dyson* equation:[10]

$$2\mathcal{P}\int_a^b \frac{\rho(y)dy}{x-y} + S'(x) = 0\,. \tag{B.3}$$

Moreover, the solution[26,27] to this equation can be expressed in purely algebraic terms[o]

$$\rho(x) = -\frac{1}{2\pi}\theta(a \leq x \leq b)\sqrt{[(x-a)(b-x)]}\left\lfloor \frac{S'(x)}{\sqrt{(x-a)(x-b)}} \right\rfloor . \tag{B.4}$$

The numbers a and b are solutions of the algebraic equations

$$\begin{aligned} \sum_{r,s=0} [r+s+1]S_{r+s+1}\frac{\left(\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_s}{r!s!}a^r b^s = 0\,, \\ \sum_{r,s=0} [r+s]S_{r+s}\frac{\left(\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_s}{r!s!}a^r b^s = 2\,, \end{aligned} \tag{B.5}$$

where $\left(\frac{1}{2}\right)_r = \frac{1}{2}\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+r-1\right)$. The simplest example is the case of the Wigner distribution. It is the analog of the Gaussian in the world of noncommutative probability distributions. For, if we choose the matrix elements of A to be independent Gaussians, $S(A) = -\frac{1}{2}\operatorname{tr} A^2$, we get the distribution function for the eigenvalues of A to be (in the large N limit)

$$\rho_0(x) = \frac{1}{2\pi}\sqrt{[4-x^2]}\theta(|x|<2)\,. \tag{B.6}$$

The odd moments vanish; the even moments are then given by the *Catalan numbers*

$$G_{2k} = C_k = \frac{1}{k+1}\binom{2k}{k}\,. \tag{B.7}$$

The Mehta–Dyson equation follows from maximizing the "action"

$$\Omega(\rho) = \int \rho(x)S(x)dx + \mathcal{P}\int \log|x-y|\rho(x)\rho(y)dx\,dy \tag{B.8}$$

with respect to $\rho(x)$. Then generating function $\log Z(S)$ is the maximum of this functional over all probability distributions ρ. The physical meaning of the first term is clear: It is just the expectation value of the action of the original matrix model:

$$\int \rho(x)S(x)dx = \sum_n G_n S_n\,. \tag{B.9}$$

The second term can be thought of as the "entropy" which arises because we have lost the information about the angular variables in the matrix variable: The function $\rho(x)$ is the density of the distribution of the eigenvalues of A. Indeed, $\sum_{i\neq j}\log|a_i - a_j|$ is (up to a constant depending only on N) the log of the volume of the space of all Hermitian matrices with spectrum $a_1, a_2\cdots a_N$.The entropy $\mathcal{P}\int\log|x-y|\rho(x)\rho(y)dx\,dy$ is the large N limit of this quantity. Note that the

[o] For a Laurent series $X(z) = \sum_{k=-\infty}^{m} X_k z^k$ around infinity, $\lfloor X(z)\rfloor = \sum_{k=0}^{m} X_k z^k$ denotes the part that is a polynomial in z. This is analogous to the "integer part" of a real number, which explains the notation.

entropy is independent of the choice of the matrix model action: It is a universal property of all one-matrix models. The meaning of the variational principle is now clear: We seek the probability distribution of maximum entropy that has a given set of moments G_r for $r = 1 \cdots n$. The coefficients of the polynomial S are just the Lagrange multipliers that enforce this condition. Thus we found a variational principle, but indirectly in terms of the function $\rho(x)$ rather than the moments G_n themselves. The entropy could not be expressed explicitly in terms of the moments. Indeed, in a sense, this is impossible:

The entropy cannot be expressed as a formal power series in G_n. This is surprising since there appears to be a linear relation between $\rho(x)$ and G_n, since $G_n = \int x^n \rho(x) dx$; also the entropy is a quadratic function of $\rho(x)$. So one might think that entropy is a quadratic function of the G_n as well. But if we try to compute this function we will get a divergent answer. Indeed, we claim that even if we do not require the series to converge, the entropy cannot be expressed as a power series in G_n.

By thinking in terms of the change of variables that bring the probability distribution we seek to a standard one, we can find an explicit formula for entropy. Since we are interested in polynomial actions $S(A)$, which are modifications of the quadratic action $\frac{1}{2}A^2$, the right choice of this reference distribution is the Wigner distribution

$$\rho_0(x) = \frac{1}{2\pi}\sqrt{[4 - x^2]}\theta(|x| < 2)\,. \tag{B.10}$$

There should thus be a change of variable $\phi(x)$ such that

$$G_k = \int x^k \rho(x) dx = \int \phi(x)^k \rho_0(x) dx\,; \tag{B.11}$$

in other words,

$$\rho(\phi(x))\phi'(x) = \rho_0(x)\,. \tag{B.12}$$

Then we get

$$\Omega(\phi) = \int S(\phi(x))\rho_0(x)dx + \int \log\left[\frac{\phi(x) - \phi(y)}{x - y}\right]\rho_0(x)dx\rho_0(y)dy\,. \tag{B.13}$$

We have dropped a constant term — the entropy of the reference distribution itself. Also it will be convenient to choose the constant of integration such that $\phi(0) = 0$. Now we can regard the diffeomorphism as parametrized by its Taylor coefficients

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n x^n\,, \quad \phi_1 > 0\,. \tag{B.14}$$

Although we cannot express the entropy in terms of the moments G_n themselves, we will be able to express both the entropy and the moments in terms of the parameters ϕ_n. Thus we have a "parametric form" of the variational problem. It is this parametric form that we can extend to the case of multi-matrix models. Indeed,

$$\phi^k(x) = \sum_{n=1}^{\infty} x^n \sum_{l_1+l_2+\cdots+l_k=n} \phi_{l_1} \cdots \phi_{l_k} \tag{B.15}$$

so that

$$G_k = \sum_{n=1}^{\infty} \Gamma_n \sum_{l_1+l_2+\cdots+l_k=n} \phi_{l_1} \cdots \phi_{l_k}\,. \tag{B.16}$$

It is convenient to factor out the linear transformation $\phi(x) = \phi_1[x + \tilde{\phi}(x)]$, where $\tilde{\phi}(x) = \sum_{k=2}^{\infty} \tilde{\phi}_k x^k$, with $\tilde{\phi}_k = \phi_k[\phi_1]^{-1}$. Then

$$\log\left[\frac{\phi(x)-\phi(y)}{x-y}\right] = \log\phi_1 + \log\left[1 + \sum_{m=2}^{\infty} \tilde{\phi}_m \frac{x^m - y^m}{x-y}\right]. \tag{B.17}$$

Using

$$\frac{x^m - y^m}{x-y} = \sum_{k+1+l=m;\ k,l\geq 0} x^k y^l \tag{B.18}$$

and expanding the logarithm we get

$$\log\left[\frac{\phi(x)-\phi(y)}{x-y}\right] = \log\phi_1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \times \sum_{k_1,l_1,\ldots,k_n,l_n} \tilde{\phi}_{k_1+1+l_1} \cdots \tilde{\phi}_{k_n+1+l_n} x^{k_1+\cdots+k_n} y^{l_1+\cdots+l_n}\,. \tag{B.19}$$

It follows then that

$$\Omega(\phi) = \sum_{k,n=1}^{\infty} S_k \Gamma_n \sum_{l_1+l_2+\cdots+l_k=n} \phi_{l_1} \cdots \phi_{l_k} + \log\phi_1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{k_1,l_1,\ldots,k_n,l_n} \tilde{\phi}_{k_1+1+l_1} \cdots \tilde{\phi}_{k_n+1+l_n} \Gamma_{k_1+\cdots+k_n} \Gamma_{l_1+\cdots+l_n}\,. \tag{B.20}$$

While this formula may not be particularly transparent, it does accomplish our goal of finding a variational principle that determines the moments. The parameters ϕ_k characterize the probability distribution of the eigenvalues: They determine the moments G_n by the above series. By extremizing the action Ω as a function of these ϕ_k, we can then determine the moments. We will be able to generalize this version of the action principle to multi-matrix models. In practice we would choose some simple function $\phi(x)$ such as a polynomial to get an approximate solution to this variational problem. Since all one-matrix models are exactly solvable, we can use them to test the accuracy of our variational approximation.

B.1. *Explicit variational calculations*

Consider the quartic one matrix model. Its exact solution in the large N limit is known from the work of Brezin *et al.*:[26]

$$\begin{aligned} Z(g) &= \int dA\, e^{N\,\mathrm{tr}\left[-\frac{1}{2}A^2 - gA^4\right]}, \\ E_{\mathrm{exact}}(g) &= -\lim_{N\to\infty}\frac{1}{N^2}\log\frac{Z(g)}{Z(0)}. \end{aligned} \tag{B.21}$$

The Gaussian with unit covariance is our reference action. Choose as a variational ansatz the linear change of variable $\phi(x) = \phi_1 x$, which merely scales the Wigner distribution. The ϕ_1 that maximizes Ω represents the Wigner distribution that best approximates the quartic matrix model.

$$\Omega(\phi_1) = \log\phi_1 - \frac{1}{2}G_2 - gG_4\,. \tag{B.22}$$

Here $G_{2k} = \phi_1^{2k}\Gamma_k$. Letting $\alpha = \phi_1^2$, $\Omega(\alpha) = \frac{1}{2}\log\alpha - \frac{\alpha}{2} - 2g\alpha^2$ is bounded above only for $g \geq 0$. It's maximum occurs at $\alpha(g) = \frac{-1+\sqrt{1+32g}}{16g}$. Notice that α is determined by a nonlinear equation. This is reminiscent of the mean field theory; we will sometimes refer to the Gaussian ansatz as mean field theory. Our variational estimates are

$$\begin{aligned} E(g) &= -\frac{1}{2}\log\frac{-1+\sqrt{1+32g}}{16g}, \\ G_{2k}(g) &= \left(\frac{-1+\sqrt{1+32g}}{16g}\right)^k C_k\,. \end{aligned} \tag{B.23}$$

The exact results from[26] are

$$\begin{aligned} E_{\mathrm{ex}}(g) &= \frac{1}{24}(a^2(g)-1)(9-a^2(g)) - \frac{1}{2}\log\left(a^2(g)\right), \\ G_{\mathrm{ex}}^{2k}(g) &= \frac{(2k)!}{k!(k+2)!}a^{2k}(g)[2k+2-ka^2(g)]\,, \end{aligned} \tag{B.24}$$

where $a^2(g) = \frac{1}{24g}[-1+\sqrt{1+48g}]$. In both cases, the vacuum energy is analytic at $g = 0$ with a square root branch point at a negative critical coupling. The mean field critical coupling $g_c^{\mathrm{MF}} = -\frac{1}{32}$ is 50% more than the exact value $g_c^{\mathrm{ex}} = -\frac{1}{48}$.

The distribution of eigenvalues of the best Gaussian approximation is given by $\rho_g(x) = \phi_1^{-1}\rho_0(\phi_1^{-1}x)$ where $\rho_0(x) = \frac{1}{2\pi}\sqrt{4-x^2}, |x| \leq 2$ is the standard Wigner distribution. The exact distribution

$$\rho_{\mathrm{ex}}(x,g) = \frac{1}{\pi}\left(\frac{1}{2} + 4ga^2(g) + 2gx^2\right)\sqrt{4a^2(g)-x^2}\,, \qquad |x| \leq 2a(g)\,. \tag{B.25}$$

is compared with the best Gaussian approximation in Fig. 1. The latter does not capture the bimodal property of the former.

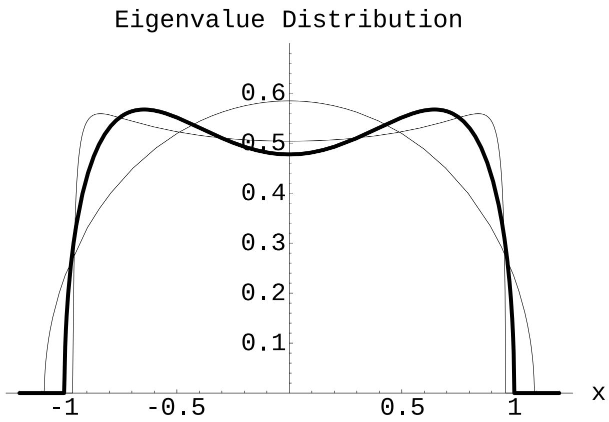

Fig. 1. Eigenvalue Distribution. The dark curve is exact, semicircle is mean field and bi-modal light curve is cubic ansatz at one-loop.

The vacuum energy estimate starts out for small g, being twice as big as the exact value. But the estimate improves and becomes exact as $g \to \infty$. Meanwhile, the estimate for $G_2(g)$ is within 10% of its exact value for all g. $G_{2k}, k \geq 2$ for the Gaussian ansatz do not have any new information. However, the higher cumulants vanish for this ansatz.

We see that a Gaussian ansatz is a reasonable first approximation, and is not restricted to small values of the coupling g. To improve on this, get a nontrivial estimate for the higher cumulants and capture the bimodal distribution of eigenvalues, we need to make a nonlinear change of variable.

B.2. *Nonlinear variational change of variables*

The simplest nonlinear ansatz for the quartic model is a cubic polynomial: $\phi(x) = \phi_1 x + \phi_3 x^3$. A quadratic ansatz will not lower the energy since $S(A)$ is even. Our reference distribution is still the standard Wigner distribution. $\phi_{1,3}$ are determined by the condition that

$$\Omega[\phi] = \left\langle \log \left| \frac{\phi(x) - \phi(y)}{x - y} \right| \right\rangle_0 - \frac{1}{2}\langle \phi^2(x) \rangle - g\langle \phi^4(x) \rangle_0 \tag{B.26}$$

be a maximum. Considering the success of the linear change of variable, we expect the deviations of $\phi_{1,3}$ from their mean field values $(\sqrt{\alpha}, 0)$ to be small, irrespective of g. Within this approximation, we get $\left(\text{with } \alpha = \frac{-1+\sqrt{1+32g}}{16g}\right)$

$$\begin{aligned} \phi_1 &= \sqrt{\alpha} - \frac{\sqrt{\alpha}(-3 + 2\alpha + (1 - 32g)\alpha^2 + 48g\alpha^3 + 144g^2\alpha^4)}{3 + 4\alpha + (1 + 96g)\alpha^2 + 48g\alpha^3 + 432g^2\alpha^4}\,, \\ \phi_3 &= \frac{8g\alpha^{\frac{5}{2}}(-2 + \alpha)}{3 + 4\alpha + (1 + 96g)\alpha^2 + 48g\alpha^3 + 432g^2\alpha^4}\,, \end{aligned} \tag{B.27}$$

from which we calculate the variational Green functions and vacuum energy. The procedure we have used to obtain $\phi_{1,3}$ can be thought of as a one-loop calculation around mean field theory. Comparing with the exact results of Ref. 26, we find the following qualitative improvements over the mean field ansatz.

In addition to the mean field branch cut from $-\infty$ to $g_c^{\rm MF}$, the vacuum energy now has a double pole at $g_c^{\rm MF} < g_c^{\text{1-loop}} = \frac{-346-25\sqrt{22}}{15138} < g_c^{\rm ex}$. We can understand this double pole as a sort of Padé approximation to a branch cut that runs all the way up to $g_c^{\rm ex}$. The vacuum energy variational estimate is lowered for all g.

Figure 1 demonstrates that the cubic ansatz is able to capture the bimodal nature of the exact eigenvalue distribution. If $\chi(x) = \phi^{-1}(x)$, then $\rho(x) = \rho_0(\chi(x))\chi'(x)$, where $\rho_0(x) = \frac{1}{2\pi}\sqrt{4-x^2}$, $|x| \le 2$.

The Green functions G_2, G_4 are now within a percent of their exact values, for all g. More significantly, the connected four-point function $G_4^c = G_4 - 2(G_2)^2$ which vanished for the Gaussian ansatz, is nontrivial, and within 10% of its exact value, across all values of g.

B.3. *Formal power series in one variable*

Given a sequence of complex numbers $(a_0, a_1, a_2, \ldots)$, with only a finite number of nonzero entries, we have a polynomial with these numbers as coefficients:[33]

$$a(z) = \sum_{n=0}^{\infty} a_n z^n \,. \tag{B.28}$$

Note that all the information in a polynomial is in its coefficients: The variable z is just a book-keeping device. In fact we could have defined a polynomial as a sequence of complex numbers $(a_0, a_1, \ldots)$ with a finite number of nonzero elements. The addition multiplication and division of polynomials can be expressed directly in terms of these coefficients:

$$[a+b]_n = a_n + b_n \,, \quad [ab]_n = \sum_{k+l=n} a_k b_l \,, \quad [Da]_n = (n+1)a_{n+1} \,. \tag{B.29}$$

A *formal power series* $(a_0, a_1, \ldots)$ is a sequence of complex numbers, with possibly an infinite number of nonzero terms. We define the sum, product and derivative as for polynomials above:

$$[a+b]_n = a_n + b_n \,, \quad [ab]_n = \sum_{k+l=n} a_k b_l \,, \quad [Da]_n = (n+1)a_{n+1} \,. \tag{B.30}$$

The set of formal power series is a ring, indeed even an integral domain. (The proof is the same as above for polynomials.) The opration D is a derivation on this ring. The ring of formal power series is often denoted by $C[[z]]$. The idea is that such a sequence can be thought of as the coefficients of a series $\sum_{n=0}^{\infty} a_n z^n$; the sum and product postulated are what you would get from this interpretation. However, the series may not make converge if z is thought of as a complex number: Hence the name *formal* power series.

The *composition* $a \circ b$ is well-defined whenever $b_0 = 0$:

$$[a \circ b]_n = \sum_{k=0}^{\infty} a_k \sum_{l_1+\cdots+l_k=n} b_{l_1} b_{l_2} \cdots b_{l_k} \,. \tag{B.31}$$

The point is that, for each n there are only a finite number of such l's so that the series on the rhs is really a finite series. In terms of series, this means we substitute one series into the other:

$$a \circ b(z) = a(b(z)) \,. \tag{B.32}$$

B.4. *The group of automorphisms*

The set of formal power series

$$\mathcal{G} = \left\{ \phi(z) = \sum_{0}^{\infty} \phi_n z^n \middle| \phi_0 = 0; \phi_1 \neq 0 \right\} \tag{B.33}$$

is a group under composition: The group of *automorphisms*. The group law is

$$[\tilde{\phi} \circ \phi]_n = \sum_{k=1}^{n} \tilde{\phi}_k \sum_{l_1+l_2+\cdots+l_k=n} \phi_{l_1} \cdots \phi_{l_k} \,. \tag{B.34}$$

The inverse of ϕ (say $\tilde{\phi}$) is determined by the recursion relations of Lagrange:

$$\tilde{\phi}_1 \phi_1 = 1 \,, \quad \tilde{\phi}_n = -\frac{1}{\phi_1^n} \sum_{k=1}^{n-1} \tilde{\phi}_k \sum_{l_1+\cdots+l_k=n} \phi_{l_1} \cdots \phi_{l_k} \,. \tag{B.35}$$

$\mathcal{G}$ is a topological group with respect to the ultrametric topology. It can be thought of as a Lie group, the coefficients ϕ_n being the coordinates. The group multiplication law can now be studied in the case where the left or the right element is infinitesimal, leading to two sets of vector fields on the group manifold. For example, if $\tilde{\phi}(x) = x + \epsilon x^{k+1}$, the change it would induce on the coordinates of ϕ is

$$[\mathcal{L}_k \phi]_n = \sum_{l_1+l_2\cdots l_{k+1}=n} \phi_{l_1} \cdots \phi_{l_{k+1}} \tag{B.36}$$

or equivalently,

$$\mathcal{L}_k \phi(x) = \phi(x)^{k+1} \,, \quad \text{for} \quad k = 0, 1, 2 \ldots \,. \tag{B.37}$$

By choosing $\tilde{\phi}(x) = x + \epsilon x^{k+1}$ in $\phi \circ \tilde{\phi}$ we get the infinitesimal right action:

$$\mathcal{R}_k \phi(x) = x^{k+1} D\phi(x) \,, \quad \text{for} \quad k = 0, 1, 2 \ldots \,. \tag{B.38}$$

Both sets satisfy the commutation relations of the Lie algebra $\underline{\mathcal{G}}$:

$$[\mathcal{L}_m, \mathcal{L}_n] = (n-m)\mathcal{L}_{m+n} \,, \quad [\mathcal{R}_m, \mathcal{R}_n] = (n-m)\mathcal{R}_{m+n} \,. \tag{B.39}$$

This Lie algebra is also called the *Virasoro* algebra by some physicists and the *Witt* algebra by some mathematicians.

There is a representation of this Lie algebra on the space of formal power series:

$$L_n a = x^{n+1} D a \,. \tag{B.40}$$

B.5. *Cohomology of $\mathcal{G}$*

Now let V be the space of formal power series with real coefficients. Then $\mathcal{G}$, the group of automorphisms has a representation on V:

$$\begin{aligned} \mathcal{G} &= \{\phi : Z_+ \to R | \phi_0 = 0, \phi_1 > 0\}\,, \\ V &= \{a : Z_+ \to R\}\,, \quad \phi^* a(x) = a(\phi^{-1}(x))\,. \end{aligned} \tag{B.41}$$

Now, $\log[\phi(x)/x]$ is a power series in x because $\phi(x)/x$ is a formal power series with positive constant term: $[\phi(x)/x]_0 = \phi_1 > 0$. We see easily that $c(\phi, x) = \log[\phi(x)/x)]$ is a one-cocycle of $\mathcal{G}$ twisted by the representation V:

$$\begin{aligned} c(\phi_1\phi_2, x) &= \log\left[\frac{\phi_1(\phi_2(x))}{x}\right] \\ &= \log\left[\frac{\phi_1(\phi_2(x))}{\phi_2(x)}\right] + \log\left[\frac{\phi_2(x)}{x}\right] \\ &= c(\phi_1\phi_2, \phi_2(x)) + c(\phi_2, x)\,. \end{aligned} \tag{B.42}$$

Of course, neither $\log\phi(x)$ nor $\log x$ are power series in x. So this cocycle is nontrivial.

The space of formal power series in two commuting variables ($\mathrm{Sym}^2\ V$) also carries a representation of $\mathcal{G}$. We again have a nontrivial one-cocycle[p] on this representation:

$$c(\phi, x, y) = \log\left[\frac{\phi(x) - \phi(y)}{x - y}\right]\,. \tag{B.44}$$

We recognize this as the entropy of the single matrix model. The same argument shows that this is a nontrivial cocycle of $\mathcal{G}$.

Now we understand that the entropy of the single matrix models has the mathematical meaning of a nontrivial one-cocycle of the group of automorphisms. It explains why we could not express the entropy as a function of the moments. This points also to a solution to the difficulty: We must think in terms of the automorphism ϕ rather than the moments as parametrizing the probability distribution.

[p] The formula

$$\frac{x^m - y^m}{x - y} = \sum_{k=1}^{m-1} x^k y^{m-k-1} \tag{B.43}$$

can be used to show that $c(\phi, x, y)$ is a formal power series in x and y.

Appendix C. Formula for Cocycle

We will now get an explicit formula for $\sigma(\tilde{\phi}, A)$. The Jacobian matrix of $\tilde{\phi}$ is obtained by differentiating the series $\tilde{\phi}(A)_i = A_i + \sum_{n=2}^{\infty} \tilde{\phi}_i^{i_1\cdots i_n} A_{i_1}\cdots A_{i_n}$:

$$J_{ib\ c}^{a\ jd}(A) = \frac{\partial \tilde{\phi}_{ib}^a(A)}{\partial A_{jd}^c} = \delta_i^j \delta_c^a \delta_b^d + \sum_{m+n\geq 1} \tilde{\phi}_i^{i_1\cdots i_m j j_1 \cdots j_n} [A_{i_1}\cdots A_{i_m}]_c^a [A_{j_1}\cdots A_{j_n}]_b^d$$

$$:= \delta_i^j \delta_c^a \delta_b^d + K_{ib\ c}^{a\ jd}(A)\,. \tag{C.1}$$

If we suppress the color indices a, b, c, d,

$$J_i^j(A) = \delta_i^j 1\otimes 1 + \tilde{\phi}_i^{IjJ} A_I \otimes A_J := \delta_i^j 1 \otimes 1 + K_i^j(A)\,. \tag{C.2}$$

We can now compute

$$\begin{aligned}
\frac{1}{N^2}\operatorname{tr} K^n(A) &= \frac{1}{N^2} K_{i_1 b_1\ a_2}^{a_1\ i_2 b_2} K_{i_2 b_2\ a_2}^{a_2\ i_3 b_3} \cdots K_{i_n b_n\ a_1}^{a_n\ i_1 b_1} \\
&= \tilde{\phi}_{i_1}^{K_1 i_2 L_1} \tilde{\phi}_{i_2}^{K_2 i_3 L_2} \cdots \tilde{\phi}_{i_n}^{K_n i_1 L_n} \\
&\quad \times \frac{1}{N}[A_{K_1}]_{a_2}^{a_1} \cdots [A_{K_n}]_{a_1}^{a_n} \frac{1}{N}[A_{L_1}]_{b_1}^{b_2} \cdots [A_{L_n}]_{b_n}^{b_1} \\
&= \tilde{\phi}_{i_1}^{K_1 i_2 L_1} \tilde{\phi}_{i_2}^{K_2 i_3 L_2} \cdots \tilde{\phi}_{i_n}^{K_n i_1 L_n} \Phi_{K_1\cdots K_n} \Phi_{L_n \cdots L_1}\,.
\end{aligned} \tag{C.3}$$

Thus,

$$\begin{aligned}
\sigma(\tilde{\phi}, A) &= \frac{1}{N^2}\log\det[1 + K(A)] \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{1}{N^2}\operatorname{tr} K^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \tilde{\phi}_{i_1}^{K_1 i_2 L_1} \tilde{\phi}_{i_2}^{K_2 i_3 L_2} \cdots \tilde{\phi}_{i_n}^{K_n i_1 L_n} \Phi_{K_1\cdots K_n} \Phi_{L_n\cdots L_1}\,.
\end{aligned}$$

This is the formula we presented earlier.

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