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On the lightest baryon and its excitations in large-N (1 + 1)-dimensional QCD

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Abstract
We study baryons in multicolour QCD_{1+1} via Rajeev's gauge-invariant reformulation as a nonlinear classical theory of a bilocal meson field constrained to lie on a Grassmannian. It is known to reproduce 't Hooft's meson spectrum via small oscillations around the vacuum, while baryons arise as topological solitons. The lightest baryon has zero mass per colour in the chiral limit; we find its form factor. It moves at the speed of light through a family of massless states. To model excitations of this baryon, we linearize equations for motion in the tangent space to the Grassmannian, parameterized by a bilocal field U. A redundancy in U is removed and an approximation is made in lieu of a consistency condition on U. The baryon spectrum is given by an eigenvalue problem for a Hermitian singular integral operator on such tangent vectors. Excited baryons are like bound states of the lightest one with a meson. Using a rank-1 ansatz for U in a variational formulation, we estimate the mass and form factor of the first excitation.

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1. Introduction and summary

An interesting problem of theoretical physics is to find the spectrum and structure of hadrons [1] from QCD. Besides direct numerical approaches, we are far from formulating this problem in (3+1)D, though there has been recent progress in the (2+1)D pure-gauge model [6, 7]. In (1+1)D, 't Hooft obtained [2] an equation for masses and form factors of mesons in the multicolour $N \to \infty$ limit of QCD. There are an infinite number of them with squared masses growing linearly $\mathcal{M}_n^2 \sim \tilde{g}^2 n$. The coupling $\tilde{g}^2 = g_{YM}^2 N$ has dimensions of mass^2, so the model is UV finite. Our aim is to do the same for the spectrum of baryons in QCD_{1+1}. Baryons are more subtle than mesons; it has not been possible to extend 't Hooft's summation of planar diagrams to find the baryon spectrum [3]. A way forward was shown in Rajeev's formulation [4, 5] of $QCD_{1+1}^{N=\infty}$ as a nonlinear classical theory of quark bilinears (*meson fields*) on a curved phase space. As $N \to \infty$, the gauge-invariant bilinears M have small fluctuations and satisfy nonlinear *classical* equations, though $\hbar = 1$. Some nonlinearities are due to a constraint on M encoding Pauli exclusion. 't Hooft's meson equation was rederived by considering oscillations around the vacuum, with masses of $\mathcal{O}(N^0)$. But the model also has large departures from the vacuum, describing baryons with masses of $\mathcal{O}(N)$. They live on a disconnected component of phase space, an infinite Grassmannian with components labelled by the baryon number. This formulation gave a qualitative picture [10, 12] of the baryon (as a soliton of the meson field and as a bound state of quarks) and estimates for the mass and form factor of the lightest baryon [9]. The latter was in reasonable agreement with numerical calculations [8]. They were also used to model the x_B-dependence of the nucleon structure function $F_3(x_B, Q^2)$ measured in deep inelastic scattering [10, 11].

Here we derive an equation for the spectrum of small oscillations around the lightest baryon, to describe excited baryons or baryon–meson bound states. For simplicity we consider one quark flavour, so these correspond to the nucleon resonances $P_{11}, D_{13}, S_{11}, D_{15}$, etc [1]. There may also exist heavier baryonic extrema of energy, analogues of Δ, Λ. Their investigation and oscillations around them are postponed. Oscillations near a baryon are harder to study than near the vacuum (section 3). To begin, we need the precise baryon ground state (g.s.). The form factor of the lightest baryon is well described by a single valence quark wavefunction ψ. In the chiral limit of massless quarks, the g.s. is exactly determined via ψ. We find ψ exactly and establish that the lightest baryon has zero mass/colour (section 4), like the lightest meson [2]. The soliton has a size $\sim P^{-1}$, where P is the mean null-momentum/colour of the baryon. Being massless, the baryon moves at the speed of light traversing a one-parameter family of even parity massless states. The probability of finding a valence quark with positive null-momentum between $[p, p + \mathrm{d}p]$ in a baryon is $P^{-1} \exp(-p/P)\, \mathrm{d}p$. Away from the chiral limit, the g.s. of the baryon is massive, containing sea and antiquarks [11]. Here we work in the simpler chiral limit. It is possible to derive [10] this soliton picture as a Hartree–Fock approximation to N quarks interacting via a linear potential, with a wavefunction antisymmetric in colour but symmetric otherwise. This is a way of seeing that the baryon is a fermion and that N is an integer.

As in 't Hooft's work, excitations around the translation-invariant Dirac vacuum were described by Rajeev [4] using a meson 'wavefunction' $\tilde{\chi}(\xi)$. Around a non-translation-invariant baryon, we need the $N \to \infty$ limit of a bilocal field $M(x, y) \sim q^{a\dagger}(x)q_a(y)/N$.[1] The vacuum is $M = 0$ while the baryon g.s. is $M_o = -2\psi\psi^\dagger$. A complication arises from a quadratic constraint $(\epsilon + M)^2 = 1$; the 'quark density matrix' must be a projection operator, up to normal ordering. We ensure that it is satisfied at all times (section 1.2), and when making approximations (section 5.4). Pleasantly, when linearized around the baryon $M = M_o + V$, the constraint $[\epsilon + M_o, V]_+ = 0$ encodes an 'orthogonality' of ground and excited states crucial for consistency of the linearized equations (section 5.5). This condition generalizes the vanishing dot product of radius $\epsilon + M_o$ and tangent V to a sphere. Roughly, V is a meson and $M_o + V$ is a meson–baryon pair. If $M_o = 0$, we return to mesonic oscillations around the Dirac vacuum. Due to translation invariance around $M_o = 0$, the bilocal field $\tilde{V}(p, q) \sim \tilde{\chi}(\xi)$ could be taken to depend only on $\xi = p/(p - q)$ and not on the 'total momentum' $p - q$. This simplification is absent near the baryon (section 5.6). So in section 5.1, we solve the constraint $[\epsilon + M_o, V]_+ = 0$

[1] We work in a gauge where the parallel transport from x to y is the identity.

via another bilocal field $V = \mathrm{i}[\epsilon + M_o, U]$. But there is a *gauge freedom* under $U \to U + U_g$ where $[\epsilon + M_o, U_g] = 0$. We gauge-fix the redundancy (section 5.2) by writing U in terms of a vector u and another bilocal field U^{+-} one-fourth the size of U. Roughly, u is a correction to the valence quarks ψ, due to the excitation. U^{+-} has the corresponding data on sea/antiquarks in the excited baryon. The gauge-fixing conditions $\psi^\dagger u = 0$ and $\psi^\dagger U^{+-} = 0$ are interpreted as orthogonality of ground and excited states. But the naively linearized equations do not preserve these conditions! The gauge freedom at each time step is used to derive linearized equations respecting the gauge conditions (section 5.7). Though the equations for U^{+-} and u are linear, we were not able to find oscillatory solutions by separation of variables. For, they couple u, U^{+-} *and* their adjoints, like a Schrödinger equation where the Hamiltonian depends on the wavefunction and its conjugate! So in section 5.8 we put $u = 0$, allowing us to separate variables and find oscillatory solutions, at the cost of a consistency condition on U^{+-} (66). Regarding V as a meson, we expect it contains a quark–antiquark sea but no valence quarks u. This motivates the $u = 0$ ansatz.

We are left with an eigenvalue problem $\hat{K}(U) = \omega U$ (68) for the form factor U^{+-}. We show that the linearized Hamiltonian $\hat{K}$ is Hermitian using the gauge condition and the ansatz $u = 0$. In the chiral limit, the mass[2] of excited baryons are $\mathcal{M}^2 = 2\omega P$, where P is the lightest baryon's momentum. But the eigenvalue problem for $\hat{K}$ is quite non-trivial. It is a singular integral operator on a 'physical subspace' of Hermitian operators. This space of physical states U^{+-} consists of Hilbert–Schmidt operators subject to the gauge and consistency conditions (appendix G). The eigenvalue problem for the baryon spectrum follows from a variational energy $\mathcal{E}$. In section 5.9 we suggest a rank-1 variational ansatz $U^{+-} = \phi\eta^\dagger$. Here ϕ, η are the sea/antiquark wavefunctions of the excited baryon. The kinetic terms in $\mathcal{E}$ differ from the naive ones due to linearization around a time-dependent g.s. The potential energy is a sum of Coulomb energy (attraction between anti- and sea-quarks) and exchange energy (between sea-partons and 'background' valence quarks ψ). In section 5.10 we obtain a crude estimate for the mass and form factor of the first excited baryon by minimizing $\mathcal{E}$ in a parameter controlling the decay of the sea quark wavefunction. But our estimate for the mass of the first excited baryon $0.3\,\tilde{g}N$ is not expected to be accurate[2] or an upper bound, as we imposed the gauge-fixing condition but not the consistency condition from the ansatz $u = 0$. In appendix G, we try to solve this consistency condition. A more careful treatment will hopefully give a quantitative understanding of the baryon spectrum.

1.1. Summary of classical hadrondynamics

We begin by recalling Rajeev's reformulation [4] of $\mathrm{QCD}_{1+1}^{N=\infty}$ as a classical theory of meson fields. In the null coordinates $x = x^1$, $t = x^0 - x^1$ we specify initial values on the null line $t = 0$. The energy $E = p_t = p_0$ and null-momentum $p = p_x = p_0 + p_1$ obey[3] $m^2 = 2Ep - p^2$. In the gauge $A_x = A_0 + A_1 = 0$, one component of quarks and the gluon A_1 are eliminated. For quarks of one flavour and N colours a, b, the action of $SU(N)$ QCD_{1+1} represents fermions χ_a interacting via a linear potential

$$S = \int \mathrm{d}t\,\mathrm{d}x\,\chi^{\dagger a}\left[-\mathrm{i}\partial_t - \frac{1}{2}\left(p + \frac{m^2}{p}\right)\right]\chi_a - \frac{g^2}{4N}\int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\chi^{\dagger a}(y)\chi_b(y)|x-y|\chi^{\dagger b}(x)\chi_a(x). \quad (1)$$

[2] From 't Hooft's work [2] the mass of the first excited meson in the chiral limit is about $1.4\tilde{g}$.
[3] Under a Lorentz boost of rapidity θ, $t \to t\mathrm{e}^\theta$ and $x \to \mathrm{e}^{-\theta}x - t\sinh\theta$.

$\hat{M}(x,y) = -\frac{2}{N} : \chi^{\dagger a}(x)\chi_a(y):$, with x, y being null-separated, defines a gauge-invariant bilocal field. Normal ordering is with respect to the Dirac vacuum. E and p have the same sign, so negative momentum states are filled in the vacuum and we split the one-particle Hilbert space $\mathcal{H} = L^2(\mathbf{R}) = \mathcal{H}_- \oplus \mathcal{H}_+$ into $\mp$ momentum states[4]. Canonical anti-commutation relations (CAR) for $\chi, \chi^\dagger$ from (1) imply commutation relations for $\hat{M}$, with fluctuations of order $1/N$. As $N \to \infty$, $\hat{M}$ tends to a classical field M, the integral kernel[5] of a Hermitian operator on $\mathcal{H}$. The Poisson brackets (PB) of M are given by

$$(\mathrm{i}/2)\{M(x,y), M(z,u)\} = \delta(z-y)\Phi(x,u) - \delta(x-u)\Phi(z,y). \tag{2}$$

$\Phi = \epsilon + M$ where ϵ is the Hilbert transform kernel $\tilde{\epsilon}(p,q) = 2\pi\delta(p-q)\mathrm{sgn}\, p$, or $\epsilon(x,y) = \frac{\mathrm{i}}{\pi}\mathcal{P}(x-y)^{-1}$. The CAR imply a constraint as $N \to \infty$, $\Phi^2 = I$; the eigenvalues of Φ are -1 (singly-occupied) or 1 (unoccupied). $\Phi = \epsilon$ is the vacuum. Thus the *phase space* is a Grassmannian [4]:

$$Gr_1 = \{M : M^\dagger = M, (\epsilon + M)^2 = I, \mathrm{tr}|[\epsilon, M]|^2 < \infty\}, \tag{3}$$

the symplectic leaf of $\Phi = \epsilon$ under the coadjoint action of a restricted unitary group [4]. The coadjoint orbit formula for Poisson brackets of linear functions of M, $f_u = -\frac{1}{2}\,\mathrm{tr}\, uM$, is

$$\{f_u, f_v\} = \frac{\mathrm{i}}{2}\mathrm{tr}\,[u,v]\Phi = f_{-\mathrm{i}[u,v]} + \frac{\mathrm{i}}{2}\mathrm{tr}[u,v]\epsilon. \tag{4}$$

The connected components of Gr_1 are labelled by an integer $B = -\frac{1}{2}\,\mathrm{tr}\, M$ (appendix E), quark number per colour, or baryon number. An analogue of parity is $\mathbf{P}\tilde{M}_{pq}(t) = \tilde{M}_{qp}(-t)$ or $\mathbf{P}M_{xy}(t) = M^*_{-x,-y}(-t)$. For example, the static real symmetric $\tilde{M}$ are even and the imaginary antisymmetric $\tilde{M}$ are odd. From (1), the energy/colour is a parity-invariant quadratic function on Gr_1:

$$E(M) = -\frac{1}{2}\int \frac{1}{2}\left(p + \frac{\mu^2}{p}\right)\tilde{M}(p,p)[\mathrm{d}p] + \frac{\tilde{g}^2}{16}\int |M(x,y)|^2|x-y|\,\mathrm{d}x\,\mathrm{d}y. \tag{5}$$

The current quark mass m is renormalized as $\mu^2 = m^2 - \frac{\tilde{g}^2}{\pi}$ while reordering quark bilinears. The kinetic energy $T = -\frac{1}{2}\,\mathrm{tr}\, hM$ is expressed in terms of the dispersion kernel

$$\tilde{h}(p,q) = 2\pi\delta(p-q)h(p), \qquad \text{where} \quad 2h(p) = p + \mu^2 p^{-1}. \tag{6}$$

Define a positive 'interaction operator' on Hermitian matrices $\hat{G} : M \mapsto G(M) \equiv G_M$ with the kernel $\hat{G}(M)_{xy} = \frac{1}{2}M_{xy}|x-y|$ (appendix C). Then the potential energy is

$$U = \frac{\tilde{g}^2}{8}\mathrm{tr}\, M\hat{G}(M) = \frac{\tilde{g}^2}{16}\int \mathrm{d}x\,\mathrm{d}y|M(x,y)|^2|x-y| \geqslant 0. \tag{7}$$

In Fourier space[6] $\tilde{G}(M)_{pq} = -\int \frac{[\mathrm{d}r]}{r^2}\tilde{M}_{p+r,q+r}$. We also associate with M a constant of motion (appendix A), its mean momentum per colour P_M. Under a boost, $P \to \mathrm{e}^\theta P$, $E \to \mathrm{e}^{-\theta}E + p\sinh\theta$:

$$P_M = -\frac{1}{2}\mathrm{tr}\,\mathrm{p}M = -\frac{1}{2}\int p\tilde{M}(p,p)[\mathrm{d}p] \qquad \text{where} \quad \mathrm{p}(p,q) = 2\pi\delta(p-q)p. \tag{8}$$

The squared-mass/colour $\mathcal{M}^2 = 2EP - P^2$ is a Lorentz-invariant constant of motion. Hamilton's equations of motion (eom) are the initial value problem (IVP)

$$\frac{\mathrm{i}}{2}\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{i}}{2}\{E(M), M\} = [E'(M), \epsilon + M]. \tag{9}$$

[4] Our convention for Fourier transforms is $\psi(x) = \int[\mathrm{d}p]\mathrm{e}^{\mathrm{i}px}\tilde{\psi}(p)$, where $2\pi[\mathrm{d}p] = \mathrm{d}p$.

[5] In Fourier space, $\tilde{M}(p,q) = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-\mathrm{i}(px-qy)}M_{xy}$. We write $\tilde{M}_{pq}$ for $\tilde{M}(p,q)$ and M_{xy} for $M(x,y)$.

[6] This uses $v(x) = \frac{1}{2}|x| = -\int \frac{[\mathrm{d}r]}{r^2}\mathrm{e}^{-\mathrm{i}rx}$ obtained by solving $v''(x) = \delta(x)$ with $v(0) = v'(0) = 0$. We used the definition of finite part integrals (appendix B) to put $\int_{-\infty}^{\infty}\frac{[\mathrm{d}r]}{r} = 0$ and $\int_{-\infty}^{\infty}\frac{[\mathrm{d}r]}{r^2} = 0$.

The PB is expressed via the commutator using the variational derivative of energy, which is inhomogeneous linear in M, $E' = T' + U' = -h/2 + (\tilde{g}^2/4)\hat{G}(M)$. Its matrix elements are

$$E'(M)_{pq} = -\pi\delta(p-q)h(p) + \frac{\tilde{g}^2}{4}\tilde{G}(M)_{pq}, \qquad \text{where} \qquad U'(M)_{xy} \equiv \frac{\delta U(M)}{\delta M_{yx}} = \frac{\tilde{g}^2}{4}\frac{|x-y|}{2}M_{xy}. \tag{10}$$

1.2. *Preservation of quadratic constraint under time evolution*

We check that (9) preserves the constraint $\Phi^2 = I$. Define the constraint matrix $C(t) = \Phi^2 - I$ and let $C(0) = 0$. We have an autonomous system of first-order nonlinear ODEs:

$$\partial_t C = \partial_t(\epsilon + M)^2 = [\epsilon + M, \partial_t M]_+ = -2\mathrm{i}[\Phi, [E', \Phi]]_+ = -2\mathrm{i}[E'(M(t)), \Phi^2(t)]. \tag{11}$$

Under suitable hypotheses, it should have a unique solution[7] given $C(0)$. Now consider the guess $C_g(t) \equiv 0$. It obeys (11) as both sides vanish: $\partial_t C_g(t) = 0$ and $-2\mathrm{i}[E', \Phi^2(t)] = -2\mathrm{i}[E', I] = 0$. Thus, $C_g(t) \equiv 0$ is the solution: constraint is always satisfied.

2. Ground state in the $B = 0$ meson sector

In the non-interacting case $\tilde{g} = 0$, $M = 0$ is a static solution since the eom are

$$\frac{\mathrm{i}}{2}\dot{M}_{pq} = \frac{1}{4}M_{pq}\left[q - p + m^2\left(\frac{1}{q} - \frac{1}{p}\right)\right] \qquad \text{when} \quad \tilde{g} \to 0. \tag{12}$$

Here, rhs $\equiv 0$ iff $M = 0$, so it is the *only* static solution if $\tilde{g} = 0$. Even with interactions, $M = 0$ is static: $\partial_t M = \{E(M), M\} = 0$ at $M = 0$ (5). But even at $M = 0$, $E'(0) = -\pi\delta(p-q)h(p)$ does not vanish! Does the gradient of energy vanish at $M = 0$? Yes. To see why, first note that $M = 0$ is a static solution as $E'(0)$ and ϵ are diagonal in momentum space. By (9)

$$\partial_t M = -2\mathrm{i}[E'(M), \epsilon + M]|_{M=0} = -2\mathrm{i}[E'(0), \epsilon] = 0. \tag{13}$$

$E'(M) = 0$ is sufficient, but not necessary for a static solution. $-2\mathrm{i}[E'(M), \Phi]$ is the symplectic gradient of energy at M. The contraction of the exterior derivative of energy with the Poisson bivector field produces the Hamiltonian vector field. So the (symplectic) gradient of energy does vanish at $M = 0$. The state $M = 0$ has zero mass $\mathcal{M}$ and qualifies as a g.s.

3. Small oscillations about vacuum and 't Hooft's meson equation

We recall the equation for mesons [2, 4] by considering small oscillations about the vacuum. Let V be a tangent vector at the translation-invariant $M = 0$. The constraint $\Phi^2 = I$ becomes[8] $[\epsilon, V]_+ = 0$ or

$$\tilde{V}_{pq}(\operatorname{sgn} p + \operatorname{sgn} q) = 0 \quad \Rightarrow \quad \tilde{V} = (0, \tilde{V}^{-+}|\tilde{V}^{+-}, 0). \tag{14}$$

$\tilde{V}_{pp} = 0$, so V has zero mean momentum P_V (8). But the generator $P_t = p - q$ of translations $M_{xy} \to M_{x+a,y+a}$, $\tilde{M}_{pq} \to \mathrm{e}^{\mathrm{i}(p-q)a}\tilde{M}_{pq}$ may be regarded as the total momentum. So we

[7] The rhs is a cubic function of Φ. Picard iteration should establish that the solution to (11) exists and is unique. We may need technical hypotheses (besides $\mathrm{tr}|[\epsilon, M(0)]|^2 < \infty$ appendix E) on $\Phi(0)$ to ensure that observables (e.g. energy) remain finite.

[8] $V^{+-}_{pq} : \mathcal{H}_- \to \mathcal{H}_+$ has entries with $p > 0 > q$, $(V^{+-})^\dagger = V^{-+}$. We separate matrix rows with $|$.

pick independent variables P_t and $\xi = p/P_t$. We write $\tilde{V}^{+-} = \tilde{\chi}(P_t, \xi, t)$. Hermiticity implies[9]

$$\tilde{V}^{-+}(p,q,t) = \tilde{\chi}(P_t,\xi,t) \qquad \text{with} \quad \tilde{\chi}^*(P_t,\xi,t) = \tilde{\chi}(-P_t, 1-\xi, t). \tag{15}$$

ξ is the quark momentum fraction. For small oscillations about $M = 0$ of energy $\omega = p_0$, we put

$$\tilde{V}^{+-}_{pq}(t) = \tilde{\chi}(P_t,\xi)\,\mathrm{e}^{\mathrm{i}\omega t} \qquad \text{and} \quad \tilde{V}^{-+}_{pq}(t) = \tilde{\chi}(P_t,\xi)\mathrm{e}^{-\mathrm{i}\omega t} \qquad \text{for} \quad \omega \in \mathbf{R}. \tag{16}$$

Parity acts as $\mathbf{P}\tilde{\chi} = \tilde{\chi}^*$. The simplest $\tilde{\chi}$ obeying (15) are independent of P_t with $\tilde{\chi}^*(\xi) = \tilde{\chi}(1-\xi)$. So even parity states are real with $\tilde{\chi}(\xi) = \tilde{\chi}(1-\xi)$ and odd parity ones imaginary with $\tilde{\chi}(\xi) = -\tilde{\chi}(1-\xi)$. The norm (appendix E) on V implies the L^2 norm on $\tilde{\chi}(\xi)$ up to a divergent constant. The linearized eom are

$$\begin{aligned}
&\frac{\mathrm{i}}{2}\dot{V} = [E'(V), \Phi] = \left[T' + \frac{1}{4}\tilde{g}^2 G(V), \Phi\right] = [T', V] + \frac{\tilde{g}^2}{4}[G(V), \epsilon] + \mathcal{O}(V^2), \\
&\frac{\mathrm{i}}{2}\partial_t \tilde{V}_{pq} = -\frac{1}{2}\{h(p) - h(q)\}\,\tilde{V}_{pq} - \frac{\tilde{g}^2}{4}(\operatorname{sgn} q - \operatorname{sgn} p)\int \frac{[\mathrm{d}s]}{s^2}\tilde{V}_{p+s,q+s}.
\end{aligned} \tag{17}$$

Put $\eta' = s/P_t$ to get an eigenvalue problem for ω. It is rewritten as 't Hooft's equation for the squared masses $\mathcal{M}^2 = 2\omega P_t - P_t^2$ with quarks of equal mass [2] $\left(\mu^2 = m^2 - \frac{\tilde{g}^2}{\pi}, \eta = \xi + \eta'\right)$. For instance, with $\mu^2 = 0$, the eigenstates alternate in parity $\tilde{\chi}_n(\xi) \approx \mathrm{i}^{n-1}\sin(n\pi\xi)$ with squared-masses $\mathcal{M}_n^2 \approx n\pi\tilde{g}^2$:

$$\begin{aligned}
&-\frac{\omega}{2}\tilde{\chi}(\xi) = -\frac{1}{4}\left[P_t + \frac{\mu^2}{\xi P_t} + \frac{\mu^2}{P_t - \xi P_t}\right]\tilde{\chi}(\xi) + \frac{\tilde{g}^2}{2}\int \frac{\tilde{\chi}(\xi+\eta')}{\eta'^2 P_t}[\mathrm{d}\eta'], \\
&\mathcal{M}^2\tilde{\chi}(\xi) = \left(\frac{\mu^2}{\xi} + \frac{\mu^2}{1-\xi}\right)\tilde{\chi}(\xi) - \frac{\tilde{g}^2}{\pi}\int_0^1 \frac{\tilde{\chi}(\eta)}{(\xi-\eta)^2}\,\mathrm{d}\eta.
\end{aligned} \tag{18}$$

4. Ground state of baryon

The trajectories $M_o(t)$ of the least mass on the $B = 1$ component are the baryonic g.s; they depend on $m, \tilde{g}$. The chiral limit is $m \to 0$ holding $\tilde{g}$ fixed, $\nu = m^2/\tilde{g}^2 \to 0$. Regarding QCD_{1+1} as an approximation to QCD_{3+1} on integrating out directions $\perp$ to hadron propagation, $\tilde{g}^{-1} \sim \mathcal{O}$ (transverse hadron size). So the chiral limit should describe u/d quarks that are much lighter than the size of hadrons. But it is hard to find the g.s. from the nonlinear eom (9). Inspired by valence partons, we found that the g.s. is approximately of rank 1 [9, 10, 12]. $M = -2\psi\psi^\dagger$ lies on the $B = 1$ component if $\tilde{\psi}$ is a positive momentum ($\epsilon\psi = \psi$) unit vector. We guessed that a minimum mass +parity state is[10]

$$\begin{aligned}
&\tilde{M}_{0\,pq} = -\frac{4\pi}{P}\mathrm{e}^{-\frac{p+q}{2P}}\theta(p)\theta(q), \qquad \tilde{\psi}_0(p) = \sqrt{\frac{2\pi}{P}}\,\mathrm{e}^{\frac{-p}{2P}}\theta(p), \\
&\psi_0(x) = \frac{1}{\sqrt{2\pi P}}\left[\frac{1}{(2P)^{-1} - \mathrm{i}x}\right].
\end{aligned} \tag{19}$$

In section 4.1 we show that (19) has zero mass as $\nu \to 0$. In section 4.2 we show that it is one of a family of degenerate massless states connected by time evolution. M_t is thus a baryon g.s.:

$$\begin{aligned}
&\tilde{M}_{t\,pq} = \tilde{M}_{0\,pq}\,\mathrm{e}^{\mathrm{i}(p-q)t/2}, \qquad \tilde{\psi}_t(p) = \mathrm{e}^{\mathrm{i}pt/2}\tilde{\psi}_0(p), \\
&\psi_t(x) = \frac{1}{\sqrt{2\pi P}}\left[\frac{1}{2P} - \mathrm{i}\left(x + \frac{t}{2}\right)\right]^{-1}.
\end{aligned} \tag{20}$$

9 $P_t \geqslant 0$ in the $+-$ block while $P_t \leqslant 0$ in the $-+$ block, but $\xi \in [0, 1]$ always.

10 $P = -\operatorname{tr} \mathrm{p}M/2$ (8) is the baryon momentum/colour; it fixes the frame. A rescaling of p and P is a boost.

$p-q$ is *not* a constant, unlike near the translation-invariant $M = 0$ (section 3). Since $M_{xx} \sim [(x+t/2)^2 + (2P)^{-2}]^{-1}$, the baryon is localized at $x = -t/2$ at time t and has a size $\sim 1/P$. As $x = x^1, t = x^0 - x^1$, the massless baryon travels at the speed of light[11] along $x^1 = -x^0$. The probability of finding a valence quark of momentum p in the baryon is $-\frac{1}{2}\tilde{M}(p,p)$[12]. So the degeneracy and time dependence are consequences of relativity: a massless soliton cannot be at rest. Time-dependent vacua are unusual[13]. Continuously connected static vacua (states of neutral equilibrium) are more common, e.g. the g.s. of a ball on a horizontal plane. There are time-dependent states of arbitrarily small energy greater than zero, where the ball adiabatically rolls between vacua. What is remarkable about M_t is that there is no 'additional kinetic energy of rolling between vacua', due to the masslessness of the quarks. But this massless baryon is special to the chiral limit. Away from $m = 0$, the g.s. of the baryon is roughly $M = -2\psi\psi^\dagger$, with $\tilde{\psi}(p) \propto p^a\, \mathrm{e}^{-p/2P}\theta(p)$, $a \approx \sqrt{3\nu/\pi}$ and $\mathcal{M}^2 \approx \tilde{g}^2\sqrt{\pi\nu/3}$ for small ν [9].

4.1. Mass of the separable exponential ansatz

To find the mass of (19), we split energy (5) as $2E = P + m^2\mathrm{KE} + \tilde{g}^2(\mathrm{SE}+\mathrm{PE})$, where $\tilde{g}^2\mathrm{SE}/2$ is a self-energy. In terms of $\nu = m^2/\tilde{g}^2$, the mass^2 $2EP - P^2$ is given by

$$\mathcal{M}^2 = \tilde{g}^2 P(\nu\mathrm{KE} + \mathrm{SE} + \mathrm{PE}) \xrightarrow{m\to 0} \tilde{g}^2 P(\mathrm{SE}+\mathrm{PE}), \qquad \text{where}$$

$$\mathrm{PE} = \frac{1}{4}\int \mathrm{d}x\,\mathrm{d}y |M_{xy}|^2 \frac{|x-y|}{2}, \qquad \mathrm{SE} = \frac{1}{2\pi}\int \tilde{M}_{pp}\frac{[\mathrm{d}p]}{p}, \qquad \mathrm{KE} = -\frac{1}{2}\int \tilde{M}_{pp}\frac{[\mathrm{d}p]}{p}. \tag{21}$$

For $M = -2\psi\psi^\dagger$, $\mathrm{PE} = \int \mathrm{d}x\, |\psi|^2 V(x)$, where $V = \frac{1}{2}\int \mathrm{d}y|\psi(y)|^2|x-y|$ obeys $V'' = |\psi|^2$,

$$V(0) = \frac{1}{2}\int \mathrm{d}y|\psi(y)|^2|y| \qquad \text{and} \qquad V'(0) = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}y|\psi(y)|^2 \mathrm{sgn}\, y. \tag{22}$$

$$\text{Thus, } \mathrm{PE} = \int [\mathrm{d}p]\,\tilde{\psi}(p)\int [\mathrm{d}r]\tilde{\psi}^*(p+r)\tilde{V}(r),$$

$$\text{where} \qquad \tilde{V} = \frac{-1}{r^2}\int [\mathrm{d}q]\tilde{\psi}(q)\tilde{\psi}^*(q-r). \tag{23}$$

Here, $|\psi_o(y)|^2 = \frac{1}{2\pi P}[(2P)^{-2} + y^2]^{-1}$ is even, so $V'(0) = 0$ and $\tilde{V}(r)$ is real and even. But $V(0)$, SE and PE are log-divergent. Yet, we will show that SE + PE= 0, regarded as a limit of regulated integrals[14]

[11] So though the null line $t = 0$ is not a Cauchy surface, the baryon trajectory intersects it.

[12] The off-forward pdfs of deeply virtual Compton scattering [13] depend on off-diagonal entries of M.

[13] They are forbidden in elementary QM: energy eigenstates must have simple-harmonic time dependence. But if the g.s. of a QFT describes a massless particle whose number is conserved, it cannot be static. Classical evolution allows more possibilities. A near example is of a pair of like charges. The *unattainable* g.s. is for them to be at rest infinitely apart. A state of finite separation cannot be static: repelling charges accelerate.

[14] To bypass the regularization, we can set up rules for manipulating these integrals based on the answers we get via the regularized calculations. From (23) the potential energy is

$$(2\pi P)\mathrm{PE} = -\int_0^\infty \mathrm{d}q\,\mathrm{e}^{-q}\int_{-q}^{\infty}\frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-\frac{s+|s|}{2}} = -\int_0^\infty \mathrm{d}q\,\mathrm{e}^{-q}\left[\int_{-q}^{0}\frac{\mathrm{d}s}{s^2} + \int_0^\infty \frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-s}\right]$$

$$= \int_0^\infty \frac{\mathrm{d}s}{s}\,\mathrm{e}^{-s} - \int_0^\infty \frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-s}. \tag{24}$$

These terms are equal by integration by parts if we ignore the boundary term. So for PE + SE = 0, we must define

$$(\pi P)\mathrm{SE} = -\int_0^\infty \mathrm{d}s s^{-1}\mathrm{e}^{-s} \equiv -\int_0^\infty \mathrm{d}s s^{-2}\,\mathrm{e}^{-s} \quad \text{or} \quad s^{-1}\,\mathrm{e}^{-s}|_{s=0} \equiv 0. \tag{25}$$

$$\tilde{V}(r) = -\frac{1}{r^2 P}\, e^{r/2P} \int_{\max(0,r)}^{\infty} e^{-q/P} dq = -\frac{1}{r^2} \exp\left(-\frac{|r|}{2P}\right);$$
$$\text{SE} = \frac{1}{2\pi} \int \tilde{M}_{pp} \frac{[dp]}{p} = \frac{-1}{\pi P} \int_0^{\infty} \frac{e^{-q}}{q}\, dq \qquad \text{and}$$
$$\text{PE} = \frac{1}{4\pi^2 P} \int dx\, dy \frac{|x-y|}{(1+x^2)(1+y^2)}. \tag{26}$$

4.1.1. Regularized/variational estimation of the baryon ground state. Let us use an IR regulator to ensure that PE and SE are finite. Let $\tilde{\psi}(p) \sim p^a\, e^{-p} \theta(p)$ so that $\tilde{\psi}$ is continuous at $p = 0$ if $a > 0$. For $a = 0$, this reduces to our ansatz ψ_o in the frame with $2P = 1$. We regard this as an ansatz for minimizing $\mathcal{M}^2$ (21). We show $\mathcal{M}^2$ vanishes as $a \to 0$ if $\nu = 0$. Let

$$\tilde{\psi}_a(p) = \frac{2^{1+a}\sqrt{\pi}}{\sqrt{\Gamma(1+2a)}} p^a\, e^{-p}\, \theta(p), \qquad \psi_a(x) = \frac{\sqrt{\Gamma(1+2a)}}{2^a \Gamma\left(\frac{1}{2}+a\right)} \frac{1}{(1-ix)^{1+a}} \qquad \text{for which}$$
$$P(a) = \int p|\tilde{\psi}_p|^2 [dp] = \frac{1}{2} + a, \qquad \text{KE} = \int |\tilde{\psi}_p|^2 \frac{[dp]}{2p} = \frac{1}{a},$$
$$\text{SE} = -\int |\tilde{\psi}_p|^2 \frac{[dp]}{\pi p} = \frac{-1}{\pi a}. \tag{27}$$

Integrating and imposing the initial condition $V_a(0) = \Gamma(a) \,/\, [2\sqrt{\pi}\Gamma(a+1/2)]$:

$$V_a'(x) = \frac{x\Gamma(a+1)\,{}_2F_1\left(\frac{1}{2}, a+1; \frac{3}{2}; -x^2\right)}{\sqrt{\pi}\Gamma\left(a+\frac{1}{2}\right)},$$
$$V_a(x) = \frac{\Gamma(a)\left(2ax^2\,{}_2F_1\left(\frac{1}{2}, a+1; \frac{3}{2}; -x^2\right) + (x^2+1)^{-a}\right)}{2\sqrt{\pi}\Gamma\left(a+\frac{1}{2}\right)}. \tag{28}$$

Note that ${}_2F_1\left(\frac{1}{2}, a+1; \frac{3}{2}; -x^2\right) \propto x^{-1}$ for large x and $a > 0$, so $V_a(x) \propto |x|$ as $|x| \to \infty$. However, we could not do the final integral to get $\text{PE} = \int dx\, V_a\, |\psi_a|^2$. It converges for $a > 0$ as $V_a\, |\psi_a|^2 \sim |x|^{-1-2a}$ as $|x| \to \infty$. On integrating for some simple values of a we find that SE + PE $\to 0$ as $a \to 0$. We fit a series[15] to the calculated PE (table 1) for several $a \in [10^{-2}, 10^{-4}]$. It is plausible that the coefficient of $1/a$ is *exactly* $1/\pi \simeq 0.3183$ and cancels SE$= -1/\pi a$ and moreover that PE + SE vanishes at $a = 0$. Encouraged by this, we calculated PE(a) using Mathematica for several round values of a^{-1}. There was a pattern and we conjectured (31), which was confirmed for hundreds of a's. We are confident that PE + SE vanishes as $a \to 0$.

[15] It is tempting to Laurent expand the integrand in a and integrate term by term. But this does not work as the operations of integration and Laurent expansion do not commute:

$$V_a(x) = (2\pi a)^{-1} + (2\pi)^{-1}(2x \arctan x - \log\{(1+x^2)/4\}) + \cdots = V_{-1}a^{-1} + V_0 + V_1 a + \cdots$$
$$|\psi_a(x)|^2 = (\pi(1+x^2))^{-1}[1 - a\log\{(1+x^2)/4\} + \cdots] = |\psi|_0^2 + |\psi|_1^2 a + \cdots. \tag{29}$$

Integrating term by term, the first converges $\frac{1}{a}\int V_{-1}|\psi|_0^2 = \frac{1}{2\pi a}$, but to half the numerical value:

$$\text{PE} = \int dx\, V(x)|\psi(x)|^2 \overset{?}{=} \frac{1}{a}\int V_{-1}|\psi|_0^2 + \int (V_{-1}|\psi|_1^2 + V_0|\psi|_0^2) + a\int (V_{-1}|\psi|_2^2 + V_0|\psi|_1^2 + V_1|\psi|_0^2) + \cdots. \tag{30}$$

The a^0 term diverges $V_0|\psi|_0^2 \sim |x|^{-1}$, $\int V_0|\psi|_1^2$ also diverges: expanding in a destroys convergence of the integral!

Table 1. Though PE $\approx 6.9\times10^{-7}+0.3183/a+1.046a-4.3a^2$ grows as $a\to0$, SE + PE $\propto \mathcal{M}^2$ decreases.

a	0.1	0.01	0.005	0.003 33	0.001 67	0.001 25	0.001	0.0001
PE	3.255	31.84	63.67	95.5	190.99	254.65	318.3	3183
SE + PE	0.072 09	0.010 03	0.005 123	0.003 44	0.001 733	0.001 302	0.001 043	0.000 105

So $M_o=-2\psi_0\psi_0^\dagger$ (19) is a massless baryon in the chiral limit: $\lim_{m\to0}\mathcal{M}^2/\tilde{g}^2=0$. From (21), its energy/colour is $E_o=P/2$, where P is its momentum/colour.

$$\mathrm{PE}(a)\overset{?}{=}\frac{\Gamma(a)\Gamma\left(\frac12+2a\right)}{4^a\Gamma\left(\frac12+a\right)^3}=\frac{1}{\pi a}+\frac{\pi}{3}a-\frac{12\zeta(3)}{\pi}a^2+\mathcal{O}(a^3). \tag{31}$$

4.2. Degeneracy and time dependence of massless baryon states in the chiral limit

We generalize the massless baryon M_o (19) to a family M_t (20). M_t clearly lie on the $B=1$ component. Further, $P(t)=-\frac12\operatorname{tr}\mathsf{p}M_0(t)=P(0)=P$, $\mathrm{KE}(t)=\mathrm{KE}(0)$, $\mathrm{SE}(t)=\mathrm{SE}(0)$ and by going to position space, $\mathrm{PE}(t)=\mathrm{PE}(0)$. So M_t is massless (21) like M_0 (19). We found M_t by time-evolving M_0 in the chiral limit, so t is time. M_0 evolves according to $\dot{M}=\{E(M),M\}$ (9):

$$\frac{\mathrm{i}}{2}\dot{M}_{pq}=\frac12\tilde{M}_{pq}[h(q)-h(p)]-\frac{\tilde{g}^2}{4}G(M)_{pq}[\operatorname{sgn}p-\operatorname{sgn}q]-\frac{\tilde{g}^2}{4}[M,G_M]_{pq}. \tag{32}$$

We must show that M_t obeys the eom $\frac{\mathrm{i}}{2}\dot{M}_{pq}=\frac14(q-p)M_{pq}+\frac{\tilde{g}^2}{4}Z(M)_{pq}$, where

$$Z(M)_{pq}=\frac{1}{\pi}\left(\frac1p-\frac1q\right)\tilde{M}_{pq}-G(M)_{pq}\{\operatorname{sgn}p-\operatorname{sgn}q\}-[M,G_M]_{pq}. \tag{33}$$

In appendix D we show that $Z(M(t))\equiv0$ for all t, so the interactions cancel out! Now

$$M(t)_{pq}=M_{pq}(0)\,\mathrm{e}^{\frac{\mathrm{i}}{2}(p-q)t}\quad\Rightarrow\quad\frac{\mathrm{i}}{2}\dot{M}_{pq}(t)=\frac14(q-p)M_{pq}(t). \tag{34}$$

So M_0 evolves to M_t with energy $P/2$, describing a baryon moving at the speed of light.

5. Small oscillations about the lightest baryon

5.1. Linearization and solution of constraint on perturbation V

Suppose $M_o(t)$ is the g.s. for $B=1$ with momentum $P_o=-\frac12\operatorname{tr}\mathsf{p}M_o$. Write $M=M_o+V$, where V is a small perturbation tangent to Gr_1 at $M_o(t)$. Then $V^\dagger=V$ and $\operatorname{tr}V=0$. V is a meson and M_o+V a baryon–meson pair. What are the masses and form factors of excited baryons? The constraint $\Phi^2=1$ linearizes to $[\epsilon+M_o,V]_+=0$. This generalizes $v\cdot\phi+\phi\cdot v=0$ for tangent vectors to S^2. Now

$$\epsilon+M_o=\begin{pmatrix}-1&0\\0&1+M_o^{++}\end{pmatrix}\quad\Rightarrow\quad[\epsilon+M_o,V]_+=\begin{pmatrix}-2V^{--}&V^{-+}M_o^{++}\\M_o^{++}V^{+-}&2V^{++}+[M_o^{++},V^{++}]_+\end{pmatrix}=0. \tag{35}$$

In particular, $V^{--}=0$. Roughly, $V^{-+}M_o^{++}=0$ expresses orthogonality of the ground and excited states. Equation (35) is solved[16] by introducing a Hermitian matrix U and a 'potential'

[16] We have not shown that this is the *most general* solution of (35). By analogy with the sphere, we suspect that the anti-commutant of Φ is the image of the adjoint action $\mathrm{i}\,\mathrm{ad}_\Phi$ on Hermitian matrices.

for V. $V = \mathrm{i}[\Phi_o, U]$ is automatically traceless, Hermitian and anti-commutes with Φ_o. This generalizes $v = \phi \times u$ for a tangent vector to $\phi \cdot \phi = 1$. Motivated by (20), let $M_o(t) = -2\psi\psi^\dagger$ be a separable baryon state; then

$$V = \mathrm{i}\begin{pmatrix} 0 & -U^{-+}\left(2 + M_o^{++}\right) \\ \left(2 + M_o^{++}\right)U^{+-} & \left[M_o^{++}, U^{++}\right] \end{pmatrix} = 2\mathrm{i}\begin{pmatrix} 0 & -U^{-+}(1 - \psi\psi^\dagger) \\ (1 - \psi\psi^\dagger)U^{+-} & [U^{++}, \psi\psi^\dagger] \end{pmatrix}. \tag{36}$$

Here, $1 = I^{++}$ is the identity on $\mathcal{H}_+$. We let $U^{--} = 0$ since it does not contribute. U^{++} and U^{+-} are the unknowns. Recall that for mesonic oscillations around $M = 0$, the constraint implied $V^{++} = 0 = U^{++}$.

5.2. *Gauge-fixing freedom in the choice of U for fixed $V = \mathrm{i}[\Phi_o, U]$*

Our solution $V = \mathrm{i}[\Phi_o, U]$ to constraint (35) is unchanged under $U \mapsto U + U_g$, if $[U_g, \Phi_o] = 0$. This generalizes the fact that if $\phi \times u = v$ is tangent to $S^2_{\phi\cdot\phi=1}$ at ϕ, then so is $\phi \times (u + u_g)$ for any u_g parallel to ϕ. We eliminate this redundancy by imposing a gauge condition picking out one member from each equivalence class $U \sim U + U_g$. A convenient condition can be used to kill some entries of U. To understand the extent of the gauge freedom, we first find the commutant $\{\Phi_o\}'$, i.e. the pure-gauge matrices $[\Phi_o, U_g] = 0$. For $M_o = -2\psi\psi^\dagger$ with $\epsilon\psi = \psi$ and $\psi^\dagger\psi = 1$, this becomes

$$\text{(i)} \quad \left[P_\psi, U_g^{++}\right] = 0 \qquad \text{and} \qquad \text{(ii)} \quad P_\psi U_g^{+-} = U_g^{+-}. \tag{37}$$

$P_\psi = \psi\psi^\dagger$ projects to span(ψ) in $\mathcal{H}_+$. (i) states that $U_g^{++} \in \{P_\psi\}'$, which we characterize by extending $\psi_0 \equiv \psi$ to an orthonormal basis for $\mathcal{H}_+$: $\{\psi_k\}_0^\infty$. The commutant of P_ψ consists of the Hermitian matrices

$$U_g^{++} = a_{00}\psi_0\psi_0^\dagger + \sum_{k,l\geqslant 1} a_{kl}\psi_k\psi_l^\dagger = (a_{00}, 0|0, A) \qquad \text{with} \quad a_{00} \in \mathbf{R}. \tag{38}$$

Here $A : \mathrm{span}_\psi^\perp \to \mathrm{span}_\psi^\perp$. To find U_g^{+-}, let $\{\eta_k\}_0^\infty$ be an orthonormal basis for $\mathcal{H}_-$ and write (37) (ii) as

$$U_g^{+-} = \sum_{k,l\geqslant 0} u_{kl}\psi_k\eta_l^\dagger = P_\psi U_g^{+-} = \sum_{l\geqslant 0} u_{0l}\psi_0\eta_l^\dagger. \tag{39}$$

The solution is $u_{kl} = 0$ for $k \neq 0$ and u_{0l} is arbitrary. Equations (38) and (40) characterize the pure-gauge U_g:

$$U_g^{+-} = \sum_{l\geqslant 0} u_{0l}\psi_0\eta_l^\dagger = \begin{pmatrix} u_{00} & u_{01} & \cdots & u_{0l} & \cdots \\ & & \mathbf{0} & & \end{pmatrix} \qquad \text{with} \quad u_{0l} \text{ being arbitrary.} \tag{40}$$

`Gauge-fixing conditions:` the gauge freedom (40) is used to kill the first row of U^{+-}. This is equivalent to imposing $P_\psi U^{+-} = 0$ or $\psi^\dagger U^{+-} = 0$. Similarly, the pure-gauge U_g^{++}'s (38) can be used to kill the 00 entry and all but the first row and column of U^{++}. So most of U^{++} is pure gauge. Thus in the *mostly zero gauge*, U may be taken in the form ($\vec{0}$ and $\vec{u}$ represent column vectors)

$$U^{--} = 0, \qquad U^{-+} = (\vec{0}\,\mathbf{W}), \qquad U^{++} = (0, \vec{u}^\dagger|\vec{u}, \mathbf{0}) = u\psi^\dagger + \psi u^\dagger, \qquad \text{where}$$
$$\mathbf{W} : \mathrm{span}_\psi^\perp \to \mathcal{H}_-; \qquad \vec{u} = (u_1 u_2 \cdots)^t, \qquad \psi_0 \perp u = \sum_{k\geqslant 1} u_k\psi_k \in \mathcal{H}_+. \tag{41}$$

For mesonic oscillations $V^{++} = U^{++} = 0$ (14) but around a baryon, U^{++} can be taken of rank 2. The physical degrees of freedom are encoded in a vector $u \in \mathcal{H}_+$ and a matrix $(U^{-+})^\dagger = U^{+-}$ in the

$$\text{mostly zero gauge:} \qquad \psi^\dagger u = 0, \qquad \psi^\dagger U^{+-} = 0 \qquad \text{and} \quad U^{--} = 0. \tag{42}$$

So ψ is $\perp$ to the excitation U. For example[17] the rank-1 ansatz $U^{-+} = \eta\phi^\dagger$ with $\phi, \eta \in \mathcal{H}_\pm$ and $\phi^\dagger\psi = 0$. The g.s. time dependence is simple, $\tilde\psi_t(p) = \tilde\psi_0(p)\,e^{ipt/2}$ (20). So if at $t = 0$, $\phi_0^\dagger\psi_0 = 0$, then orthogonality is maintained if $\tilde\phi_t(p) = \tilde\phi_0(p)\,e^{-ipt/2}$. To *summarize*, if U is picked in gauge (42), then by (36)

$$V = \begin{pmatrix} 0 & V^{-+} \\ V^{+-} & V^{++} \end{pmatrix} = i\begin{pmatrix} 0 & -U^{-+}(2+M_o) \\ (2+M_o)U^{+-} & [M_o, U^{++}] \end{pmatrix} = 2i\begin{pmatrix} 0 & -U^{-+} \\ U^{+-} & u\psi^\dagger - \psi u^\dagger \end{pmatrix}. \tag{43}$$

Conversely, $U(V)$ is defined up to addition of a pure-gauge U_g. Given V, we can find a convenient representative in the equivalence class of U's that it corresponds to. In the mostly zero gauge, upon using $u^\dagger\psi = 0$, we get $u = \frac{1}{2i}V^{++}\psi$.[18] Thus, $U^{++} = u\psi^\dagger + \psi u^\dagger = -\frac{i}{4}[V^{++}, 2\psi\psi^\dagger]$. In this gauge, $U^{+-} \propto V^{+-}$. Given V, the most general corresponding U is the sum of any $U_g \in \{\Phi_o\}'$ ((38) and (40)) and

$$U_{\text{mostly zero gauge}} = \begin{pmatrix} 0 & U^{-+} \\ U^{+-} & U^{++} \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} 0 & -V^{+-} \\ V^{+-} & [V^{++}, \psi\psi^\dagger] \end{pmatrix}. \tag{44}$$

5.3. Linearized equations of motion for perturbation V

For $M(t) = M_o(t) + V(t)$, (9) becomes $i\partial_t(M_o + V) = 2\big[E'_{M_o+V}, \Phi_o + V\big]$. The solution describes a curve $M(t)$ on the $B = 1$ component of phase space. Our g.s. is time dependent, so this is like the effect of Jupiter on the motion of Mercury. For the nucleon, we refer to resonances created by scattering a π, e^- or ν off the proton. From (10), $E'(M_o + V) = E'(M_o) + \frac{\tilde g^2}{4}G_V$, so linearizing,

$$\frac{i}{2}\dot V = \left(-\frac{i}{2}\partial_t M_o + [E'(M_o), \Phi_o]\right) + [E'(M_o), V] + \frac{\tilde g^2}{4}[G_V, \Phi_o] + \mathcal{O}(V^2). \tag{45}$$

The terms in round brackets add to zero if $M_o(t)$ satisfies the eom, as does our baryon g.s. (20). So

$$\frac{1}{2}\dot V = i[V, E'(M_o)] - \frac{i\tilde g^2}{4}[G_V, \Phi_o] = i[V, T'] - \frac{i\tilde g^2}{4}\{[G_{M_o}, V] + [G_V, \Phi_o]\}. \tag{46}$$

Here $T' = -h/2$. To see the departure from 't Hooft's meson equation write $\dot V = H = H_1 + H_2$ with

$$H_1 = i[h, V] - \frac{i\tilde g^2}{2}[G_V, \epsilon] \qquad \text{and} \qquad H_2 = -\frac{i\tilde g^2}{2}\{[G_{M_o}, V] + [G_V, M_o]\}. \tag{47}$$

H_1 is independent of M_o and leads to 't Hooft's meson equation (17) if $M_o = 0$. H_2 has 'baryon–meson' interactions leading to many complications. In blocks, the eom are

$$\begin{pmatrix} 0 & \dot V^{-+} \\ \dot V^{+-} & \dot V^{++} \end{pmatrix} = i\begin{pmatrix} 0 & [h, V^{-+}] \\ [h, V^{+-}] & [h, V^{++}] \end{pmatrix} - \frac{i\tilde g^2}{2}\begin{pmatrix} [G_M, V]^{--} & [G_M, V]^{-+} + G_V^{-+}(2 + M^{++}) \\ -\text{h.c.} & [G_M, V]^{++} + \big[G_V^{++}, M^{++}\big] \end{pmatrix}. \tag{48}$$

[17] A more general example of a matrix with ψ in its kernel is $U^{-+} = \underline{U}^{-+}(1 - P_\psi)$ for *any* matrix $\underline{U}^{-+}$.

[18] Since $u \perp \psi$, this is consistent only if $V^{++}\psi \perp \psi$, i.e. $\psi^\dagger V^{++}\psi = 0$, which is the same as the condition $\operatorname{tr} M_o^{++}V^{++} = 0$. But this is guaranteed by constraint (35) $2V^{++} = [V^{++}, M_o^{++}]$ upon multiplying by M_o^{++} and taking a trace.

5.4. Linearized time evolution preserves constraints

Equation (46) describes the motion of a point $V(t)$ in the tangent bundle of the Grassmannian restricted to the base $M_o(t)$. To establish this, we show that (46) preserves hermiticity of V and the linear constraint (35). If V is Hermitian at time t, then so are G_V, G_{M_o} and $H(V)$. By (46), $V(t+\delta t)$ is also Hermitian. As for the linear constraint, suppose $\Phi_o(t)$ is the solution of (9) about which we perturb by $V(t)$, and define a constraint function $C(t) = [\Phi_o(t), V(t)]_+$, which satisfies $C(0) = 0$. Then using (46)

$$\frac{\mathrm{i}}{2}\dot{C} = \frac{\mathrm{i}}{2}\{[\dot{\Phi}_o, V]_+ + [\Phi_o, \dot{V}]_+\} = \left[\left[E'_{M_o}, \Phi_o\right], V\right]_+ + \left[\Phi_o, \left[E'_{M_o}, V\right]\right]_+ + \frac{\tilde{g}^2}{4}[\Phi_o, [G_V, \Phi_o]]_+. \tag{49}$$

To find the unique solution of this autonomous linear system of first-order ODEs, we make the guess $C(t) \equiv 0$ which annihilates the lhs. On the rhs, the first two terms cancel as $[\Phi_o, V] = 0$. The third term vanishes as $\Phi_o^2 = I$ (section 1.2). So $C(t) = 0$ is the unique solution and (46) preserves the linear constraint. `Corollary:` As both $V(t)$ and $V(t+\delta t)$ satisfy the constraint, so does the difference quotient $H(V(t))$. And when H is split as in (46), both $[E'(M_o), V]$ and $[G(V), \Phi_o]$ satisfy the linear constraint if V does. But if H is split as in (47), H_1 and H_2 *do not* each satisfy (35), except at $M_o = 0$.

5.5. Equation of motion in '$--$' block: orthogonality of excited states

The $--$ block of the eom (48) is simplest as it is non-dynamical, $[G(M_t), V(t)]^{--} = 0$. This is necessary for consistency of the eom. It states that $V^{-+}G_M^{+-} : \mathcal{H}_- \to \mathcal{H}_-$ is always Hermitian:

$$G(M_t)^{-+}V_t^{+-} = V_t^{-+}G(M_t)^{+-}. \tag{50}$$

Using the constraint $V^{-+}M^{++} = 0$ (35), we show that $V^{-+}G_M^{+-} \equiv 0$! Our argument uses the exponential form of the g.s. $M_o(t)$ (20), but there may be a more general proof. We simplify (50) using the fact that the g.s. interaction operator (C.9) is always of rank 1. Putting

$$\tilde{G}(M_t)_{pr}^{-+} = (2/P)\,\mathrm{e}^{-r/2P}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}rt}\,\mathrm{e}^{-p/2P}\,\mathrm{e}^{\frac{\mathrm{i}}{2}pt}\mathrm{I}_2(-p) \quad \text{in} \quad (50)$$

$$\Rightarrow \int_0^\infty [\mathrm{d}r]\,\mathrm{e}^{\frac{\mathrm{i}}{2}(p-r)t}\,\mathrm{e}^{-\frac{p+r}{2P}}\,I_2(-p)V_{rq}^{+-} = \int_0^\infty [\mathrm{d}r]V_{pr}^{-+}\,\mathrm{e}^{\frac{\mathrm{i}}{2}(r-q)t}\,\mathrm{e}^{-\frac{r+q}{2P}}\,I_2(-q) \tag{51}$$

for all $p, q < 0$. Dividing by $I_2 \neq 0$ (C.4) and using $\tilde{\psi}_t(r) \propto \theta(r)\,\mathrm{e}^{-r(1/P-\mathrm{i}t)/2}$ (20), we get

$$\frac{\int_0^\infty [\mathrm{d}r]\tilde{\psi}_t(r)V_{rq}^{+-}}{I_2(-q)\mathrm{e}^{-\frac{q}{2}(\frac{1}{P}-\mathrm{i}t)}} = \frac{\int_0^\infty [\mathrm{d}r]V_{pr}^{-+}\psi_t(r)}{I_2(-p)\,\mathrm{e}^{-\frac{p}{2}(\frac{1}{P}-\mathrm{i}t)}} = c(t), \qquad \forall p, q < 0. \tag{52}$$

The lhs and rhs depend on q and p, respectively, so they must be equal! $c(t) \in \mathbf{R}$ by hermiticity. So (50) becomes

$$\int_0^\infty [\mathrm{d}r]\tilde{V}_{pr}^{-+}\tilde{\psi}_t(r) = c(t)\,\mathrm{e}^{-\frac{p}{2}(\frac{1}{P}-\mathrm{i}t)}I_2(-p), \qquad \forall p < 0. \tag{53}$$

V^{-+} maps the g.s. to $c(t)\times$ a vector in $\mathcal{H}_-$. But V annihilates the g.s: $V^{-+}M_o^{++} = 0$ (35)! So $c(t) \equiv 0$, $V^{-+}G_{M_o}^{+-} = 0$ and $[G_{M_o}, V]^{--} \equiv 0$. It states that the excited states are $\perp$ to the g.s.

5.6. Lack of translation invariance: failure of the ansatz $V_{pq}^{+-}(t) = \tilde{\chi}_t(\xi)$

In the $+-$ block of the eom (48), let us try what worked for mesons (section 3). Around the translation-invariant $M = 0$ vacuum, $V_{pq}^{+-}(t) = \tilde{\chi}_t(\xi, P_t)$ could be taken independent of

$P_t = p - q$ (16). For oscillations around a *non*-translation-invariant baryon M_o (19), such an ansatz does not work; P_t cannot be regarded as the momentum of $\tilde{V}$. The *orthogonality* constraint $V^{-+}M_o^{++} = 0$ (35) is violated if $\tilde{\chi}$ is independent of $p-q$. To see this, $V^{-+}M_o^{++} = 0$ is expressed using $\tilde{M} = -2\psi\psi^\dagger$ as

$$\int_0^\infty \tilde{\chi}(\xi, t)\tilde{\psi}_t(q)\,\mathrm{d}q = 0, \qquad \forall p < 0 \Leftrightarrow \int_0^1 \tilde{\chi}(\xi, t)\tilde{\psi}_t(p(1-\xi^{-1}))\frac{\mathrm{d}\xi}{\xi^2} = 0, \quad \forall p < 0. \tag{54}$$

$\tilde{\chi}_t$ must be $\perp$ to each of $f_p(\xi; t) = \tilde{\psi}_t(p(1 - 1/\xi))/\xi^2$ for $p < 0$ at all times t. For example, at $t = 0$,

$$f_p(\xi) = \xi^{-2}\psi_o(p(1 - 1/\xi)) \sim \xi^{-2}\exp\{-p(1 - 1/\xi)\} \quad \text{for} \quad p < 0. \tag{55}$$

$f_p(\xi)$ are linearly independent positive functions going from $f_p(0) = 0$ to $f_p(1) = 1$ with maxima shifting rightwards as $0 \geqslant p \geqslant -\infty$. Plausibly, for $\tilde{\chi}$ to be $\perp$ (in $L^2(0, 1)$) to all of them requires $\tilde{\chi} \equiv 0$. So non-trivial $\tilde{V}^{+-}_{pq}$ must depend on $p-q$. It seems prudent to work instead with the unconstrained U.

5.7. *Linearized evolution of the unconstrained perturbation U*

To find the linearized evolution of U, we put $V = \mathrm{i}[\Phi_o, U]$ in (46)

$$\mathrm{i}[\Phi_o, \dot{U}] = [[\Phi_o, U], h] + \frac{\tilde{g}^2}{2}[G_{[\Phi_o,U]}, \epsilon] + \frac{\tilde{g}^2}{2}\{[G_{M_o}, [\Phi_o, U]] + [G_{[\Phi_o,U]}, M_o]\}. \tag{56}$$

Some entries of U are redundant due to gauge freedom. So we derive the eom in the mostly zero gauge in terms of the vector u and matrix U^{+-} (41). This requires some care. The eom do not know our gauge choice, and we must not expect them to preserve the gauge conditions (42) $\psi^\dagger u = 0$ and $\psi^\dagger U^{+-} = 0$. Using (43), we begin by writing (the tentative nature of this evolution is conveyed by $\doteq$)

$$2\mathrm{i}\dot{u} \doteq V^{++}\dot{\psi} + \dot{V}^{++}\psi, \qquad 2\mathrm{i}\dot{U}^{+-} \doteq \dot{V}^{+-}. \tag{57}$$

Here, $\dot{\psi}_t(p) = \frac{1}{2}\mathrm{i}p\tilde{\psi}_t(p)$, if ψ is chosen as the g.s. valence quark wavefunction in the chiral limit (20). We use the eom (48) for V and (43) to express the rhs in terms of u, U^{+-}. For example,

$$\begin{aligned}2\mathrm{i}\dot{u} \doteq{}& 2\mathrm{i}(u\psi^\dagger - \psi u^\dagger)\dot{\psi} + 2[u\psi^\dagger - \psi u^\dagger, h]\psi - \tilde{g}^2\big\{\big[u\psi^\dagger - \psi u^\dagger, G_M^{++}\big] \\ &+ G_M^{+-}U^{-+} + U^{+-}G_M^{-+}\big\}\psi - \mathrm{i}\tilde{g}^2\big[\psi\psi^\dagger, G_V^{++}\big]\psi.\end{aligned} \tag{58}$$

G_V is given in appendix C.1. We regard these as equations for $(u, U^{+-})(t+\delta t)$ given $(u, U)^{+-}(t)$ satisfying the gauge conditions (42). So on the rhs we can use (42) to simplify

$$\begin{aligned}\mathrm{i}\dot{u} \doteq{}& \mathrm{i}(u\psi^\dagger - \psi u^\dagger)\dot{\psi} + (u\psi^\dagger - \psi u^\dagger)h\psi - hu \\ &+ \frac{\tilde{g}^2}{2}\big\{G_M^{++}u - (u\psi^\dagger - \psi u^\dagger)G_M^{++}\psi - U^{+-}G_M^{-+}\psi - \mathrm{i}P_\psi G_V^{++}\psi + \mathrm{i}G_V^{++}\psi\big\}, \\ \mathrm{i}\dot{U}^{+-} \doteq{}& [U^{+-}, h] + \frac{\tilde{g}^2}{2}\big\{(\psi u^\dagger - u\psi^\dagger)G_M^{+-} - U^{+-}G_M^{--} + G_M^{++}U^{+-} + \mathrm{i}(1 - P_\psi)G_V^{+-}\big\}.\end{aligned} \tag{59}$$

But we have a problem. This evolution does not preserve the gauge-fixing conditions:

$$\begin{aligned}&\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}(\psi^\dagger u) \doteq \mathrm{i}\dot{\psi}^\dagger u - u^\dagger h\psi - \psi^\dagger hu + \frac{\tilde{g}^2}{2}\big\{\psi^\dagger G_M^{++}u + u^\dagger G_M^{++}\psi\big\} \neq 0, \\ &\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}(\psi^\dagger U^{+-}) \doteq \mathrm{i}\dot{\psi}^\dagger U^{+-} - 2\psi^\dagger hU^{+-} + \tilde{g}^2\big\{u^\dagger G_M^{+-} + \psi^\dagger G_M^{++}U^{+-}\big\} \neq 0.\end{aligned} \tag{60}$$

But at each time step, we may add to $U(t+\delta t)$ a pure-gauge $U_g(t+\delta t)$ to bring it to the mostly zero gauge, so that at $t+\delta t$, $\psi^\dagger u = 0$ and $\psi^\dagger U^{+-} = 0$. This corresponds to subtracting out the instantaneous projections on ψ and defining a new time evolution that preserves (42)

$$\mathrm{i}\dot{u} := \tfrac{1}{2}(1-P_\psi)(V^{++}\dot{\psi} + \dot{V}^{++}\psi) \qquad \text{and} \quad \mathrm{i}\dot{U}^{+-} := \tfrac{1}{2}(1-P_\psi)\dot{V}^{+-}. \tag{61}$$

This projection involves no approximation. We use (42) to simplify the rhs to get[19]

$$\begin{aligned}
\mathrm{i}\dot{u} &\equiv -l = \mathrm{i}u\psi^\dagger\dot{\psi} + \{\psi^\dagger h\psi - [1-P_\psi]h\}u \\
&\quad - \frac{\tilde{g}^2}{2}\left\{\psi^\dagger G_M^{++}\psi u + U^{+-}G_M^{-+}\psi - [1-P_\psi]\big(G_M^{++}u + \mathrm{i}G_V^{++}\psi\big)\right\} \\
\mathrm{i}\dot{U}^{+-} &\equiv -L^{+-} = U^{+-}h - [1-P_\psi]hU^{+-} \\
&\quad - \frac{\tilde{g}^2}{2}\left\{u\psi^\dagger G_M^{+-} + U^{+-}G_M^{--} - [1-P_\psi]\big(G_M^{++}U^{+-} + \mathrm{i}G_V^{+-}\big)\right\}.
\end{aligned} \tag{62}$$

Our goal is small oscillations around the baryon. We write (62) as a Schrödinger equation, where the wavefunction consists of a vector u and a matrix U^{+-} and the Hamiltonian is the pair (l, L^{+-}):

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ U^{+-}\end{pmatrix} = \begin{pmatrix} l(u, u^\dagger, U^{+-}, U^{-+}) \\ L^{+-}(u, u^\dagger, U^{+-})\end{pmatrix}. \tag{63}$$

However, (l, L^{+-}) depend on u, U^{+-} and $u^\dagger$, U^{-+} through G_V in (62). Indeed, from appendix C.1,

$$G_V^{+-} = 2\mathrm{i}G(u\psi^\dagger - \psi u^\dagger + U^{+-})^{+-}, \qquad \frac{1}{2i}G_V^{++} = G^{++}_{u\psi^\dagger - \psi u^\dagger} - G^{++}_{U^{-+}} + G^{++}_{U^{+-}}. \tag{64}$$

So the time dependence does not factorize under separation of variables[20]. This prevented us from finding oscillatory solutions to the full system (62) using (ω is complex *a priori*)

$$\tilde{u}_p(t) = \tilde{u}_p\,\mathrm{e}^{\mathrm{i}(\omega+p/2)t} \qquad \text{and} \quad \tilde{U}^{+-}_{pq} = \tilde{U}^{+-}_{pq}\,\mathrm{e}^{\mathrm{i}(\omega+(p-q)/2)t}. \tag{65}$$

5.8. *Eigenvalue problem for oscillations in approximation $u = 0$*

We make an ansatz that permits us to find oscillations around the baryon. V is a meson bound to M_o whose valence-quark wavefunction is ψ. u and U^{+-} represent valence and sea/antiquarks in V, respectively. Mesons are usually described as a quark–antiquark sea. This suggests putting $u = 0$. Moreover, for mesons around the vacuum, $V^{+-} \propto U^{+-} \neq 0$ (section 3), and our analysis should reduce to that far from the baryon. For u to remain zero under time evolution (62), a `consistency condition` must hold for $\tilde{g} \neq 0$:

$$\mathrm{i}\dot{u} = -\frac{\tilde{g}^2}{2}\left\{U^{+-}G_M^{-+} - \mathrm{i}(1-P_\psi)G_V^{++}\right\}\psi = 0, \qquad \text{where} \quad G_V^{++} = 2\mathrm{i}\left\{G^{++}_{U^{+-}} - G^{++}_{U^{-+}}\right\}. \tag{66}$$

It states that ψ is in the kernel of a certain operator. Equation (66) is studied in appendix G. Hilbert–Schmidt U^{+-} obeying (66) and $\psi^\dagger U^{+-} = 0$ form the *physical* subspace

[19] Signs of l, L^{+-} are chosen so that the Hamiltonian in section 5.8 is positive. Some integrals are IR divergent if $\tilde{\psi}(p) \propto \mathrm{e}^{-p/2P}\theta(p)$ is the exact chiral g.s. For example for the regulator of section 4.1.1, $\psi^\dagger h\psi = \frac{1}{2}(\frac{1}{2}+a+\frac{\mu^2}{a})$. We suspect that all divergences cancel in physical quantities, as for the lightest baryon. Also, most of these divergences disappear for the ansatz $u = 0$ studied in sections 5.8–5.10.

[20] We are looking for vibrations about a time-dependent state $\tilde{\psi}_t(p) = \tilde{\psi}_o(p)\mathrm{e}^{\mathrm{i}pt/2}$. The momentum-dependent phases in u and U^{+-} guarantee that the gauge conditions $\psi_t^\dagger u_t = 0$ and $\psi_t^\dagger U_t^{+-} = 0$ remain satisfied if they initially were.

for the ansatz $u = 0$. Now we assume oscillatory behaviour about the time-dependent g.s. The time dependence in the eom (62) factorizes

$$U^{+-}_{pq}(t) = U^{+-}_{pq}\, e^{i(\omega+\frac{p-q}{2})t} \quad \Rightarrow \quad \left(\omega + \frac{p-q}{2}\right) U^{+-}_{pq}\, e^{i(\omega+\frac{p-q}{2})t} = L^{+-}(U^{+-})_{pq}\, e^{i\left(\omega+\frac{p-q}{2}\right)t}. \tag{67}$$

Let $K^{+-}(U^{+-}) = L^{+-}(U^{+-}) + \left[U^{+-}, \frac{\mathrm{p}}{2}\right]$. We get an eigenvalue problem for the excitation energies ω above the g.s. of the baryon[21]. The *correction* $\left[U^{+-}, \frac{\mathrm{p}}{2}\right]$ accounts for the time dependence of the g.s.:

$$K^{+-}(U^{+-}) = \left[U^{+-}, \frac{\mathrm{p}}{2}\right] + (1 - P_\psi) h U^{+-} - U^{+-} h$$
$$+ \frac{\tilde{g}^2}{2}\left\{U^{+-} G^{--}_M - (1 - P_\psi)\left(G^{++}_M U^{+-} - 2G^{+-}_{U^{+-}}\right)\right\} = \omega U^{+-}. \tag{68}$$

The *eigenvector is a matrix* U^{+-} with ψ in its left nullspace and constrained by (66). Similarly,

$$K^{-+}(U^{-+}) = \left[\frac{\mathrm{p}}{2}, U^{-+}\right] + U^{-+} h (1 - P_\psi) - h U^{-+}$$
$$+ \frac{\tilde{g}^2}{2}\left\{G^{--}_M U^{-+} - \left(U^{-+} G^{++}_M - 2G^{-+}_{U^{-+}}\right)(1 - P_\psi)\right\} = \omega^* U^{-+}$$
$$\Rightarrow \quad \hat{K}(U) = \begin{pmatrix} 0 & K^{-+}(U^{-+}) \\ K^{+-}(U^{+-}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega^* U^{-+} \\ \omega U^{+-} & 0 \end{pmatrix}. \tag{69}$$

An advantage of the ansatz $u = 0$ is that K^{+-} depends only on U^{+-}. $\hat{K}$ is Hermitian with respect to the Hilbert–Schmidt inner-product defined in appendix E:

$$(U, \hat{K}(\underline{U})) = (\hat{K}(U), \underline{U}) \text{ i.e. } \Re\, \mathrm{tr}\, U^{-+} \hat{K}(\underline{U})^{+-} = \Re\, \mathrm{tr}\, \hat{K}(U)^{-+} \underline{U}^{+-}. \tag{70}$$

Indeed, cyclicity of tr, the gauge condition $U^{-+}\psi = 0$ and self-adjointness[22] of $\hat{G}$ (C.1) imply

$$\mathrm{tr}\, U^{-+} \hat{K}(\underline{U})^{+-} = \mathrm{tr}\Bigg[U^{-+}\left[\underline{U}^{+-}, \frac{\mathrm{p}}{2}\right] + U^{-+}(1 - P_\psi) h \underline{U}^{+-} - U^{-+}\underline{U}^{+-} h$$
$$+ \frac{\tilde{g}^2}{2}\left\{U^{-+}\underline{U}^{+-} G^{--}_M - U^{-+}(1 - P_\psi)\left(G^{++}_M \underline{U}^{+-} - 2G^{+-}_{\underline{U}^{+-}}\right)\right\}\Bigg]$$
$$= \mathrm{tr}\Bigg[\left[\frac{\mathrm{p}}{2}, U^{-+}\right]\underline{U}^{+-} + U^{-+} h \underline{U}^{+-} - h U^{-+} \underline{U}^{+-}$$
$$+ \frac{\tilde{g}^2}{2}\left\{G^{--}_M U^{-+} \underline{U}^{+-} - U^{-+} G^{++}_M \underline{U}^{+-} + 2G^{-+}_{U^{-+}} \underline{U}^{+-}\right\}\Bigg]$$
$$= \mathrm{tr}\, \hat{K}(U)^{-+} \underline{U}^{+-}. \tag{71}$$

The original linearized $H(V)$ (47) is *not* self-adjoint. By passing from $V \mapsto U$, eliminating redundant variables and imposing $u = 0$, we isolated a subspace on which the linearized evolution admits harmonic time dependance and is formally self-adjoint. $\hat{K}^\dagger = \hat{K} \Rightarrow \omega = \omega^*$. The eigenmodes U^{+-} thus describe *oscillations* about the baryon. Without translation invariance, we use $P_M = -\mathrm{tr}\,\mathrm{p}(M_o + V)/2 = P + P_V$ (appendix A) as the excitation momentum instead of P_t (section 5.6). So the mass^2 per colour is $\mathcal{M}^2_M = P_M(2E_M - P_M)$. For small oscillations, $E_{M_o+V} \approx E_o + \omega$ where E_o is the g.s. energy. $2E_o \geqslant P$ where P is the g.s. momentum. In the chiral limit, $2E_o = P$ (section 4.1.1), so

$$\mathcal{M}^2_{M_o+V} = P_M(2E_M - P_M)$$
$$\approx (P + P_V)(2E_o + 2\omega - P - P_V) \xrightarrow{m\to 0} (P + P_V)(2\omega - P_V). \tag{72}$$

[21] Recall (8) that p is the Hermitian operator with kernel $\mathrm{p}_{pq} = 2\pi\delta(p - q)p$.
[22] This means that $\mathrm{tr}\, U^{-+} G^{+-}_{U^{+-}} = \mathrm{tr}\, G^{-+}_{U^{-+}} U^{+-}$, which follows from the definition of $\tilde{G}(U)_{pq}$.

Since $V \ll M_o$, we expect $|P_V| \ll P$, so $P + P_V \approx P > 0$. To ensure[23] $\mathcal{M}^2_{M_o+V} \geqslant 0$, we need $2\omega \geqslant P_V - (2E_o - P)$ or in the chiral limit, $2\omega \geqslant P_V$. But for $u = 0$, $P_V = 0$ by (43). So

$$u = 0 \quad \Rightarrow \quad \mathcal{M}^2 = P(2E_M - P) \approx P(2E_o + 2\omega - P) \stackrel{m\to 0}{\longrightarrow} 2\omega P. \tag{73}$$

So $\hat{K}$ and ω should be $\geqslant 0$ in the chiral limit. Define the parity of *meson* V as even if $\tilde{V}_{pq}$ is real symmetric and odd if it is imaginary antisymmetric. For the ansatz $u = 0$, the eigenvalue equation (68) and (69) follows from a variational principle. If we extremize $\mathcal{E} = (U, \hat{K}(U)) = \operatorname{tr} U^{-+}\hat{K}(U)^{+-}$,

$$\mathcal{E} = \operatorname{tr}\bigg[\left(h - \frac{\mathrm{p}}{2}\right)\{U^{+-}U^{-+} - U^{-+}U^{+-}\} + \frac{\tilde{g}^2}{2}\left\{G^{--}_M U^{-+}U^{+-} - G^{++}_M U^{+-}U^{-+} + 2G^{-+}_{U^{-+}}U^{+-}\right\}\bigg], \tag{74}$$

holding $\|U\|^2 = (U, U) = \operatorname{tr} U^{-+}U^{+-}$ fixed via the Lagrange multiplier ω, we get (68)

$$\frac{\delta}{\delta U^{-+}_{qp}}\left\{\operatorname{tr} U^{-+}_{rs}\hat{K}(U^{+-})^{+-}_{sr} - \omega \operatorname{tr} U^{-+}_{rs}U^{+-}_{sr}\right\} = 0 \quad \Rightarrow \quad \hat{K}(U)^{+-}_{pq} = \omega U^{+-}_{pq}. \tag{75}$$

We treated $U^{+-}_{sr} = U^{-+*}_{rs}$ and $U^{-+}_{rs} = U^{+-*}_{sr}$ as independent variables and used the fact that $\hat{K}^{+-}$ depends only on U^{+-}. We must solve the eigenvalue problem (68) on a space of U^{+-} examined in appendix G. In section 5.9 we interpret the terms in the variational energy $\mathcal{E}$, and approximately minimize it in section 5.10.

5.9. *Rank-1 ansatz* $U^{+-} = \phi\eta^\dagger$*: sea quarks and antiquarks*

Let $U^{+-} = \phi\eta^\dagger$, with $\phi, \eta \in \mathcal{H}_\pm$ being the sea/antiquark wavefunctions of the excited baryon. They have antiquarks even if the lightest one does not, just as mesons have antiquarks though the vacuum does not. Equation (75) states to hold $\operatorname{tr} U^{+-}U^{-+} = \|\phi\|^2\|\eta\|^2$ fixed and extremize the linearized energy $(U, \hat{K}(U))$

$$\mathcal{E}(U) = \operatorname{tr}\left(h - \frac{\mathrm{p}}{2}\right)[\|\eta\|^2\phi\phi^\dagger - \|\phi\|^2\eta\eta^\dagger] + \frac{\tilde{g}^2}{2}\operatorname{tr}\left[\|\phi\|^2 G^{--}_M \eta\eta^\dagger - \|\eta\|^2 G^{++}_M \phi\phi^\dagger + 2G^{-+}_{\eta\phi^\dagger}\phi\eta^\dagger\right] \tag{76}$$

on the physical subspace. If we factor out $\|U\|^2 = \|\phi\|^2\|\eta\|^2$ and work with *unit vectors* ϕ and η,

$$\mathcal{E}(U)/\|U\|^2 = \operatorname{tr}\left[\left(h - \frac{\mathrm{p}}{2}\right)(P_\phi - P_\eta) + \tilde{g}^2\left(G^{-+}_{\eta\phi^\dagger}\phi\eta^\dagger + \frac{1}{2}P_\eta G^{--}_M - \frac{1}{2}P_\phi G^{++}_M\right)\right]. \tag{77}$$

Here $P_\eta = \eta\eta^\dagger$ and $P_\phi = \phi\phi^\dagger$. The variational principle cannot determine $\|\phi\|$ or $\|\eta\|$. Recall that $2h = p + \mu^2/p$ with $\mu^2 = m^2 - \tilde{g}^2/\pi$, so the kinetic and self-energies $\mathcal{T}$ of sea-partons are

$$\mathcal{T} = \operatorname{tr}\left(h - \frac{\mathrm{p}}{2}\right)(P_\phi - P_\eta) = \frac{\mu^2}{2}\int \frac{[\mathrm{d}p]}{p}[|\tilde{\phi}_p|^2 - |\tilde{\eta}_p|^2]. \tag{78}$$

In the chiral limit $\mathcal{T} < 0$ is purely self-energy. Equation (78) is valid for excitations around the massless $M_o(t)$ (20). If the lightest baryon were static, then $h - \mathrm{p}/2 \mapsto h$. Interactions are simply interpreted in position space. As $\phi, \eta \in \mathcal{H}_\pm$ the block designations in (77) are

[23] $2\omega \geqslant P_t$ for 't Hooft's meson operator (18), since meson mass^2's were $\geqslant 0$ if $m \geqslant 0$ [2].

automatic ($\mathrm{tr}\, P_\eta G_M^{--} = \mathrm{tr}\, P_\eta G_M$, etc). Thus, the Coulomb energy $\tilde{g}^2\mathcal{V}_c$ of the sea quarks ϕ interacting with antiquarks η is positive

$$\mathcal{V}_c = \mathrm{tr}\, G_{\eta\phi^\dagger}\phi\eta^\dagger = \int \mathrm{d}x\, \mathrm{d}y |\phi(x)|^2 \frac{1}{2}|x-y||\eta(y)|^2 = \int \mathrm{d}x |\phi_x|^2 \mathrm{v}(x) > 0. \tag{79}$$

Here $\mathrm{v}(x) = \frac{1}{2}\int |\eta_y|^2|x-y|\mathrm{d}y$ obeys Poisson's equation. The exchange interaction of sea-partons and 'background' valence quarks ψ is $\tilde{g}^2\mathcal{V}_e = \tilde{g}^2(\mathcal{V}_{e\eta} + \mathcal{V}_{e\phi})$:

$$\mathcal{V}_e = \frac{1}{2}\mathrm{tr}[P_\eta G_M - P_\phi G_M] = \int \mathrm{d}x\, \mathrm{d}y \psi^*(x)\psi(y)\frac{1}{2}|x-y|\{\phi(x)\phi^*(y) - \eta(x)\eta^*(y)\}. \tag{80}$$

Now $v(x) = \frac{1}{2}\int \psi_y\phi_y^*|x-y|\,\mathrm{d}y$ and $w(x) = \frac{1}{2}\int \psi_y\eta_y^*|x-y|\,\mathrm{d}y$ both obey Poisson's equation. Then $V_{e\eta} = \int |w'(x)|^2\,\mathrm{d}x > 0$ and $V_{e\phi} = -\int |v'(x)|^2\,\mathrm{d}x < 0$. However, $\mathrm{sgn}\,\mathcal{V}_e$ is not clear *a priori*. Thus, the energy $\mathcal{E} = \mathcal{T} + \tilde{g}^2(\mathcal{V}_c + \mathcal{V}_e)$ has a simple relativistic potential-model meaning. In the chiral limit, the mass of an excited baryon is $\mathcal{M}^2 = 2P\omega$, where P is the g.s. momentum and $\omega = \min \mathcal{E}$ (73).

5.10. Crude estimate for mass and shape of the first excited baryon in the chiral limit

To estimate the mass and form factor $U^{+-} = \phi\eta^\dagger$ of the first excited baryon (19), we must extremize $\mathcal{E}$ (77) holding $\|U\| = 1$ and restrict to U^{+-}, satisfying the gauge and consistency conditions (appendix G). We have not yet solved the consistency condition (G.2), an intricate orthogonality condition. But even without it, the interacting parton model derived in section 5.9 may be postulated as a mean-field description of excited baryons. So as an approximation, we impose $\psi^\dagger\phi = 0$ but ignore (G.2). Our ansatz for the unit norm η, ϕ contains two parameters a, b controlling the decay of sea-parton wavefunctions[24]

$$\tilde{\psi}_p = \sqrt{4\pi c}\, \mathrm{e}^{-cp}\theta(p), \qquad \tilde{\phi}_p = \frac{\sqrt{8\pi b}b^2(b+c)}{\sqrt{b^2+3c^2}}p\left(p - \frac{2}{b+c}\right)\mathrm{e}^{-bp}\theta(p),$$
$$\tilde{\eta}_p = -ap\sqrt{8\pi a}\,\mathrm{e}^{ap}\theta(-p). \tag{81}$$

A boost rescales p. We choose our frame by fixing the momentum $P = 1/2c$ of the g.s. Since $\tilde{\phi}, \tilde{\eta}$ have been chosen real, $\tilde{V} = \mathrm{i}[\tilde{\Phi}_o, \tilde{U}] = 2\mathrm{i}(0, -\tilde{\eta}\tilde{\phi}^T|\tilde{\phi}\tilde{\eta}^T, 0)$ has odd parity, $\tilde{V}^T = -\tilde{V}$. The minimum of $\mathcal{E} = \mathcal{T} + \tilde{g}^2(\mathcal{V}_c + \mathcal{V}_e)$ among (81) is the (approx.) energy of the first excited baryon. But it is not an upper-bound, as we ignored (G.2). In the chiral limit, the self-energy is $\mathcal{T} = \mathcal{T}_\phi + \mathcal{T}_\eta$:

$$\mathcal{T}_\phi = \mathrm{tr}\left(h - \frac{\mathrm{p}}{2}\right)P_\phi = -\frac{\tilde{g}^2(3b^2 - 2bc + 3c^2)}{4\pi(b^2+3c^2)/b}, \qquad \mathcal{T}_\eta = \mathrm{tr}\left(\frac{\mathrm{p}}{2} - h\right)P_\eta = -\frac{\tilde{g}^2 a}{2\pi}. \tag{82}$$

$\mathcal{T}_\eta, \mathcal{T}_\phi$ are minimized as $a, b \to \infty$. By real symmetry of $G(M)$ (appendix C) and P_η, the exchange integral

$$\mathcal{V}_{e\eta} = \frac{1}{2}\mathrm{tr}\, P_\eta G_M^{--} = \int [\mathrm{d}p]\tilde{\eta}_p \int [\mathrm{d}q]\tilde{\eta}_q G(M)^{--}_{p>q}$$
$$= \frac{4a^2P}{\pi(1-2aP)^4}\left\{(1-2aP)^2 + 8aP\log\frac{8aP}{(1+2aP)^2}\right\}. \tag{83}$$

[24] To be accurate in the chiral limit $m \to 0$, $\tilde{\phi}_p$ and $\tilde{\eta}_p$ should probably vanish like small positive powers of p as $p \to 0^\pm$, just as the valence quark wavefunction ψ does. But to keep the calculation of $\mathcal{E}$ simple, we chose the smallest integer powers ($\tilde{\phi}_p \sim p^2$ and $\tilde{\eta}_p \sim p$) that ensure the absence of IR divergences and orthogonality $\psi^\dagger\phi = 0$.

$\mathcal{V}_{e\eta} > 0$ since $G(M)^{--}_{pq}$ and $\tilde{\eta}_q$ are positive. $\mathcal{V}_{e\eta}$ increases with a; it vanishes at $a = 0$. We cross-checked this using $\mathcal{V}_{e\eta} = \int |w'(x)|^2 \, dx$ (80). $V_{e\phi} = \int dx \, v(x) v''(x)^*$ (80) is minimized as $b \to \infty$:

$$\mathcal{V}_{e\phi} = -\frac{1}{2} \text{tr} \, P_\phi G^{++}_M = -\int_0^\infty [dp] \tilde{\phi}_p \int_0^p [dq] \tilde{\phi}_q G(M)^{++}_{p>q} = -\frac{2b^2 P}{\pi(3 + 4b^2 P^2)} < 0. \tag{84}$$

So the exchange energy is the difference of two positive quantities $\tilde{g}^2 \mathcal{V}_e = \tilde{g}^2(\mathcal{V}_{e\eta} + \mathcal{V}_{e\phi})$. As for the Coulomb energy (79), $\mathcal{V}_c = \int |\phi(x)|^2 \mathrm{v}(x) \, dx$, with $\mathrm{v}(x) = \frac{1}{\pi}\left(a + x \arctan \frac{x}{a}\right)$:

$$\mathcal{V}_c = \frac{a^2(a + 2b)(b^2 + 3c^2) + 2b^2(2a + b)(b^2 + c^2)}{\pi(a + b)^2(b^2 + 3c^2)}, \qquad \text{where} \quad 2Pc = 1. \tag{85}$$

So $\mathcal{T}$ and $\mathcal{V}_{e\phi}$ prefer large, while $\mathcal{V}_c$ and $\mathcal{V}_{e\eta}$ prefer small values of a and b. What about $\mathcal{E} = \mathcal{T} + \tilde{g}^2(\mathcal{V}_{e\phi} + \mathcal{V}_{e\eta} + \mathcal{V}_c)$? a and b are lengths, so define dimensionless parameters $\alpha = aP$ and $\beta = bP$. In the chiral limit, the minimum $\mathcal{M}_1^2$ of $2\mathcal{E}P$ is the mass^2 of the first excited baryon (73), so it must be Lorentz invariant: independent of P. $\tilde{g}$ is the only other dimensional quantity, so $\mathcal{E} = \tilde{g}^2 \mathtt{e}(\alpha, \beta)/P$, where $\mathtt{e}$ is a function of the dimensionless variational parameters. We find

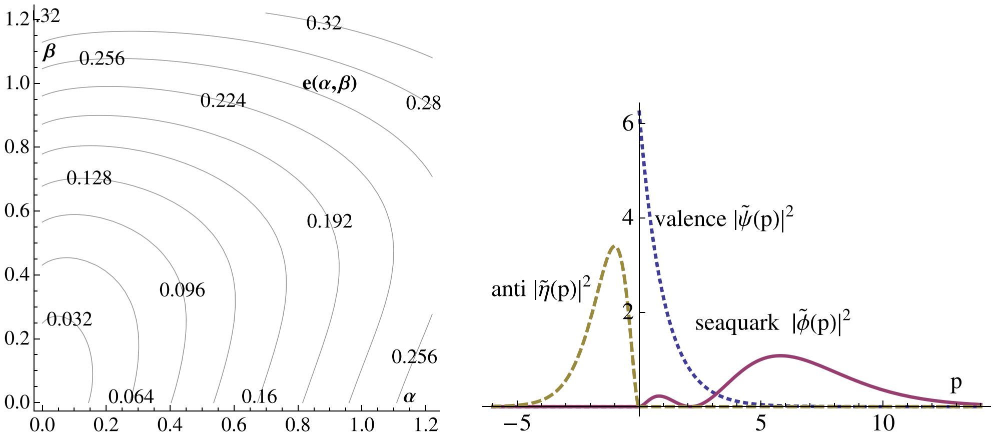
$$\pi \mathtt{e} = \frac{\alpha}{2} - \frac{12\beta^3 - 4\beta^2 + 3\beta}{4(4\beta^2 + 3)} + \frac{\alpha + 2\beta + 12\alpha\beta^2 + 8\beta^3}{\beta^{-2}(\alpha + \beta)^2(4\beta^2 + 3)} - \frac{2\beta^2}{4\beta^2 + 3} + \frac{(1 - 2\alpha)^2 + 8\alpha \log \frac{8\alpha}{(2\alpha+1)^2}}{(4\alpha)^{-2}(1 - 2\alpha)^4}. \tag{86}$$

As there is no other scale, the minimum of $\mathtt{e}$ should be at $\alpha, \beta \sim \mathcal{O}(1)$. But as figure 1(*a*) of level curves of $\mathtt{e}$ indicates, the minimum is $\mathtt{e} = 0$ as $\alpha, \beta \to 0^+$, corresponding to the pathological state where both $\tilde{\phi}$ and $\tilde{\eta}$ (81) tend point-wise to zero! If both α and β are the free parameters, the minimum occurs on the boundary of the space of rank-1 states $U^{+-} = \phi\eta^\dagger$ obeying the gauge condition. Perhaps this was to be expected: without imposing (G.2), we are exploring unphysical states! In the spirit of getting a crude estimate sans imposing (G.2), we put $\alpha = 1$, and minimize in β to find $\beta_{\min} = 0.445$ with $\mathtt{e}(1, \beta_{\min}) = 0.205$. So our crude estimate[25] for the mass/colour of the first excited baryon in the chiral limit is $\mathcal{M}_1 = 0.29\tilde{g}$. Figure 1(*b*) has the approximate valence, sea and antiquark densities (81) with the parameters $aP = 1$, $bP = \beta_{\min}$ and $2cP = 1$. The momentum/colour P of the lightest baryon sets the frame of reference. However, this is not an upper-bound on the mass gap, $\mathcal{M}_1$ could be an underestimate as we did not impose (G.2). There is still the unlikely possibility of zero modes other than the one-parameter family of states associated with the motion of the lightest baryon (section 4).

6. Discussion

We found that the lightest baryon has zero mass/colour in the chiral limit of large-N QCD$_{1+1}$. There is no spontaneous chiral symmetry breaking in this sense. Being massless, it evolves at the speed of light into a family of massless even parity states (section 2). They have the same quark distributions $\tilde{M}(p, p)$, differing only in off-diagonal form factors $\tilde{M}_0(p, q) \, e^{i(p-q)t/2}$. The other *modulus* of the baryon is its size $1/P$. P is its mean momentum/colour, fixed by the frame. Excited baryons (small oscillations around M_o) are like bound states of a meson V with M_o. On eliminating redundant variables, we derived an approximate eigenvalue problem for a

[25] As the plot shows, if we set $\beta = 1$ and minimize in α, then $\alpha_{\min} = 0.212$ with $\mathcal{M} = 0.32\tilde{g}$, which is roughly the same.



(*a*) (*b*)

Figure 1. (*a*) Level curves of the dimensionless energy $\mathfrak{e}(\alpha, \beta)$. (*b*) Valence, sea and anti-quark densities in the excited baryon for $P = 1, \alpha = 1, \beta = .445$. The orthogonality of sea and valence ($\phi^\dagger \psi = 0$ gauge condition) implies that $\tilde{\phi}(p)$ has a node. The normalization of anti/sea distributions is arbitrary and small compared to the valence distribution. One may contrast these with the first excited meson for which $|\tilde{\chi}(\xi)|^2 \approx \sin^2 \pi\xi$ where $\xi, 1-\xi$ are the quark and antiquark momentum fractions.

singular integral operator to determine the form factors U^{+-} and masses of excited baryons[26]. Based on the ansatz $U^{+-} = \phi\eta^\dagger$, we derived an interacting mean-field parton model for the structure of excited baryons (section 5.9). Using simple trial anti/seaquark wavefunctions η, ϕ, we estimated the mass and shape of the first excited baryon for which V has odd parity (analogue of Roper resonance). The baryon M_o breaks translation invariance, deforms the vacuum and consequently deforms the shape of the meson V. Unlike the mesons $\tilde{\chi}(\xi)$ near the Dirac vacuum, where $\xi \leftrightarrow 1-\xi$ relates quark and antiquark distributions; the distribution of quarks $|\tilde{\phi}_p|^2$ and antiquarks $|\tilde{\eta}_p|^2$ in V is not simply related. By linearizing around M_o, we approximated these excited baryons as non-interacting and stable. The nonlinear/linear treatment of M_o/V also prevented us from assigning a parity to excited baryons. But their nonlinear time evolution (9) should contain information on interactions and decay. Our approach is summarized in figure 2.

Acknowledgments

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Appendix A. Conservation of the mean momentum $P_M = -\frac{1}{2}\,\mathrm{tr}\,\mathrm{p}M$

E_M and P_M were used to define the mass (21) of the baryon and of oscillations above a non-translation-invariant $\tilde{M}_t(p,q)$, where the other concept of momentum $P_t = p - q$ is not meaningful (see section 3). Here we show that $P_M = -\frac{1}{2}\int p\tilde{M}_{pp}[\mathrm{d}p]$ is conserved even if $M(x, y; t)$ is not static, as long as it decays sufficiently fast: $|M_{xy}|^2 \sim |x|^{-1-\delta}$ for some $\delta > 0$

[26] However, we have not quite solved the consistency condition for the approximation $u = 0$ (appendix G) which restricts the space of physical states U^{+-}. It is also of interest to find a way of proceeding without this approximation.

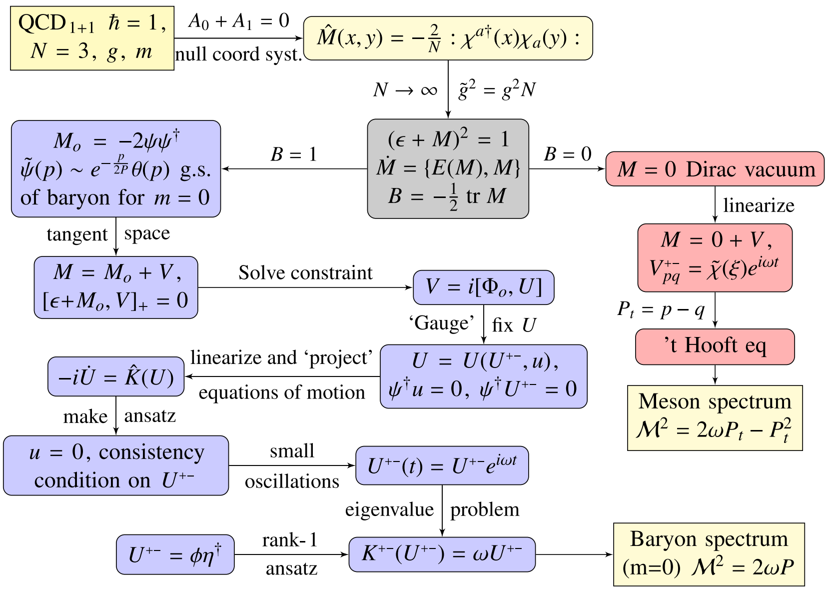

Figure 2. Flowchart of our approach to the baryon spectrum of large-N QCD$_{1+1}$.

as $|x| \to \infty$ for each y, t. When $\tilde{g} = 0$, energy $T = -\frac{1}{2}\text{tr}\, hM$ is linear. Also, $\tilde{\mathrm{p}}$, $\tilde{h}$ and $\tilde{\epsilon}$ are diagonal, so their commutators vanish. From (4),

$$\partial_t P = \{T(M), P\} = \{f_h, f_{\mathrm{p}}\} = f_{-\mathrm{i}[h,\mathrm{p}]} + \frac{\mathrm{i}}{2}\text{tr}[h, \mathrm{p}]\epsilon = 0. \tag{A.1}$$

So for $g \neq 0$ only U (7) contributes to $\partial_t P_M$. U is simpler in position space, so write

$$P_M = -\frac{1}{2}\int [\mathrm{d}p]\,\mathrm{d}x\,\mathrm{d}y\,p\,\mathrm{e}^{-\mathrm{i}p(x-y)}M_{xy} = -\frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y M_{xy}D_{xy}, \tag{A.2}$$

where $D_{xy} = \int [\mathrm{d}p] p \mathrm{e}^{-\mathrm{i}p(x-y)} = \mathrm{i}\partial_x \delta(x-y)$ is Hermitian. So we have a quadruple integral

$$\partial_t P = \{E(M), P\} = \{U, P\} = -\frac{\tilde{g}^2}{16}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}u \frac{|x-y|}{2} D_{zu}\{M_{xy}M_{yx}, M_{zu}\}. \tag{A.3}$$

We do two integrals and integrate by parts elsewhere to show $\partial_t P = 0$! By (2), the PB is

$$\mathrm{i}\{M_{xy}M_{yx}, M_{zu}\} = \delta_{yz}M_{yx}\Phi_{xu} - \delta_{xu}M_{yx}\Phi_{zy} + (x \leftrightarrow y). \tag{A.4}$$

After one integration and relabelling variables, $\partial_t P = -\frac{\tilde{g}^2}{8}\Im I$, where $I = \mathrm{i}\int \mathrm{d}y\,\mathrm{d}z \Phi_{yz}\int \mathrm{d}x|x-y|M_{xy}\partial_x\delta_{xz}$. Integrate by parts on x noting that the boundary term $B_1(y,z) = [|x-y|\delta_{xz}M_{xy}]_{-\infty}^{\infty} = 0$,

$$I = -\mathrm{i}\int \mathrm{d}y\,\mathrm{d}z \Phi_{yz}|z-y|\partial_z M_{zy} - \mathrm{i}\int \mathrm{d}y\,\mathrm{d}z \Phi_{yz}\,\text{sgn}\,(z-y)M_{zy}. \tag{A.5}$$

The second term is real and does not contribute to $\Im I$. So

$$\partial_t P = \frac{\tilde{g}^2}{8}\Re\int \mathrm{d}x\,\mathrm{d}y \Phi(y,x)|x-y|\partial_x M(x,y) \equiv \frac{\tilde{g}^2}{8}\Re J. \tag{A.6}$$

Integrating by parts, the boundary term vanishes if M falls off sufficiently fast[27]

$$J = B_2 - \int \mathrm{d}x\, \mathrm{d}y M_{xy} \Phi_{yx} \mathrm{sgn}\,(x-y) - \int \mathrm{d}x\, \mathrm{d}y M_{xy} |x-y| \partial_x \epsilon_{yx} - \int \mathrm{d}x\, \mathrm{d}y M_{xy} |x-y| \partial_x M_{yx}. \qquad \text{(A.8)}$$

The first two integrals are imaginary and do not contribute to $\Re J$, so

$$\partial_t P = -\frac{1}{8} \tilde{g}^2 \Re K, \qquad \text{where} \quad K = \int \mathrm{d}x\, \mathrm{d}y M_{xy} |x-y| \partial_x M_{yx}. \qquad \text{(A.9)}$$

Integrating by parts we express $L = K + K^* = 2\Re K = -\int \mathrm{d}x\, \mathrm{d}y |M_{xy}|^2 \mathrm{sgn}\,(x-y) + B_3$. $B_3 = \int \mathrm{d}y [|M_{xy}|^2 |x-y|]_{-\infty}^{\infty}$ is familiar to B_2 (A.7), and vanishes under the same hypothesis. Finally, sgn is odd, so $\partial_t P = -\tilde{g}^2 L/16 = 0$. So P_M is conserved if $|M_{xy}|^2$ decays as $x^{-1-\delta}$ for some $\delta > 0$.

Appendix B. Finite part integrals (Hadamard's *partie finie*)

A finite part integral is like an ODE; rules to integrate the singular measure are like boundary conditions (b.c.). Here we define the $1/p^2$ singular integrals appearing in the potential energy. In position space this is manifested in the linearly rising $|x-y|$ potential. 't Hooft [2] defines them by averaging over contours that go above/below the singularity. Here we formulate them via real integrals and physically motivate and justify the definition by showing that it satisfies the relevant b.c. Both methods use analytic continuation. Consider the rank-1 baryon section 4.1 and suppose support $\tilde{\psi} \subseteq [0, P]$,

$$PE = \int [\mathrm{d}p] \tilde{\psi}(p) ⨍ [\mathrm{d}s] \tilde{\psi}^*(p+s) \tilde{V}(s), \qquad \text{where} \quad \tilde{V}(s) = -s^{-2} \tilde{W}(s). \qquad \text{(B.1)}$$

Recall that $V'' = |\psi|^2$ with two b.c. (22). So $\tilde{V}(s) = -s^{-2} \int [\mathrm{d}q] \tilde{\psi}(s+q) \tilde{\psi}^*(q)$ is singular at $s = 0$. Here $\tilde{W}^*(s) = \tilde{W}(-s)$, i.e. $\Re \tilde{W}(s)$ is even and $\Im \tilde{W}(s)$ is odd[28]. Now, the two b.c. imply

$$V(0) = -⨍ \Re \tilde{W}(s) \frac{[\mathrm{d}s]}{s^2} = \int |\psi(y)|^2 \frac{|y|}{2} \mathrm{d}y, \qquad V'(0) = ⨍ \Im \tilde{W}(s) \frac{[\mathrm{d}s]}{s} = -\frac{1}{2} \int \mathrm{d}y |\psi(y)|^2 \mathrm{sgn}\, y. \qquad \text{(B.2)}$$

The lhs of (B.2) do not exist as Riemann integrals since $\tilde{W}(0) = 1$. But the rhs exists quite often and can be used to define the lhs. For example, the rhs of $V(0)$ makes sense if ψ decays faster than $1/y$. The rhs of $V'(0)$ makes sense as long as $\psi(y)$ decays faster than $|y|^{-\frac{1}{2}}$. This includes $|\psi(y)| \sim 1/|y|$ as $|y| \to \infty$ corresponding to $\tilde{\psi}(p)$ having a jump discontinuity. In particular, it can be used to define $⨍ \tilde{W}(s) s^{-1} \mathrm{d}s$ even when $\tilde{W}'(p)$ is discontinuous at $p = 0$. Now we eliminate ψ and express singular integrals of W in terms of Riemann integrals of

[27] From (1.1) $\epsilon_{yx} \sim \mathrm{i}(\pi x)^{-1}$ as $|x| \to \infty$ for any fixed y. So the first term in B_2 vanishes if $M_{xy} \to 0$ as $|x| \to \infty$:

$$B_2 = \int \mathrm{d}y [\{\epsilon_{yx} + M_{yx}\} |x-y| M_{xy}]_{-\infty}^{\infty}. \qquad \text{(A.7)}$$

The second term in B_2 vanishes iff $\lim_{|x| \to \infty} |M_{xy}|^2 |x-y| = 0$, for any fixed y. This second condition subsumes the first. So $B_2 = 0$ provided $|M_{xy}|^2 \sim |x|^{-1-\delta}$ for some $\delta > 0$. This is easily satisfied by our ansatz $M_o(x, y)$ (19) for the baryon g.s.

[28] From section 4.1.1, if $\tilde{\psi}(p)$ is (dis)continuous at $p = 0$, then so is $\tilde{W}'(s)$ at $s = 0$. If $\tilde{\psi}(p) \sim p^a$, then $\tilde{W}(s) - 1 \sim |s|^{1+2a}$.

W. For simplicity, suppose $\tilde{\psi}(p) \in \mathbf{R}$. Then, $\psi(-x) = \psi^*(x)$, and $\tilde{W}$ is real and even. The $V'(0)$ b.c. (B.2) is satisfied. Let us also restrict our attention to wavefunctions such that $\tilde{\psi}(p) \sim p^a$, $a > 0$ as $p \to 0$. Our aim is to define $-\!\!\!\!\!\int \frac{1}{s^2}\tilde{W}(s)[\mathrm{d}s]$, so as to satisfy the first b.c. The rule should reduce to the Riemann integral, when this quantity is finite to begin with.

Claim. *Let $\tilde{W}(s)$ be even and $\tilde{W}'(0) = 0$ For $P > 0$, if we define*

$$\int_{-P}^{P} \frac{1}{s^2}\tilde{W}(s)[\mathrm{d}s] := \int_{-P}^{P} \frac{\tilde{W}(s) - \tilde{W}(0)}{s^2}[\mathrm{d}s] - \frac{\tilde{W}(0)}{\pi P}, \qquad \text{then}$$
$$\int_{-P}^{P} \frac{1}{s^2}\tilde{W}(s)[\mathrm{d}s] = -\int_{-\infty}^{\infty} |\psi(x)|^2 \frac{|x|}{2}\,\mathrm{d}x. \tag{B.3}$$

Proof. We subtracted divergent terms and analytically continued what we would have got if $\tilde{W}(s)$ vanished sufficiently fast at the origin (i.e. $W(s) \sim s^{1+\epsilon}$, $\epsilon > 0$) to make the integral converge. The main point is that this definition satisfies the $V(0)$ b.c. (B.2). Recall that W is the charge density:

$$\tilde{W}(s) = \int_{-\infty}^{\infty} |\psi(x)|^2 \mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x, \qquad \text{so that} \quad \tilde{W}(s) - \tilde{W}(0) = \int_{-\infty}^{\infty} \mathrm{d}x|\psi(x)|^2(\mathrm{e}^{-\mathrm{i}sx} - 1). \tag{B.4}$$

Moreover, $\tilde{W}'(0) = -\mathrm{i}\int_{-\infty}^{\infty} x|\psi(x)|^2\,\mathrm{d}x = 0$ as the integrand is odd. Therefore, $\tilde{W}(s) - \tilde{W}(0)$ vanishes at least as fast as $s^{1+\epsilon}$, $\epsilon > 0$ as $s \to 0$. For example, for $\tilde{\psi}(p) \propto p^a \mathrm{e}^{-p}$, $\tilde{W}(s) - 1 \propto -s^{2a+1} + O(s^2)$. Therefore, $\int_{-P}^{P}\{\tilde{W}(s) - \tilde{W}(0)\}s^{-2}\,[\mathrm{d}s] < \infty$. As the integrand is even it suffices to consider

$$\int_0^P \frac{\tilde{W}(s) - \tilde{W}(0)}{s^2}[\mathrm{d}s] = \int_0^P \frac{\mathrm{d}s}{2\pi s^2}\int_{-\infty}^{\infty} \mathrm{d}x|\psi(x)|^2(\mathrm{e}^{-\mathrm{i}sx} - 1). \tag{B.5}$$

Only the even part of $(\mathrm{e}^{-\mathrm{i}sx} - 1)$ contributes to the integral on x. Reversing the integrals,

$$\int_0^P \frac{\tilde{W}(s) - \tilde{W}(0)}{s^2}[\mathrm{d}s] = \int_{-\infty}^{\infty} \mathrm{d}x|\psi(x)|^2\left(\frac{1}{2\pi P} - \nu(x)\right). \tag{B.6}$$

This involves the sine integral $2\pi P\nu(x) = Px\mathrm{Si}(Px) + \cos(Px)$. Now $\tilde{W}(0) = 1$, so

$$\int_0^P \frac{\tilde{W}(s) - \tilde{W}(0)}{s^2}[\mathrm{d}s] - \frac{\tilde{W}(0)}{2\pi P} = -\int_{-\infty}^{\infty} \mathrm{d}x|\psi(x)|^2\nu(x). \tag{B.7}$$

We must show that $\nu(x)$ may be replaced by $|x|/4$ under the integral. Since $\mathrm{Si}(t)$ is odd, we have

$$\nu(x) = \frac{|x|}{4} + \frac{1}{2\pi P}\left(Px\mathrm{Si}(Px) - \frac{P|x|\pi}{2} + \cos(Px)\right) = \frac{|x|}{4} + \frac{R(Px)}{2\pi P}, \tag{B.8}$$

where $R(t) = t\mathrm{Si}(t) - |t|\pi/2 + \cos t$. We have the desired result except for a remainder term

$$\int_{-P}^{P} \frac{\tilde{W}(s) - \tilde{W}(0)}{s^2}[\mathrm{d}s] - \frac{\tilde{W}(0)}{\pi P} =$$
$$- \int_{-\infty}^{\infty} |\psi(x)|^2 \frac{|x|}{2}\,\mathrm{d}x - \frac{1}{\pi P}\int_{-\infty}^{\infty} |\psi(x)|^2 R(Px)\,\mathrm{d}x. \tag{B.9}$$

When $P \to \infty$, the remainder term $\to 0$ as $|R(t)| \leqslant 1$. For finite P, $R(t) \sim \frac{-\sin t}{t}$, $|t| \to \infty$ is oscillatory[29], so we expect the remainder term to be small. But it is zero. Consider

$$\int_{-\infty}^{\infty} \mathrm{d}x |\psi(x)|^2 R(Px) = \int_0^P [\mathrm{d}q] \int_{-q}^{P-q} [\mathrm{d}r] \tilde{\psi}(q+r)\tilde{\psi}^*(q) \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}rx} R(Px).$$
$$R(t) \quad \text{is even and} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}rx} R(Px) = 2\int_0^{\infty} \mathrm{d}x \cos(rx) R(Px) = 0, \tag{B.10}$$

from the properties of Si, provided $|r| < P$, which is the region of interest. Thus the remainder term vanishes, and we have shown that our definition of the 'finite part' integral satisfies the b.c. This justifies our definition (B.3) when $\tilde{W}(s)$ is even and $\tilde{W}'(0) = 0$. □

According to (B.3), $⨍_{-P}^{P} \frac{\mathrm{d}r}{r^2} = -\frac{2}{P}$. Moreover, it makes sense to define $⨍_{-P}^{P} \frac{\mathrm{d}r}{r} := 0$ since the integrand is odd. We use these to extend the definition to functions on an even interval $[-P, P]$ but with $W'(0)$ possibly non-zero. Suppose $W(s)$ is continuously differentiable at $s = 0$ with $W(s) - W(0) - sW'(0) \sim s^{1+\epsilon}$ for some $\epsilon > 0$ and s sufficiently small. Then we define

$$⨍_{-P}^{P} \frac{\mathrm{d}s}{s^2} W(s) := \int_{-P}^{P} \frac{\mathrm{d}s}{s^2}[W(s) - W(0) - sW'(0)] - \frac{2}{P} W(0). \tag{B.11}$$

This is used to evaluate $\hat{G}(M_o)$ in appendix C. In general, this rule is applied in a small neighbourhood $[-\epsilon, \epsilon]$ of the singularity. The first term on the rhs of (B.11) vanishes as $\epsilon \to 0$ giving

$$⨍_{-\infty}^{\infty} W(s) \frac{\mathrm{d}s}{s^2} := \lim_{\epsilon \to 0} \left[\left\{ \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right\} W(s) \frac{\mathrm{d}s}{s^2} - \frac{2}{\epsilon} W(0) \right]. \tag{B.12}$$

Appendix C. Interaction operator $\hat{G}$ and $\hat{G}(M)$ for baryonic vacua

$\hat{G}$ is the operator on Hermitian M defining (7) the potential energy $8U = \tilde{g}^2 \operatorname{tr} M\hat{G}(M)$. $\hat{G}(M)$ is a Hermitian matrix with the kernel $G(M)_{xy} = \frac{1}{2} M_{xy}|x-y|$ or $\tilde{G}(M)_{pq} = -⨍ \frac{[\mathrm{d}r]}{r^2} \tilde{M}_{p+r,q+r}$. The null-space of $\hat{G}$ consists of the diagonal $M_{xy} = m(x)\delta(x-y)$, which do not lie on the phase space (3) except for $M = 0$. U is positive definite. The matrix elements of $\hat{G}$ are real

$$\hat{G}_{xy}^{zw} = \frac{1}{2}|x-y|\delta(x-z)\delta(w-y), \qquad \text{where} \quad G(M)_{xy} = \int \mathrm{d}z\, \mathrm{d}w \hat{G}_{xy}^{zw} M_{zw}. \tag{C.1}$$

The entries $\hat{G}_{xy}^{zw}$ are symmetric under a left–right flip $\hat{G}_{xy}^{zw} = \hat{G}_{yx}^{wz}$, which means $M \mapsto G(M)$ preserves hermiticity. Moreover $\hat{G}_{xy}^{zw} = \hat{G}_{wz}^{yx}$, which implies that $\hat{G}$ is Hermitian as an operator on Hermitian matrices (appendix F). In momentum space, $\tilde{G}_{pq}^{rs} = \tilde{G}_{qp}^{sr} = \tilde{G}_{rs}^{pq}$ are real, with $\tilde{G}(M)_{pq} = \int [\mathrm{d}r\, \mathrm{d}s] \tilde{G}_{pq}^{rs} \tilde{M}_{rs}$. Here $\tilde{G}_{pq}^{rs} = -⨍ \frac{[\mathrm{d}t]}{t^2} \delta_{p+t}^{r} \delta_{q+t}^{s}$ and $\delta_p^q \equiv 2\pi\delta(p-q)$. $G(M)_{xy}$ is simple, but the Fourier transform $\tilde{G}(M)_{pq}$ is sometimes more convenient to solve the eom (e.g. section 4.2, (68)). At the baryon vacua $M(\tau)$ (20):

$$\tilde{G}(M(\tau))_{pq} = -\mathrm{e}^{\frac{\mathrm{i}}{2}(p-q)\tau} ⨍ \frac{[\mathrm{d}r]}{r^2} \tilde{M}(0)_{p+r,q+r} = \mathrm{e}^{\frac{\mathrm{i}}{2}(p-q)\tau} G(M_o)_{pq}. \tag{C.2}$$

So it suffices to take $\tau = 0$. For $M_o = -2\psi_o\psi_o^\dagger$ (19) with $\tilde{\psi}_o$ real, $\tilde{G}_{M_o}$ is symmetric. $G(M)_{xy}$ is not of rank 1. But $\psi_o(p+r) \sim \mathrm{e}^{-p}\mathrm{e}^{-r}\theta(p+r)$ factorizes, ensuring that $G(M_o)^{\pm\mp}$ are of rank 1. In general,

$$\tilde{G}(M_o)_{pq} = \frac{2}{P} \exp\left(-\frac{p+q}{2P}\right) ⨍_{\max(-p,-q)}^{\infty} \frac{\mathrm{d}r}{r^2} \mathrm{e}^{-r/P}. \tag{C.3}$$

[29] The asymptotic expansion of Si(t) for large t is $\mathrm{Si}(t) \sim \frac{\pi}{2} + (-\frac{1}{t} + \mathcal{O}(t^{-3}))\cos t + (-\frac{1}{t^2} + \mathcal{O}(t^{-4}))\sin t$.

If p or $q < 0$, then $t \equiv \max(-p, -q) = -\min(p, q) > 0$ and there is no singularity:

$$I_2(t) = \int_t^\infty \frac{\mathrm{d}r}{r^2}\,\mathrm{e}^{-r/P} = \frac{\mathrm{e}^{-t/P}}{t} + \frac{1}{P}\mathrm{Ei}\left(-\frac{t}{P}\right) > 0, \qquad \text{for} \quad t > 0. \quad \text{(C.4)}$$

Here $\mathrm{Ei}(z) = -\int_{-z}^{\infty} \frac{\mathrm{e}^{-u}}{u}\,\mathrm{d}u$. $I_2(t)$ monotonically decays from ∞ to 0 exponentially, as t goes from 0 to ∞. Thus, in the $(p, q) = (-+), (+-)$ and $(--)$ quadrants,

$$\tilde{G}(M_o)_{pq} = \frac{2}{P}\exp\left(-\frac{p+q}{2P}\right)\left(\frac{\mathrm{e}^{-t/P}}{t} + \frac{1}{P}\mathrm{Ei}\left(-\frac{t}{P}\right)\right), \qquad \text{where} \quad t = -\min(p, q) > 0. \quad \text{(C.5)}$$

In the ++ quadrant, $s = \min(p, q) > 0$ so we may write

$$\tilde{G}(M_o)^{++}_{pq} = -\frac{1}{2\pi}\tilde{M}_o(p, q)\mathrm{I}(s), \qquad \text{where} \quad I(s) = \left[\int\!\!\!\!\!\!-_{-s}^{s} + \int_s^\infty\right]\frac{\mathrm{d}r}{r^2}\,\mathrm{e}^{-r/P} = I_1 + I_2. \quad \text{(C.6)}$$

Here $I_1(s)$ is a finite part integral defined in (B.11), and expressed via the sinh integral

$$I_1(s) = \int\!\!\!\!\!\!-_{-s}^{s} \frac{\mathrm{d}r}{r^2}\,\mathrm{e}^{-r/P} := -\frac{2}{s} + \int_{-s}^{s}\frac{\mathrm{d}r}{r^2}\left\{\mathrm{e}^{-r/P} - 1 + \frac{r}{P}\right\} = -\frac{2}{s}\cosh\left(\frac{s}{P}\right) + \frac{2}{P}\mathrm{Shi}\left(\frac{s}{P}\right). \quad \text{(C.7)}$$

Here, $\mathrm{Shi}(z) = \int_0^z \frac{\sinh(t)}{t}\,\mathrm{d}t$. Combining with the previously encountered $I_2(s)$ (C.4),

$$\begin{aligned} I(s) = I_1 + I_2 &= -\frac{1}{s}\,\mathrm{e}^{s/P} + \frac{2}{P}\mathrm{Shi}\left(\frac{s}{P}\right) + \frac{1}{P}\mathrm{Ei}\left(-\frac{s}{P}\right) \\ &= -\frac{1}{s}\,\mathrm{e}^{s/P} + \frac{1}{P}\left(\mathrm{Chi}\left(\frac{s}{P}\right) + \mathrm{Shi}\left(\frac{s}{P}\right)\right). \end{aligned} \quad \text{(C.8)}$$

$\mathrm{Chi}(z) = \gamma + \log z + \int_0^z \frac{\cosh t - 1}{t}\,\mathrm{d}t$. Now we summarize $\tilde{G}(M_o)_{pq}$ in all blocks. Let $s = \min(p, q)$, then

$$\begin{aligned} \tilde{G}(M_o)_{pq} &= \frac{2}{P}\exp\left(-\frac{p+q}{2P}\right)\begin{cases} I_2(-s) = -\frac{1}{s}\,\mathrm{e}^{s/P} + \frac{1}{P}\mathrm{Ei}\left(\frac{s}{P}\right) & \text{if } s < 0 \\ I(s) = -\frac{1}{s}\,\mathrm{e}^{s/P} + \frac{1}{P}\left(\mathrm{Chi}\left(\frac{s}{P}\right) + \mathrm{Shi}\left(\frac{s}{P}\right)\right) & \text{if } s > 0. \end{cases} \\ &= \frac{2}{P}\exp\left(-\frac{p+q}{2P}\right)\begin{pmatrix} I_2(-\min(p,q)) & I_2(-p) \\ I_2(-q) & I(\min(p,q)) \end{pmatrix}. \end{aligned} \quad \text{(C.9)}$$

$I_2(t)$ monotonically decays from ∞ to 0 exponentially, as t goes from 0 to ∞. $I(s)$ monotonically grows from $-\infty$ to ∞ for $0 < s < \infty$. The factor $\frac{2}{P}\exp\left(-\frac{p+q}{2P}\right) = -\frac{1}{2\pi}\tilde{M}_o(p, q)$, but only for $p, q > 0$. $G(M_o)$ inherits some properties of M_o: $G(M_o)^{+-}_{pq} = f(p)g(q)$ is of rank 1 like M_o and $V^{-+}M_o^{++} = 0$ implies that $V^{-+}G(M_o)^{+-} = 0$ (section 5.5). But $G(M_o)$ *does not* commute with M_o, ϵ or Φ_o.

What if $s = \min(p, q) = 0$, which is the boundary of the ++ quadrant? From (C.3), when $s = 0$, $\tilde{G}(M_o)_{pq} \propto \int\!\!\!\!\!-_0^\infty \frac{\mathrm{d}t}{t^2}\mathrm{e}^{-t}$, which cannot be prescribed a finite value[30]. $\tilde{G}(M_o)_{pq}$ is continuous everywhere except along $s = 0$. It approaches $\pm\infty$ as $s \to 0^\pm$. However, its derivative is discontinuous across the line $p = q$. It decays exponentially to zero in all directions except along the positive p- or q-axes.

[30] Recall that $\tilde{G}(M_o)_{pq} = \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y M_o(x, y)|x - y|\mathrm{e}^{-\mathrm{i}(px-qy)}$. For $s = 0$, an oscillatory phase is absent. As $M_o(x, y)|x - y| \sim x^0$, the integral diverges. The divergence is absent on a space of finite length or for $M(x, y)$ decaying faster at infinity.

C.1. Interaction operator $\hat{G}(V)$ in terms of U

Since $V^{--} = 0$ (35) for a tangent to the phase space at the lightest baryon $M_o(t)$, there are some simplifications in $G_V(t)$. Let $s = \max(p, q)$; then $G(V)_{pq} = -\int_{-s}^{\infty} \frac{[dr]}{r^2} \tilde{V}_{p+r,q+r}$. Due to the positive support of $\tilde{M}_{pq}$, G_V^{--} never appears in the eom. In the mostly zero gauge (43)

$$\tilde{G}(V)_{p>q} = 2i\tilde{G}(u\psi^\dagger - \psi u^\dagger + U^{+-})_{p>q}, \qquad \tilde{G}(V)_{p<q} = 2i\tilde{G}(u\psi^\dagger - \psi u^\dagger - U^{-+})_{p<q}. \tag{C.10}$$

Of course, u, ψ, V and U are all time dependent. In particular, if $u = 0$ as in section 5.8, we write compactly

$$G_V^{+-} = 2iG_{U^{+-}}^{+-}, \quad G_V^{-+} = -2iG_{U^{-+}}^{-+} \quad \text{and} \quad G_V^{++} = 2i\{G_{U^{+-}}^{++} - G_{U^{-+}}^{++}\} = 2iG_{U^{+-}}^{++} + \text{h.c.} \tag{C.11}$$

Appendix D. Completing proof that $M(t)$ solves equations of motion

In section 4.2 we studied the time evolution of the baryon states $M(t)$ (20). In the chiral limit, the eom is $\frac{i}{2}\dot{M}_{pq} = \frac{1}{4}M_{pq}(q-p) + \frac{\tilde{g}^2}{4}Z(M(t))_{pq}$ (33). We show here that the interaction terms $\propto Z(t)$ identically vanish for our massless states $M(t)$ (Z stands for zero). Recall that

$$\begin{aligned} Z(t)_{pq} &= \frac{1}{\pi}\left(\frac{1}{p} - \frac{1}{q}\right)\tilde{M}_t(p,q) - G(M_t)_{pq}\{\operatorname{sgn} p - \operatorname{sgn} q\} + [G(M_t), M_t]_{pq} \\ &= Z_1 + Z_2 + Z_3. \end{aligned} \tag{D.1}$$

It is seen that $Z(t)_{pq} = Z(0)_{pq} \exp\left[\frac{i}{2}(p-q)t\right]$. We show here that $Z_{pq} \equiv Z(0)_{pq} = 0$. Now M_o and $G(M_o)$ (appendix C) are real symmetric, so $Z_{1,2,3}(p,q)$ are real antisymmetric. Z_1 is simplest

$$Z_1(p,s) = \pi\left(\frac{1}{p} - \frac{1}{s}\right)\tilde{M}_{ps} = \frac{4}{P}\,e^{-(p+s)/2P}\left(\frac{1}{s} - \frac{1}{p}\right)\theta(p)\theta(s). \tag{D.2}$$

$Z_2(p,s) = -G(M)_{ps}\{\operatorname{sgn} p - \operatorname{sgn} s\}$ vanishes in the $ps = ++, --$ quadrants while

$$(Z_2)_{ps}^{+-} = -2G(M)_{ps}^{+-} \quad \text{and} \quad (Z_2)_{ps}^{-+} = 2G(M)_{ps}^{-+}. \tag{D.3}$$

Since $\tilde{M}_o$ has positive support, $Z_3^{--} = [G(M_o), M_o]^{--} = 0$. So $Z^{--} = 0$. What about the other quadrants? To proceed, we need $G(M_o)_{pq}$, from (C.5). In the $-+, --, +-$ quadrants

$$G(M_o)_{pq} = \frac{2}{P}\,e^{\frac{-(p+q)}{2P}} \begin{cases} -\frac{1}{p}\,e^{p/P} + \frac{1}{P}\text{Ei}(p/P) & \text{if } p < 0,\ p < q \\ -\frac{1}{q}\,e^{q/P} + \frac{1}{P}\text{Ei}(q/P) & \text{if } q < 0,\ q < p. \end{cases} \tag{D.4}$$

This is enough to evaluate Z_2^{+-}, (antisymmetry determines Z_2^{-+}, while $Z_2^{++} = Z_2^{--} = 0$)

$$Z_2^{+-}(p,s) = \frac{4}{P}\,e^{-(p+s)/2P}\left(\frac{1}{s}\,e^{s/P} - \frac{1}{P}\text{Ei}\left(\frac{s}{P}\right)\right) \qquad \text{for} \quad p > 0 > s. \tag{D.5}$$

This is also adequate to find Z_3^{+-} and Z_3^{-+}. For example,

$$Z_3^{+-}(p,s) = -\int_0^\infty [dq]\tilde{M}_{pq}^{++}G(M_o)_{qs}^{+-} = -\frac{4}{P}\,e^{-(p+s)/2P}\left(\frac{1}{s}\,e^{s/P} - \frac{1}{P}\text{Ei}\left(\frac{s}{P}\right)\right). \tag{D.6}$$

We see that $Z_2^{+-} + Z_3^{+-} = 0$. As $Z_1^{+-} = 0$, we conclude $Z^{+-} = 0$. By antisymmetry, $Z^{-+} = 0$.

`++ Block:` here $Z^{++} = Z_1^{++} + Z_3^{++}$ with $Z_3^{++} = \left[G(M_o)^{++}, M_o^{++}\right]$. For Z_3^{++} we need $G(M_o)^{++}_{pq} = \frac{2}{P}\,\mathrm{e}^{-(p+q)/2P}\mathrm{I}[\min(p,q)]$ (C.9). Antisymmetry allows us to consider $0 < p \leqslant s$,

$$Z_3^{++}(p,s) = \frac{4}{P^2}\,\mathrm{e}^{-\frac{p+s}{2P}}\int_0^\infty \mathrm{d}q\,\mathrm{e}^{-\frac{q}{P}}\{\mathrm{I}[\min(q,s)] - \mathrm{I}[\min(p,q)]\}$$
$$= \frac{4}{P^2}\,\mathrm{e}^{-\frac{p+s}{2P}}\left[P\left\{\mathrm{I}(s)\,\mathrm{e}^{-\frac{s}{P}} - \mathrm{I}(p)\,\mathrm{e}^{-\frac{p}{P}}\right\} + \int_p^s \mathrm{d}q\,\mathrm{e}^{-\frac{q}{P}}\mathrm{I}(q)\right]. \qquad \text{(D.7)}$$

This is antisymmetric in p and s, so it is valid for all $p, s > 0$. The integral is expressed as

$$\int_p^s \mathrm{d}q\,\mathrm{e}^{-q/P}\mathrm{I}(q) = \mathrm{e}^{-p/P}\mathrm{Ei}\left(\frac{p}{P}\right) - \mathrm{e}^{-s/P}\mathrm{Ei}\left(\frac{s}{P}\right), \qquad \text{so}$$

$$Z_3^{++}(p,s) = \frac{4}{P^2}\mathrm{e}^{-\frac{p+s}{2P}}\left[\mathrm{e}^{-\frac{p}{P}}\left\{\mathrm{Ei}\left(\frac{p}{P}\right) - P\mathrm{I}(p)\right\} - (s \leftrightarrow p)\right] = \frac{4}{P}\,\mathrm{e}^{-\frac{p+s}{2P}}\left(\frac{1}{p} - \frac{1}{s}\right). \qquad \text{(D.8)}$$

From (D.2) and (D.8), $Z^{++} = Z_1^{++} + Z_3^{++} = 0$. So $Z(t) \equiv 0$ and $M(t)$ (20) solves the chiral eom.

Appendix E. Convergence conditions and inner product on perturbations

The phase space of $\mathrm{QCD}_{1+1}^{N=\infty}$ is the Grassmannian Gr_1 (3, [4]). To define an integer-valued baryon number labelling components of Gr_1, we need the convergence condition $\mathrm{tr}[\epsilon, M]^\dagger[\epsilon, M] < \infty$, i.e. $[\epsilon, M]$ is Hilbert–Schmidt. Applying this to $M = M_o + V$, the condition on a tangent vector V is

$$2\mathrm{tr}[\epsilon, V]^\dagger[\epsilon, M_o] + \mathrm{tr}|[\epsilon, V]|^2 < \infty. \qquad \text{(E.1)}$$

The first term is 0 for the g.s. $M_o = -2\psi\psi^\dagger$ with $\epsilon\psi = \psi$, since $[\epsilon, M_o] = 0$. Decomposing V in blocks (35), (E.1) becomes $\mathrm{tr}\,V^{+-}V^{-+} < \infty$, i.e. V^{+-} is H-S. Also, $\mathrm{tr}\,V^{++} < \infty$ must be trace class (section 4.1 of [4]). There is a natural positive-definite symmetric inner product $(V, \underline{V}) = \mathrm{tr}\,V\underline{V}$ on the tangent space to Gr_1, if we further assume that V^{--} and V^{++} are H-S. We use it to define self-adjointness of the Hamiltonian for linearized evolution in (71). At the baryon g.s., $M_o = -2\psi\psi^\dagger$, $V^{--} = 0$, so writing $V = \mathrm{i}[\Phi_o, U]$ and expressing U in the mostly zero gauge (43), the inner product is

$$(V, \underline{V}) = \mathrm{tr}\,V\underline{V} = 2\Re\,\mathrm{tr}\,V^{-+}\underline{V}^{+-} + \mathrm{tr}\,V^{++}\underline{V}^{++} = 4(U, \underline{U}) = 8\Re\,\mathrm{tr}(U^{-+}\underline{U}^{+-} + u\underline{u}^\dagger). \qquad \text{(E.2)}$$

Appendix F. Hermiticity of a linear operator on Hermitian matrices

A transformation $U \mapsto K(U)$ on Hermitian matrices must preserve hermiticity. If $K(U)_{pq} = \hat{K}_{pq}^{rs}U_{rs}$, this becomes $\left(\hat{K}_{pq}^{rs} - \hat{K}_{qp}^{sr*}\right)U_{rs} = 0\ \forall$ Hermitian U. We cannot conclude $\hat{K}_{pq}^{rs} = \hat{K}_{qp}^{sr*}$, this is not necessary as $U_{rs} = U_{sr}^*$ are not independent. We go to a basis for Hermitian matrices

$$[R_{ab}]_{pq} = \delta_{ap}\delta_{bq} + \delta_{aq}\delta_{bp}, \qquad [I_{ab}]_{pq} = \mathrm{i}(\delta_{ap}\delta_{bq} - \delta_{aq}\delta_{bp}), \qquad \text{(F.1)}$$

and deduce the necessary and sufficient conditions[31] for $\hat{K}$ to preserve hermiticity of U

$$\hat{K}_{pq}^{[rs]} = \hat{K}_{qp}^{[rs]*} \qquad \text{and} \qquad \hat{K}_{pq}^{\{rs\}} = -\hat{K}_{qp}^{\{rs\}*}. \qquad \text{(F.2)}$$

[31] Here, $K_?^{[rs]} = K_?^{rs} + K_?^{sr}$ and $K_?^{\{rs\}} = K_?^{rs} - K_?^{sr}$ while ? is held fixed.

What does it mean for such a $\hat{K}$ to be formally self-adjoint? The space of Hermitian matrices has the inner-product $(U, U') = \mathrm{tr}\, U U'$. So self/skew-adjointness is the condition

$$(\hat{K}U, U') = \pm(U, \hat{K}U') \qquad \text{or} \qquad \mathrm{tr}\, K(U)U' = \pm\mathrm{tr}\, U K(U')\ \forall U, U' \text{ Hermitian.} \tag{F.3}$$

So $\forall$ Hermitian U, U': $\hat{K}^{qp}_{sr} U_{qp} U'_{rs} = \pm\hat{K}^{rs}_{pq} U_{qp} U'_{rs}$. A sufficient condition for $\hat{K}$ to be self/skew-adjoint is (anti-)symmetry under left–right *and* up–down flips of indices: $\hat{K}^{qp}_{sr} = \pm\hat{K}^{rs}_{pq}$. Using (F.1), necessary and sufficient conditions for self/skew-adjointness of $\hat{K}$ are

$$\hat{K}^{[ab]}_{[cd]} = \pm\hat{K}^{[cd]}_{[ab]}, \qquad \hat{K}^{\{ab\}}_{\{cd\}} = \pm\hat{K}^{\{cd\}}_{\{ab\}} \qquad \text{and} \qquad \hat{K}^{\{ab\}}_{[cd]} = \mp\hat{K}^{[cd]}_{\{ab\}}. \tag{F.4}$$

Appendix G. Space of physical states consistent with $u = 0$ ansatz

The physically motivated (section 5.8) ansatz $u = 0$ led to a Hermitian eigenvalue problem for the baryon spectrum (68). We imposed it so that the equation for perturbations around the g.s. (62) admits oscillatory solutions via variable separation, by removing simultaneous dependence on both U^{+-} and U^{-+}. $U^{+-} : \mathcal{H}_- \to \mathcal{H}_+$ must be H-S (appendix E) and respect the gauge $\psi^\dagger U^{+-} = 0$ and consistency condition (66) for $u(t)$ to remain 0. Here we examine (66). Momentum-dependent phases (67) cancel, leaving

$$\mathrm{e}^{\mathrm{i}\omega t} U^{+-} G^{-+}_M \psi + 2(1 - P_\psi)\left(\mathrm{e}^{\mathrm{i}\omega t} G^{++}_{U^{+-}}\psi - \mathrm{e}^{-\mathrm{i}\omega t} G^{++}_{U^{-+}}\psi\right) = 0. \tag{G.1}$$

So the coefficients of $\mathrm{e}^{\pm\mathrm{i}\omega t}$ must vanish, leaving two time-independent vector conditions

$$(\mathrm{A}) : \left\{U^{+-}G^{-+}_M + 2(1^{++} - P_\psi)G^{++}_{U^{+-}}\right\}\psi = 0 \quad \text{and} \quad (\mathrm{B}) : (1^{++} - P_\psi)G^{++}_{U^{-+}}\psi = 0, \tag{G.2}$$

on a whole operator U^{+-}. We expect a large space of solutions U^{+-}. Equation (G.2) states that ψ is annihilated by a pair of operators built from U^{+-}: another type of orthogonality between the ground/excited states. (B) is simpler than (A). Introducing an arbitrary $n \in \mathcal{H}_-$ and $\lambda \in \mathbf{C}$,

$$(B) : (1^{++} - P_\psi)G^{++}_{U^{-+}}\psi = 0 \quad \Leftrightarrow \quad G_{U^{-+}}\psi = \lambda\psi + n. \tag{G.3}$$

Let us look for rank-1 solutions $U^{+-} = \phi\eta^\dagger$ with $\phi, \eta \in \mathcal{H}_\pm$, the sea and antiquark wavefunctions of the meson V bound to the baryon M_o. We solve for $\phi^*(x) = \frac{1}{\psi}\left(\frac{\lambda\psi+n}{\eta}\right)''$. For ϕ to lie in $\mathcal{H}_+$, $\phi^*(x)$ must necessarily be analytic in $\mathbf{C}^-$.[32] We argue that this requires $\lambda = 0$. $\psi(x) \propto (c - \mathrm{i}x)^{-1}$ does not have zeros (20), but it has a pole in $\mathbf{C}^-$, which cannot be cancelled by either $\eta(x)$ or $n(x)$, both of which are analytic in $\mathbf{C}^-$. Thus $\lambda = 0$, and in particular $G(\eta\phi^\dagger)^{++}\psi = 0$: an interaction operator built from U annihilates the g.s. So rank-1 solutions of (B) are of the form $\phi^*(x) = \frac{1}{\psi}(n/\eta)''$, parameterized by vectors $n, \eta \in \mathcal{H}_-$.[33] For e.g., $\eta = (a+\mathrm{i}x)^{-2} \in \mathcal{H}_-$, $n = (a+\mathrm{i}x)^{-m} \in \mathcal{H}_-$, $m > 2$, and $\phi \propto (2Px - \mathrm{i})(a - \mathrm{i}x)^{-m} \in \mathcal{H}_+$ is a family of solutions of (B) with $P, a > 0$.

We have not yet solved (A) in such generality. Here we give a restricted class of solutions of (A), where each term of (A) is zero. For $U^{+-} = \phi\eta^\dagger$ we get two conditions on ϕ and η:

$$(\mathrm{A1})\ \phi\left(\eta^\dagger G^{-+}_M\psi\right) = 0 \qquad \text{and} \qquad (\mathrm{A2})\ (1^{++} - P_\psi)G^{++}_{\phi\eta^\dagger}\psi = 0. \tag{G.4}$$

[32] A necessary (but not sufficient) condition for $\tilde{\psi}(p)$ to be a positive momentum function ($\psi \in \mathcal{H}_+$), is for $\psi(x)$ to be the boundary value of a function holomorphic in the upper half of the complex x plane $\mathbf{C}^+$.

[33] We have not *proved* that $\phi \in \mathcal{H}_+$. There may be more conditions on n, η to *guarantee* $\phi \in \mathcal{H}_+$.

(A1) $\Rightarrow$ $\eta^\dagger G_M^{-+}\psi = 0$: the antiquark wavefunction must be $\perp$ to $G_M^{-+}\psi$.[34] For $P = 1$, $\tilde{\eta}_p = p(p + 0.474)\mathrm{e}^p\theta(-p)$ is such a function. (A2) $\Leftrightarrow$ $G_{\phi\eta^\dagger}\psi = \lambda'\psi + m$ for arbitrary $\lambda' \in \mathbf{C}$ and $m \in \mathcal{H}_-$. (A2) resembles (B), but they are not the same though ${G_{U^{-+}}}^\dagger = G_{U^{+-}}$. We solve (A2) for $\eta^*(x) = \frac{1}{\psi}\left(\frac{\lambda'\psi+m}{\phi}\right)''$. As before, there are conditions for this η to lie in $\mathcal{H}_-$. But it is possible that (A1) and (A2) form too small a class of solutions of (A). We have not yet combined (A) and (B) to find U^{+-} obeying (G.2). We hope to remedy this in the future.

References

[1] Amsler C *et al* (Particle Data Group) 2008 *Phys. Lett.* B **667** 1
[2] 't Hooft G 1974 *Nucl. Phys.* B **72** 461
't Hooft G 1974 *Nucl. Phys.* B **75** 461
[3] Witten E 1979 *Nucl. Phys.* B **160** 57
[4] Rajeev S G 1994 *Int. J. Mod. Phys.* A **9** 5583 (arXiv:hep-th/9401115)
[5] Rajeev S G 1999 Derivation of the hadronic structure functions from quantum chromodynamics arXiv:hep-th/9905072
[6] Karabali D, Kim C J and Nair V P 1998 *Phys. Lett.* B **434** 103 (arXiv:hep-th/9804132)
[7] Rajeev S G 2004 Three-dimensional simplicial Yang–Mills theory: an approach to the mass gap arXiv:hep-th/0401202
[8] Hornbostel K, Brodsky S J and Pauli H C 1990 *Phys. Rev.* D **41** 3814
[9] Krishnaswami G S 2004 Large-*N* limit as a classical limit: baryon in two-dimensional QCD and multi-matrix models arXiv:hep-th/0409279
[10] Krishnaswami G S and Rajeev S G 1998 *Phys. Lett.* B **441** 429 (arXiv:hep-ph/9807345)
[11] John V, Krishnaswami G S and Rajeev S G 2000 *Phys. Lett.* B **487** 125 (arXiv:hep-ph/0310027)
[12] John V, Krishnaswami G S and Rajeev S G 2000 *Phys. Lett.* B **492** 63 (arXiv:hep-th/0310014)
[13] Belitsky A V and Radyushkin A V 2005 *Phys. Rep.* **418** 1 (arXiv:hep-ph/0504030)

[34] If ψ_o is the baryon g.s. (19), η must be $\perp$ to $(G_M^{-+}\psi)_{p<0} = \sqrt{\frac{2}{\pi P}}\mathrm{e}^{-\frac{p}{2P}}\left\{-\frac{\mathrm{e}^{\frac{p}{P}}}{p} + \frac{1}{P}\mathrm{Ei}(\frac{p}{P})\right\}$. $(G_M^{-+}\psi)_p$ is positive and exponentially decays monotonically from ∞ to 0 as p goes from 0 to $-\infty$. $(G_M^{-+}\psi)_p \sim -\frac{1}{p}\sqrt{2/\pi P}$ as $p \to 0^-$. To avoid IR divergences, $\tilde{\eta}(p) \sim (-p)^\gamma$ for some $\gamma > 0$ as $p \to 0^-$.