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Naturalness via scale invariance and non-trivial UV fixed points in a 4d $O(N)$ scalar field model in the large-N limit

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Abstract

We try to use scale-invariance and the $1/N$ expansion to construct a non-trivial 4d $O(N)$ scalar field model with controlled UV behavior and naturally light scalar excitations. The principle is to fix interactions at each order in $1/N$ by requiring the effective action for arbitrary background fields to be scale-invariant. We find a line of non-trivial UV fixed-points in the large-N limit, parameterized by a dimensionless coupling. Nether action nor measure is scale invariant, but the effective action is. Scale invariance makes it natural to set a mass deformation to zero. The model has phases where $O(N)$ invariance is unbroken or spontaneously broken. Masses of the lightest excitations above the unbroken vacuum are found. Slowly varying quantum fluctuations are incorporated at order $1/N$. We find the $1/N$ correction to the potential, beta function of mass and anomalous dimensions of fields that preserve a line of fixed points for constant backgrounds.

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Contents

1 Introduction

We investigate the naturalness concept of 't Hooft [1] applied to 4d $O(N)$ scalar fields. If there are scalar particles very light compared to the Planck mass, it must be due to a symmetry. We observe that one non-trivial scale-invariant RG fixed point in *quantum* scalar field theory would be enough to make small masses natural. For, setting masses to zero would buy us symmetry under scale transformations. We try to implement this idea by developing a method due to Rajeev [2] for constructing a fixed point in the large-N limit.

1.1 Background and motivations

Many discussions of 4d quantum field theory begin with massive $\lambda\phi^4$ theory. However, this model most likely does not have a non-trivial continuum limit [3, 4, 5, 6]. Our first motivation is to answer the question: 'Can one construct a non-trivial 4d scalar field theory?'

Our second motivation comes from particle physics. The 2004 Nobel prize in physics for the discovery of asymptotic freedom in QCD has reminded us about the physical importance of quantum field theories with well-controlled ultraviolet behavior. Indeed, Yang-Mills theories, which have a gaussian high energy fixed point are at the heart of our best models for both the strong and weak interactions. In equilibrium statistical mechanics of $O(N)$ magnets, the gaussian fixed point controls high energy behavior while the lower energy dynamics is governed by a crossover to the non-trivial Wilson-Fisher fixed point [7].

However, the situation in 4d massive $\lambda\phi^4$ theory, which is the simplest and currently favored (but experimentally unconfirmed) model for W and Z mass generation, is less satisfactory for two reasons. First, $\lambda\phi^4$ theory is based on the gaussian infrared fixed point, but does not flow to any UV fixed point to control the high energy behavior. Perturbatively, interactions become very strong at a finite energy $\sim m \exp\left[\frac{16\pi^2}{3\lambda}\right]$ where m, λ are the parameters of the model in the infrared (Landau pole). This is in contrast with asymptotically free theories or theories based on an interacting UV fixed point which might (at least in principle) be defined in the UV. Both numerical [3, 4] and analytical[6, 5] calculations suggest that in the absence of a UV cutoff, the theory is 'trivial'[1]. Unfortunately, the non-trivial Wilson-Fisher fixed point in quartic scalar field theory in $4-\epsilon$ dimensions merges with the gaussian fixed point in 4d. As a practical matter, the lack of a UV fixed point in $\lambda\phi^4$ theory does not prevent us from using it as an effective theory with a cutoff or as a perturbatively defined model like QED, over a range of relatively low energies.

The second problem with $4d$ $\lambda\phi^4$ theory, is the naturalness problem[2]. In QED, a small electron mass (compared to m_{Planck}) is natural because setting it equal to zero gives QED an additional symmetry, chiral symmetry, which is not broken by quantum effects. On the other hand, in massive $\lambda\phi^4$ theory, setting $m=0$ makes the classical theory free of any dimensional parameter. But scale-invariance is broken in the quantum theory due to the absence of any regularization and renormalization scheme that preserves scale-invariance, there is a scale anomaly. In the absence of any symmetry to explain a small scalar mass in the quantum theory, naturalness suggests that the lightest scalar excitation, the Higgs particle should have a mass of order the Planck mass. A very large Higgs mass, however, leads to other problems since (perturbative,

[1] The renormalized coupling constant vanishes identically and correlations satisfy Wick's formula.
[2] See appendix A for an explanation with examples.

essentially tree level) unitarity would then be violated [8]. The perturbative unitarity bound from a partial wave analysis of scalar exchange in W-boson scattering is estimated to be of the order of 1 TeV. Moreover, the likely triviality of the continuum theory implies a 'triviality bound' on the Higgs mass, which is also of the same order [9]. There does not seem to be any non-perturbative cure for these problems, they also arise on the lattice [3, 4] and in other analytical approaches [5, 6].

Another issue is that the 1-loop correction to the square of the bare Higgs mass is quadratic in the momentum cutoff and leads to the fine-tuning problem. If a large cutoff is to be maintained, either the effective Higgs mass is of the order of the cutoff or the bare mass must be fine-tuned to cancel the radiative correction (at each order). We already mentioned the difficulties with a very large Higgs mass. However, the fine-tuning problem needs to be treated with some care. In a renormalizable QFT (eg. $\lambda\phi^4$), the use of a cutoff as a regulator is a matter of convenience. The quadratic divergence is absent in dimensional or ζ-function regularization or Epstein-Glazer renormalization. In the end of a calculation, one sends all regulators to their limiting values before making physical conclusions. From this standpoint, a quadratically divergent self energy is not a deficiency of the model.

On the other hand, one may be of the opinion that the QFTs of particle physics are effective field theories that come with a physical large momentum cutoff at either the Planck or other scale where new effects render the standard model inaccurate. This is similar to ones attitude in some (but not all) condensed matter physics contexts, where the crystal lattice is physical and not merely a convenient regulator. In this view, renormalizability is expendable. The cutoff is not to be sent to infinity, and the fine tuning problem mentioned above is indeed present.

In QCD or other renormalizable models that are based on a UV fixed point (gaussian or not) to control high energy behavior, there is no need for the 'effective field theory-physical cutoff' view point. For, these models self-consistently predict low energy behavior irrespective of what the physics beyond the standard model may be or at what scale it may kick in. But in models such as $\lambda\phi^4$, which (despite being renormalizable) seem to be non-trivially defined only in the presence of a cutoff, the latter viewpoint cannot be ignored. This leads us to an important distinction between the naturalness concept and fine tuning. The former requires any model to have an extra symmetry to explain a small parameter. But fine tuning, for its very definition, needs the model to have a physical cutoff. Fine tuning is potentially an issue for non-renormalizable models or models which need a cutoff in order to be non-trivial (eg. $\lambda\phi^4$).

There is an empirical relation between naturalness and degree of fine tuning. Suppose we dogmatically insist on working with a cutoff even though our model may be renormalizable. Assume also that the model is natural, i.e. gains a symmetry if masses are set to zero. Then it is often the case that radiative corrections are 'protected by the symmetry': self energies are only logarithmically divergent rather than as a power of the cutoff. An example is QED. Due to chiral symmetry, the renormalization of the electron mass[2] is not quadratically divergent as one might naively expect, but only logarithmically divergent (the quadratic divergence persists in scalar QED, which does not have chiral symmetry).

Our conclusions are (1) it is worth looking for a non-trivial scalar field model based on a UV fixed point; (2) it is worthwhile to look for a symmetry that would ensure naturally light scalar excitations and (3) If (1) and (2) are achieved, it is less important to worry about fine-tuning in the presence of a cutoff.

Despite our conceptual criticisms of $\lambda\phi^4$ theory, as long as a light Higgs is discovered, it will be possible to use the model to predict scattering amplitudes at relatively low energies. Thus it remains the default mechanism for giving masses to the weak gauge bosons. It may turn out to be an effective description of some more intricate framework devised by nature.

Many alternatives have been proposed. Most studied is the use of supersymmetry to ensure light scalars [10]. The challenge here is to break SUSY without introducing new naturalness problems. Technicolor models try to realize scalars as fermion composites [11]. Little Higgs models [12] attempt to use the Nambu-Goldstone mechanism in a novel way. In the Coleman-Weinberg mechanism [13], classical conformal invariance of massless $\lambda\phi^4$ theory is used to explain small scalar masses. But this is not a naturalness explanation in the strict sense since conformal invariance is broken quantum mechanically. An idea to use scale invariance in a manner analogous to SUSY and soft SUSY breaking, has been proposed [14]. Other approaches include higher derivative models to expel one-particle scalar excitations as asymptotic states [15]; and the possibility of the gaussian fixed point being UV with respect to some non-polynomial potentials [16].

Richter [17] has criticized naturalness. However, our definition of naturalness is not quite the same as the one he uses (in particular, a quadratic divergence in self-energy is not unnatural by our definition). Moreover, his criticisms seem to have more to do with the large number of parameters in the MSSM, than with naturalness as we understand it. We hope the examples in appendix A will help to improve the dismal score he gives the concept! Finally, one hopes that experiments at the LHC will add empirical discrimination to discussions on naturalness and electro-weak symmetry-breaking.

1.2 Reasoning and summary

Motivated by these considerations, we investigate whether there is any resolution to the difficulties of $\lambda\phi^4$ theory that does not require adding new parameters or degrees of freedom beyond those of the standard model[3]. Can we construct an interacting scalar field theory in 4d, which has good UV behavior and supports naturally light scalar excitations? We argue that one possible scenario is that such a model should be built around a nontrivial UV fixed point. Existing work does not indicate (indeed, almost rules out) a non-trivial fixed point in the neighborhood of the gaussian fixed point[4], so we will look farther afield.

The two key concepts of this paper are (1) naturally light scalars from scaling symmetry in the quantum theory and (2) non-trivial quantum effective actions via cancelation between 'action' and 'quantum fluctuations'.

To find non-trivial fixed points far from the gaussian one, we give up thinking of a QFT as being defined by a classical action. The reason the classical action is a useful concept is that it provides a first approximation to the quantum effective action in the $\hbar \to 0$ limit. Indeed, the contribution of the path integral measure is suppressed in this limit. By contrast, in the zeroth order of the approximation method we propose, both the 'action' and the quantum fluctuations from the 'measure' are comparable. In fact, both are strictly infinite prior to regularization and neither is scale invariant if regulated. However, their combined effect at the zeroth order of our scheme is to produce a finite 1-parameter family of scale invariant (quantum) effective

[3] In this paper we work in an approximation where gauge and Yukawa couplings vanish.

[4] Halpern and Huang [16] have argued that there may be potentials with respect to which the gaussian fixed point is UV. But this scenario is quite different from what we propose.

actions when regulators are removed. It is the latter that is physically observable and defines the theory. Such a scenario is not amenable to analysis via the loop expansion, traditional perturbation theory or a weak field expansion in powers of the scalar field. In particular, we give up perturbative renormalizability, but require it in a non-perturbative sense so as to ensure predictive power. We hope this is excusable since despite its great success elsewhere, insisting on perturbative renormalizability in scalar field theory leads to a model that is unnatural, lacking in good UV behavior and most likely trivial.

To embark on the formidable task of constructing a non-trivial fixed point far from the gaussian one, it will help to have a small parameter to expand in. We investigate whether there is a 4d scale-invariant interacting $O(N)$ scalar field theory in the $1/N$ expansion. The case of interest in particle physics is $N = 4$, since in the absence of gauge and Yukawa couplings, the symmetry group of the standard model is $O(4)$. Our approach is inspired by work on Yang-Mills theory, where we have learned that the theory has two different "classical" limits in which quantum fluctuations of some observables are small. Both the $\hbar \to 0$ and $N \to \infty$ limits (holding the other parameter fixed) can be used as starting points for studying the full quantum theory. Might the large-N limit be of use in constructing a scale-invariant 4d scalar field theory? This possibility was pointed out by Rajeev [2].

If scale-invariance could be maintained even after including quantum effects, then such a scale-invariant model would define a fixed point. The way we achieve this, is to pick an action which is not scale-invariant, so as to cancel the 'scale anomaly' from quantum fluctuations. In the large-N limit, we actually find a line of fixed points[5] parameterized by λ. λ is the dimensionless coupling constant of a $\lambda\phi^4$-type term which is marginally irrelevant when considered around the gaussian fixed-point but whose beta function vanishes in the large-N limit when considered around the non-trivial fixed point. In the spirit of Wilsonian renormalization, we should study the most relevant deformation of the line of fixed points. This corresponds to a mass term. Setting mass to zero should be natural, since we would gain scale-invariance by doing so. This line of scale-invariant theories would be UV fixed points with respect to the mass deformation and thus ensure good high energy behavior. For naturally light scalars via scaling symmetry, it is sufficient to have one fixed point. It may well be that when $1/N$ corrections are included, scale invariance can be maintained only for one[6] value of $\lambda = \lambda_0$. This would be acceptable, since $m = 0, \lambda = \lambda_0$ would be natural due to scale invariance at that point. It would be a bonus if scale-invariance reduces the degree of divergence of self energy in the presence of a cutoff, just as chiral symmetry does in QED. But as a matter of principle, this is not necessary as long as the model is renormalizable and non-trivial when regulators are removed. Even more importantly, we would like locality, causality and unitarity. Locality is likely to be a subtle issue. We must also investigate the low energy behavior as we flow away from one of the UV fixed points along a mass deformation.

Our model is to be constructed order by order in the $1/N$ expansion by requiring that the effective action for arbitrary backgrounds be scale-invariant. The physical output of this procedure is the effective action, not the classical action. Our starting point is a 4d $N+1$ component[7] Euclidean scalar field model whose partition function can be written formally in

[5]A line of fixed points has appeared previously in 4d QFT. In $\mathcal{N} = 4$ super Yang-Mills theory, g_{YM}^2 is believed to parameterize a line of *quantum* fixed points. In a sense, things are simpler in $\mathcal{N} = 4$ SYM since $g_{YM}^2 = 0$ is the trivial fixed point and the line of fixed points may be studied perturbatively.

[6]There may be no non-trivial fixed point when $1/N$ corrections are incorporated, then our scenario would fail.

[7]It is convenient to work with $N+1$ components $\phi_0, \phi_1, \cdots \phi_N$ since we will integrate out all but ϕ_0.

terms of an $O(N+1)$ invariant potential V

$$Z = \int [D\phi] \exp\Big[-\frac{1}{\hbar}\int d^4x \Big\{\frac{1}{2}|\nabla\phi_i|^2 + NV(\frac{|\phi|^2}{N})\Big\}\Big]. \quad (1)$$

It is analyzed by introducing an additional field σ via a Laplace transform so that the action is quadratic in ϕ_i. $\sigma(x)$ is the Laplace conjugate[8] of the $O(N+1)$ singlet $\eta = \frac{\phi_i\phi_i}{N}$ (section 2). The reason to work with σ instead of ϕ_i is that in the large-N limit holding $\hbar \neq 0$ fixed, σ has small quantum fluctuations, while ϕ_i continue to have large fluctuations. The functional integral over all but one of the ϕ_i is performed, leaving only $b = \frac{\phi_0}{\sqrt{N}}$ and σ as dynamical fields. Keeping b facilitates discussion of $O(N+1)$ symmetry breaking.

The goal of a QFT is the determination of the (quantum) effective action. In section 3.2 we show that in the large-N limit holding $\hbar$ fixed, there is a one parameter family of finite and scale-invariant effective actions when regulators are removed. For background fields $B(x), \Sigma(x)$, it is obtained in an expansion around a constant background Σ_o. In zeta-function regularization,

$$\Gamma_0(B,\Sigma) = \frac{1}{2}\int d^4x \Big[(\nabla B)^2 + \Sigma B^2 + \lambda\Sigma^2 - \frac{\hbar}{16\pi^2}\Big\{(\Sigma-\Sigma_o)\Pi(\Delta)(\Sigma-\Sigma_o) + \mathcal{O}(\Sigma-\Sigma_o)^3\Big\}\Big]. \quad (2)$$

Here $\Delta = -\nabla^2/\Sigma_o$ and $\Pi(\Delta)$ is a specific function determined in appendix C.3. Σ_o is arbitrary[9], it is *not* a parameter of the theory. It appears merely because we are studying the theory around a constant background field. $\Gamma_0(B,\Sigma)$ depends on the regularization scheme. For instance, to relate two schemes for constant backgrounds, λ must be shifted by a finite additive constant which we have determined for zeta function, cutoff (sec. 3.1.1) and dimensional regularization (appendix B.1). The principle that the effective action be scale-free leaves the dimensionless coupling λ undetermined, and it parameterizes a line of fixed points. In section 3.4 we show that Γ_0 is free of scale anomalies. Effective actions in QFT usually involve proper vertices of all orders and are very complicated, ours is no exception. What is perhaps unusual is that unlike in the zeroth order of the loop expansion of $\lambda\phi^4$ theory, the effective action of our model in the zeroth order of the $1/N$ expansion already involves vertices of all orders. This merely reflects that in the large-N limit, we have already incorporated *all* quantum fluctuations of $\phi_1 \cdots \phi_N$.

The potential $V(\phi^2/N)$ that leads to this effective action is not physically well-defined. The same is true about the quantum fluctuations coming from the path integral measure. Unlike the effective action, neither V nor the measure is finite when regulators are removed. Furthermore, neither the quantum fluctuations nor V is scale invariant. They depend on a scale M, which however mutually cancels to give a finite and scale free effective action. $V(\phi^2/N)$ is not in any sense an approximation to the effective potential i.e., Γ for constant backgrounds. Nevertheless, V does appear in intermediate stages of calculations and many people want to know what it is. Its 'finite part' in zeta function regularization in the large-N limit, grows as $V(\frac{|\phi|^2}{N}) \sim \frac{|\phi|^4/N^2}{\log(\lambda|\phi|^2/M^2N)}$ for large $|\phi|^2/N$ (appendix D). We have not yet determined its behavior for small $|\phi|^2/N$, though there is no indication of singular behavior. $V(\eta)$ is most easily expressed in terms of the Laplace transformed potential $W(\sigma)$. At $N=\infty$ the finite part of $W(\sigma)$ in zeta function regularization is (e is the base of natural logarithms)

$$W_0(\sigma) = \lambda\sigma^2 - \frac{\hbar}{16\pi^2}\Big\{\frac{1}{2}\sigma_o^2\log[\frac{\sigma_o e^{-3/2}}{M^2}] + \sigma_o\log[\frac{\sigma_o}{M^2 e}](\sigma-\sigma_o) + \frac{1}{2}\log[\frac{\sigma_o}{M^2}](\sigma-\sigma_o)^2\Big\}. \quad (3)$$

[8] σ is valued on a contour $\mathcal{C}$ from $-i\infty$ to $i\infty$. Within our approximations, $\mathcal{C}$ must avoid $\mathbf{R}^-$.

[9] When Σ_o is a non-positive real number there are further divergences, which we have not yet treated.

σ_o is the constant background value of the field $\sigma(x)$ appearing in the effective action. Though Γ_0 involves arbitrarily high derivatives, W_0 does not, indicating a form of locality. It is interesting to know whether this remains true in other regularization schemes and after including $1/N$ corrections.

Next, we deform this line of fixed points by adding a mass term $-m^2\sigma$ to $W(\sigma)$, which explicitly breaks scale-invariance. In section 4.1 we determine constant extrema of the large-N effective action and find a phase $b = 0, \sigma = \frac{m^2}{2\lambda}$ where $O(N+1)$ is unbroken and another phase $b = \pm m, \sigma = 0$, where $O(N+1)$ symmetry is spontaneously broken to $O(N)$. In section 4.2 we determine the masses of the lightest scalar excitations in the $O(N+1)$ symmetric phase

$$M_b = \frac{m}{\sqrt{2\lambda}} \qquad \text{and} \qquad M_\sigma = \frac{4\sqrt{3}\pi m}{\sqrt{\hbar}}. \tag{4}$$

We are not yet able to study the broken phase; it occurs at $\sigma = 0$. Though $W(\sigma)$ is singular at $\sigma = 0$, the effective action Γ may be regular there (the effective potential is), we do not know.

In section 5 we attempt to preserve scale-invariance after including quantum fluctuations in σ and b at order $1/N$. Some quantum effects were already present at $N = \infty$, due to fluctuations of $\phi_1, \cdots, \phi_N$ which we integrated out. Determining the effective action at order $1/N$ is complicated, so we make some further approximations which we hope to relax later. In sec. 5.1.1 we assume that quantum fluctuations are slowly varying and determine the $1/N$ correction to the fixed point potential, $W_0 + \frac{1}{N}W_1$ that ensures the effective action $\Gamma(B = 0, \Sigma) = \Gamma_0 + \frac{1}{N}\Gamma_1$ is finite & scale-invariant for general background fields $\Sigma(x)$. But there could be further divergences not canceled by $W_1(\Sigma)$, since it is independent of B. In sec. 5.1.2 we drop the assumption $B = 0$, but assume that background fields are constant and derive a Callan-Symanzik renormalization group equation (RGE) for the effective action in zeta function regularization. Imposing the condition that the β function of λ vanishes and using the previous result for $W_1(\Sigma)$ leads to a unique solution for the β-function of the mass and anomalous dimensions of b and σ fields. Within our (drastic) approximations, the model is renormalizable and the line of fixed points is maintained at order $1/N$. It remains to analyze the phase in which $O(N+1)$ symmetry is broken, and also couple our model to fermions and gauge fields. Finally, we wonder whether there might be a dual/holographic/string description of our scale-invariant model by analogy with the AdS/CFT conjecture [18].

2 Lagrangian and change of field variables

Consider an $N+1$ component real scalar field $\phi_i, i = 0, 2, \cdots, N$ in 4 Euclidean space-time dimensions. We require the action to be globally $O(N+1)$ invariant. The partition function is

$$Z = \int [D\phi] e^{-\frac{1}{\hbar}\int d^4x \frac{1}{2}\{|\nabla\phi_i|^2 + NV(\phi^2/N)\}}. \tag{5}$$

The factors of N have been chosen to facilitate a meaningful large-N limit. ϕ^2 is short for $\sum_{i=0}^N \phi_i\phi_i$. $V(\frac{\phi^2}{N})$ will be determined by requiring the quantum effective action to be scale-invariant. We will find the effective action in an expansion around $N = \infty$ holding $\hbar$ fixed. In this limit, $\eta(x) = \phi^2/N \geq 0$ has small fluctuations and behaves classically but, ϕ_i continue to have large fluctuations. We will integrate out all the ϕ_i except ϕ_0 and study the model in an expansion around $N = \infty$, where η and $b = \phi_0/\sqrt{N}$ have small fluctuations. We impose $\eta = \phi^2/N$ via a delta function

$$Z = \int [D\phi] \int_0^\infty [D\eta] e^{-\frac{1}{\hbar}\int d^4x \frac{1}{2}\{|\nabla\phi|^2 + NV(\eta)\}} \prod_x \delta(N\eta(x) - \phi^2(x)). \tag{6}$$

Now insert the integral representation

$$\delta(N\eta - \phi^2) = \int_{\mathcal{C}} \frac{d\sigma}{2\pi i} e^{\sigma(N\eta - \phi^2)}. \tag{7}$$

$\mathcal{C}$ can be any contour from $-i\infty$ to $i\infty$ since the integrand is entire. Up to an overall constant that cancels from normalized correlations, (5) becomes

$$Z = \int [D\phi] \int_0^\infty [D\eta] \int_{\mathcal{C}} [D\sigma] e^{-\frac{1}{\hbar}\int d^4x \frac{1}{2}\{|\nabla\phi|^2 + \sigma\phi^2 + NV(\eta) - N\sigma\eta\}}. \tag{8}$$

σ is a Lagrange multiplier enforcing the constraint $\eta = \phi^2/N$. Though $\mathcal{C}$ is not the real line, Z is real by construction. We separate $b = \phi_0/\sqrt{N}$ in anticipation of integrating out $\phi_1, \ldots, \phi_N$,

$$Z = \int [Db] \int [D\phi] \int_0^\infty [D\eta] \int_{\mathcal{C}} [D\sigma] e^{-\frac{1}{2\hbar}\int d^4x \left[\sum_{i=1}^N \{(\nabla\phi_i)^2 + \sigma\phi_i^2\} + N(\nabla b)^2 + N\sigma b^2 + NV(\eta) - N\sigma\eta\right]}. \tag{9}$$

Henceforth $[D\phi]$ does not include ϕ_0. Now, we reverse the order of η and σ integrals and observe that the η integral is a Laplace transform at each space-time point x,

$$\int_0^\infty [D\eta] e^{-(N/2\hbar)\int d^4x [V(\eta) - \sigma\eta]} = e^{-(N/2\hbar)\int d^4x W(\sigma)}. \tag{10}$$

Then

$$Z = \int [Db] \int [D\phi] \int_{\mathcal{C}} [D\sigma] e^{-(1/2\hbar)\int d^4x \left[\sum_{i=1}^N ((\nabla\phi_i)^2 + \sigma\phi_i^2) + N(\nabla b)^2 + N\sigma b^2 + NW(\sigma)\right]}. \tag{11}$$

Reversal of σ and η integrals is allowed if $W(\sigma)$ is non-singular along contour $\mathcal{C}$. Now we reverse the order of the ϕ and σ integrals. The ϕ integral is a gaussian

$$\int [D\phi] e^{-(1/2\hbar)\sum_{i=1}^N \int d^4x \phi_i(-\nabla^2 + \sigma)\phi_i} = \left[\det\left(\frac{-\nabla^2 + \sigma}{2\pi\hbar}\right)\right]^{-N/2}. \tag{12}$$

This integral converges if $-\nabla^2 + \sigma$ has eigenvalues with positive real part. Since $-\nabla^2$ is a positive operator, this is ensured if $\Re\sigma > 0$ (though the answer possesses an analytic continuation even to σ with negative real part, as long as it stays off the negative real axis). Thus we get (up to an irrelevant overall constant)

$$Z = \int_{-\infty}^\infty [Db] \int_{\mathcal{C}} [D\sigma] e^{-NS(b,\sigma)} \tag{13}$$

where

$$S(b,\sigma) = \frac{1}{2\hbar}\left[\hbar \text{ tr } \log(-\nabla^2 + \sigma) + \int d^4x \left\{(\nabla b)^2 + \sigma b^2 + W(\sigma)\right\}\right]. \tag{14}$$

Here, $W(\sigma)$ is obtained from $V(\eta)$ via a Laplace transform (10), at each x

$$e^{-(N/2\hbar)W(\sigma(x))} = \int_0^\infty e^{-(N/2\hbar)(V(\eta(x))-\sigma(x)\eta(x))} d\eta(x). \tag{15}$$

Conversely, $V(\eta)$ is obtained from $W(\sigma)$ by an inverse Laplace transform along contour $\mathcal{C}$ to the right of all singularities of $W(\sigma)$

$$\int_{\mathcal{C}} \frac{d\sigma}{2\pi i} e^{-(N/2\hbar)(W(\sigma)+\sigma\eta)} = e^{-(N/2\hbar)V(\eta)}. \tag{16}$$

σ is a dynamical field, it carries space-time derivatives and complicated self interactions. We are interested in correlation functions of the σ and b fields, which are defined as

$$\langle \sigma(x_1)\cdots\sigma(x_n)b(y_1)\cdots b(y_m)\rangle = \frac{1}{Z}\int [Db]\int_{\mathcal{C}} [D\sigma] e^{-NS(b,\sigma)}\sigma(x_1)\cdots\sigma(x_n)b(y_1)\cdots b(y_m). \tag{17}$$

Positivity of $\eta = \phi^2/N$ implies that $\sigma(x)$ is valued on a contour $\mathcal{C}$ from the south pole to north pole of the complex plane. On the other hand, the contour of integration for the b field is the real line. σ has dimensions of mass^2 while b has dimensions of mass.

3 Scale-invariance of the effective action at $N = \infty$

The interaction $W(\sigma)$ appearing in the action (14) is expanded in inverse powers of N

$$W(\sigma) = W_0(\sigma) + \frac{1}{N}W_1(\sigma) + \frac{1}{N^2}W_2(\sigma) + \cdots. \tag{18}$$

We do not assume analyticity of $W(\sigma)$ at $\sigma = 0$, so we do not expand it in powers of σ. $W(\sigma)$ is to be determined by the principle that the theory be scale-invariant at each order in $1/N$. Of course, the action $S(b,\sigma)$ is also expanded in inverse powers of N

$$\begin{aligned} S(b,\sigma) &= \frac{1}{\hbar}\Big[S_0 + \frac{1}{N}S_1 + \frac{1}{N^2}S_2 + \cdots\Big] \\ \text{where}\quad S_0 &= \frac{1}{2}\Big[\hbar \,\mathrm{tr}\,\log[-\nabla^2+\sigma] + \int d^4x\Big\{(\nabla b)^2 + \sigma b^2 + W_0(\sigma)\Big\}\Big], \\ S_1 &= \frac{1}{2}\int d^4x W_1(\sigma), \quad S_2 = \frac{1}{2}\int d^4x W_2(\sigma) \quad \text{etc.} \end{aligned} \tag{19}$$

$W_{1,2,3\cdots}$ must be chosen to cancel divergences and scale anomalies coming from fluctuations in b and σ while W_0 is chosen to cancel those from fluctuations in $\phi_1\cdots\phi_N$. The possible choice(s) of $W_{0,1,2\cdots}$ define the scale invariant fixed point(s) just as $\frac{1}{2}|\partial\phi|^2$ defines the trivial fixed point. W_n are not counter terms in the perturbative sense. For a given choice of W_n that does the job, there may be interactions with arbitrary coupling constants we can add to W_n and preserve scale invariance. We want this to be a finite parameter family. W_n are not restricted to be of any particular form. For locality, we wish to avoid arbitrarily high derivatives of σ.

N and $\hbar$ appear differently in $S(b,\sigma)$. $N \to \infty$ is a 'classical limit' in which b,σ have small fluctuations and is governed by the action $S_0(b,\sigma)$. $\hbar \to 0$ is also a classical limit, one in which the original fields ϕ_i have small fluctuations. It is governed by the original action $\int d^4x[|\nabla\phi|^2 + NV(\phi^2/N)]$. These two classical limits potentially capture different features of the full quantum theory for a given $W(\sigma)$. There is *a priori* no reason for the two limits $\hbar \to 0$ and $N \to \infty$ to commute. $\hbar$ tr $\log[-\nabla^2+\sigma]$ is a quantum correction to the action in the $\hbar \to 0$ limit. But it is part of the 'classical' action in the $N \to \infty$ limit.

A theory is scale-invariant if its effective action Γ (Legendre transform of the generating series of connected correlations, which generates all 1PI or proper vertices, see appendix B), is scale-invariant. Such an effective action defines a fixed point of the renormalization group flow. Γ is obtained by averaging over fluctuations in b and σ and is defined implicitly by

$$e^{-N\Gamma(B,\Sigma)} = \int [D\beta] \int_{\mathcal{C}} [D\varsigma] \exp\left[-N\left\{ S(B+\beta, \Sigma+\varsigma) - \beta\frac{\delta\Gamma}{\delta B} - \varsigma\frac{\delta\Gamma}{\delta\Sigma}\right\}\right]. \tag{20}$$

$B(x)$ and $\Sigma(x)$ are arbitrary background fields while β and ς ('varsigma') are the fluctuating fields, $b = B+\beta$, $\sigma = \Sigma+\varsigma$. $\Gamma(B,\Sigma)$ is calculated in a series in powers of $1/N$ holding $\hbar$ fixed

$$\Gamma(B,\Sigma) = \Gamma_0(B,\Sigma) + \frac{1}{N}\Gamma_1(B,\Sigma) + \frac{1}{N^2}\Gamma_2(B,\Sigma) + \cdots. \tag{21}$$

To zeroth order in $1/N$, the effective action

$$\Gamma_0(B,\Sigma) = S_0(B,\Sigma) = \frac{1}{2}\left[\hbar \text{ tr } \log[-\nabla^2+\Sigma] + \int d^4x \left\{ (\nabla B)^2 + \Sigma B^2 + W_0(\Sigma)\right\}\right]. \tag{22}$$

$\text{Tr}\log[-\nabla^2+\Sigma(x)]$ is divergent. We must regulate the model so that it is finite. Then we must pick $W_0(\Sigma)$ (which will depend on the regulator) in such a way that when the regulator is removed, $\Gamma_0(B,\Sigma)$ is not just finite but also scale-invariant. Similarly, W_1 is determined by the principle that Γ_1 be scale-invariant and so on. Actually, we will also have to allow for wave function renormalizations, but these appear only at order $1/N$, see sec. 5.

3.1 Effective action for constant σ at $N = \infty$

Tr $\log[-\nabla^2+\Sigma(x)]$ appearing in the large-N effective action $\Gamma_0(B,\Sigma)$ (22) is most easily evaluated for a constant background $\Sigma(x) = \Sigma_o$. This is a physically reasonable first approximation if space-time inhomogeneities are small. $\Sigma(x)$ takes values on the contour $\mathcal{C}$ from $-i\infty$ to $i\infty$. We anticipate needing to pick the contour to avoid the negative real axis, so Σ_o will be a complex number that lies off the negative real axis.

3.1.1 Momentum cutoff regularization

In momentum cutoff regularization,

$$\begin{aligned}
\text{tr } \log[-\nabla^2+\Sigma_o] &= \int_{|p|<\Lambda} d^4p\, d^4q\, \tilde{\delta}(p-q)\tilde{\delta}(p-q)\log(p^2+\Sigma_o) = \tilde{\delta}(0)\int_{|p|<\Lambda} d^4p \log[p^2+\Sigma_o] \\
&= \frac{\Omega}{(2\pi)^4}\int_0^\Lambda dp\, p^3 \log[p^2+\Sigma_o] \int d\Omega_4.
\end{aligned} \tag{23}$$

Here the space-time volume is $\int d^4x = \Omega = (2\pi)^4\tilde{\delta}(0)$ and $\int d\Omega_4 = 2\pi^2$ is the 'surface area' of a unit 3-sphere embedded in 4d Euclidean space. Thus,

$$\hbar \text{ tr } \log[-\nabla^2+\Sigma_o] = \frac{\hbar\Omega}{8\pi^2}\int_0^\Lambda dp\, p^3 \log(p^2+\Sigma_o)$$

$$
\begin{aligned}
&= \frac{\hbar\Omega}{64\pi^2}\Big[2\Lambda^4\log(\Lambda^2+\Sigma_o)-\Lambda^4+2\Lambda^2\Sigma_o-2\Sigma_o^2\log(\Lambda^2+\Sigma_o)+2\Sigma_o^2\log\Sigma_o\Big] \\
&= \frac{\hbar\Omega}{64\pi^2}\Big[2\Lambda^4\log\Lambda^2-\Lambda^4+4\Lambda^2\Sigma_o-2\Sigma_o^2\log\Lambda^2 \quad (\text{ divergent terms}) \\
&\quad +2\Sigma_o^2\log\Sigma_o \quad (\text{non scale invariant finite term}) \\
&\quad -\Sigma_o^2 \quad (\text{scale invariant finite term}) + \text{ terms that vanish as } \Lambda\to\infty\Big]. \qquad (24)
\end{aligned}
$$

We must pick $W_0(\Sigma)$ such that the large-N effective action (22) is both finite and scale-invariant when $\Lambda \to \infty$. In sec.3.2 we do this for general Σ, B. Here we get an idea of the answer by requiring that $\Gamma_0(B,\Sigma)$ be scale-invariant for constant $\Sigma = \Sigma_o$. To this end, we can pick

$$
W_0(\Sigma_o,\Lambda) = \frac{-\hbar}{64\pi^2}\Big[2\Lambda^4\log\Lambda^2-\Lambda^4+4\Lambda^2\Sigma_o-2\Sigma_o^2\log\Lambda^2\Big]-\frac{\hbar}{32\pi^2}\Sigma_o^2\log\Sigma_o. \qquad (25)
$$

This is the 'minimal subtraction' choice. Our principle that $\Gamma_0(B,\Sigma_o)$ be finite and scale-free is ambiguous. We could add to this choice of $W_0(\Sigma_o)$, any scale-free finite term of the form $\lambda\Sigma_o^2$, where λ is a dimensionless coupling constant. Other terms such as $m^2\Sigma_o$ or terms proportional to a higher power of Σ_o would involve dimensional coupling constants and would explicitly introduce a scale into the theory. The general choice leading to a scale-free $\Gamma_0(B,\Sigma_o)$ is

$$
W_0(\Sigma_o,\Lambda) = \frac{-\hbar}{64\pi^2}\Big[2\Lambda^4\log\Lambda^2-\Lambda^4+4\Lambda^2\Sigma_o-2\Sigma_o^2\log\Lambda^2\Big]-\frac{\hbar}{32\pi^2}\Sigma_o^2\log\Sigma_o+\lambda\Sigma_o^2. \qquad (26)
$$

In the large-N limit, we have a 1-parameter family of RG fixed points, parameterized by λ. The addition of $m^2\Sigma_o$ corresponds to a relevant perturbation of one of these fixed points. The addition of $c_n\Sigma_o^n$ for $n>2$ corresponds to an irrelevant deformation, since coupling c_n has a negative mass dimension. In the sequel we consider the mass deformed theory, where $W_0(\Sigma_o)$ is a two-parameter (m,λ) family

$$
W_0(\Sigma_o,\Lambda) = \frac{-\hbar}{64\pi^2}\Big[2\Lambda^4\log\Lambda^2-\Lambda^4+4\Lambda^2\Sigma_o-2\Sigma_o^2\log\Lambda^2\Big]-\frac{\hbar}{32\pi^2}\Sigma_o^2\log\Sigma_o+\lambda\Sigma_o^2-m^2\Sigma_o. \qquad (27)
$$

$W_0(\Sigma_o)$ has a branch cut along the negative Σ_o axis, consistent with our expectation that $\Sigma(x)$ is valued on a contour that misses the negative real axis. The corresponding large-N effective action (22) for constant backgrounds B_o, Σ_o is

$$
\Gamma_0(B_o,\Sigma_o) = \frac{\Omega}{2}\Big[-m^2\Sigma_o+\Big(\lambda-\frac{\hbar}{64\pi^2}\Big)\Sigma_o^2+\Sigma_o B^2\Big]. \qquad (28)
$$

Having found a line of fixed points, are they UV or IR? The answer can depend on the direction in which we flow from a fixed point. In this case, all the above fixed points are UV with respect to the mass deformation. The physical reason is that as we go to higher energies, the ratio of m to the energy scale will decrease and the RG flow will tend towards the fixed point. Recall that the gaussian fixed point (massless free scalar field theory) in 4d is UV with respect to mass deformations, but IR with respect to the quartic coupling $\lambda\phi^4$. So far, in our model, the analogue of the quartic coupling, $\lambda\Sigma_o^2$ is exactly marginal. Note that Σ has a canonical dimension of mass squared, while ϕ has dimensions of mass.

The presence of a 1-parameter family of UV fixed points is a godsend. It means that for *any* values of λ and $\hbar$, we can set $m=0$ and get an extra symmetry, scale-invariance. Thus, the mass parameter can be naturally small (at least at $N=\infty$). An interesting question is whether the line of fixed points can be maintained after including effects of quantum fluctuations in σ and b in a $1/N$ expansion (see sec. 5).

3.1.2 Zeta function regularization

We repeat the evaluation of the large-N effective potential by regularizing $\text{tr}\ \log[-\nabla^2+\Sigma_o]$ via zeta-function regularization. This method directly prescribes a finite part for $\text{tr}\ \log[-\nabla^2+\Sigma_o]$ in the unregulated limit. The finite part is not scale-invariant. We will pick $W_0(\Sigma_o)$ to cancel this non-scale-invariant quantity, so that $\Gamma_0(B,\Sigma_o)$ is both scale-invariant and finite. This procedure means we do not need to prescribe how $W_0(\Sigma_o)$ must depend on the regulator, but just the finite part of its limiting unregulated value. Such a short-cut is not possible in other regularization schemes such as momentum cutoff (sec. 3.1.1) or dimensional regularization (appendix B.1). For this and other reasons, we will use the zeta-function regularization in the rest of the paper. On the other hand, comparison of different schemes allows us to better understand what is scheme independent. We find that the presence of a 1-parameter family of fixed points is scheme independent. Moreover, the finite and scale-invariant effective potential obtained in the three schemes are the same up to a finite relabeling of λ. To define the finite part of $\text{tr}\ \log[-\nabla^2+\Sigma_o]$ by zeta-function regularization, let

$$\zeta(s) = \text{ tr } [-\nabla^2+\Sigma_o]^{-s} = \int d^4p\ d^4q\ \tilde\delta^4(p-q)\frac{\tilde\delta^4(p-q)}{[p^2+\Sigma_o]^s} = \Omega\int \frac{d^4p}{(2\pi)^4}\ \frac{1}{[p^2+\Sigma_o]^s}. \tag{29}$$

$\Omega = \int d^4x = (2\pi)^4\tilde\delta^4(0)$, appears because $\Sigma(x)$ is homogeneous and we are taking the trace of an operator that is diagonal in momentum space. $\zeta(s)$ is analytic for $\Re s > 2$. By analytic continuation, we get a meromorphic function $\zeta(s)$. Away from singularities,

$$\zeta'(s) = -\text{ tr }\left[\frac{\log[-\nabla^2+\Sigma_o]}{[-\nabla^2+\Sigma_o]^s}\right]. \tag{30}$$

So, if $\zeta(s)$ is analytic at $s=0$, $\zeta'(0) = -\text{ tr }\log[-\nabla^2+\Sigma_o]$. Now let us calculate

$$\frac{\zeta(s)}{\Omega} == \frac{1}{(2\pi)^4}\int d\Omega_4\int_0^\infty\frac{p^3dp}{(p^2+\Sigma_o)^s} = \frac{1}{(2\pi)^4}2\pi^2\frac{\Sigma_o^{2-s}}{2(s-1)(s-2)} = \frac{1}{16\pi^2}\frac{\Sigma_o^{2-s}}{(s-1)(s-2)}. \tag{31}$$

$\zeta(s)$ has simple poles at $s=1,2$ but is analytic at $s=0$ so,

$$\begin{aligned}\frac{\zeta'(0)}{\Omega} = \frac{3\Sigma_o^2}{64\pi^2} - \frac{\Sigma_o^2\log\Sigma_o}{32\pi^2} &= -\frac{\Sigma_o^2\log[\Sigma_o e^{-3/2}]}{32\pi^2}\\ \Rightarrow\quad \text{tr }\log[-\nabla^2+\Sigma_o] &= \frac{\Sigma_o^2\Omega}{32\pi^2}\log[e^{-3/2}\Sigma_o].\end{aligned} \tag{32}$$

(32) is not scale-free due to the logarithm. The effective action at $N=\infty$ for constant background fields B,Σ is

$$\Gamma_0(B_o,\Sigma_o) = \frac{\Omega}{2}\left[\frac{\hbar\Sigma_o^2}{32\pi^2}\log[e^{-3/2}\frac{\Sigma_o}{M^2}] + \Sigma_o B_o^2 + W_0(\Sigma_o)\right]. \tag{33}$$

The parameter M with dimensions of mass sets the scale for the logarithm and breaks scale-invariance. The choice of $W_0(\sigma)$ that ensures $\Gamma_0(B,\Sigma_o)$ is scale-free (for $m=0$) is

$$W_0(\sigma_o) = -m^2\sigma_o + \lambda\sigma_o^2 - \frac{\hbar\sigma_o^2}{32\pi^2}\log[e^{-3/2}\frac{\sigma_o}{M^2}]. \tag{34}$$

We added a mass term, a relevant perturbation away from the line of fixed points parameterized by λ. We will prove in sec. 3.4 that the scale anomaly in $\Gamma_0(B_o,\Sigma_o)$ vanishes for this choice of

$W_0(\sigma_o)$. We avoid cubic and higher powers of σ_o as before. Note that the terms in $W_0(\sigma_o)$ are of different orders in $\hbar$ but of the same order in $\frac{1}{N}$. For this choice of W_0, we get

$$\Gamma_0(B_o, \Sigma_o) = \frac{\Omega}{2}[-m^2\Sigma_o + \lambda\Sigma_o^2 + \Sigma_o B_o^2]. \tag{35}$$

M cancels out from the effective potential, which is scale-free for $m = 0$. Though $W_0(\sigma_o)$ has a branch cut along the negative Σ_o axis, $\Gamma_0(B_o, \Sigma_o)$ (and $S_0(b_o, \sigma_o)$) is entire. It is very interesting to know whether this is true for general backgrounds. The method of expanding in inverse powers of Σ_o that we use in the rest of this paper prevents us from answering this question here. Comparing with section 3.1.1 we see that irrespective of regularization scheme, there is a one parameter family of fixed points parameterized by λ. However, the definition of the coupling depends on regularization scheme,

$$\lambda_{\text{zeta function}} = \lambda_{\text{cutoff}} - \frac{\hbar}{64\pi^2}. \tag{36}$$

3.2 $N = \infty$ effective action expanded around a constant background

In section 3.1 we calculated the $N = \infty$ effective action (22) for constant B, Σ. Allowing for arbitrary backgrounds $B(x)$ is easy, the difficulties lie in non-constant $\Sigma(x)$. Here, we get an expansion for $\Gamma_0(B, \Sigma)$ in powers and derivatives of $\Sigma - \Sigma_o$ where Σ_o is a constant background whose value is not a negative real number or zero. It would be interesting to calculate Γ_0 by complementary methods too. From appendix C, in zeta function regularization,

$$\begin{aligned} \text{tr } \log[-\nabla^2 + \Sigma(x)] \;&=\; \frac{\Sigma_o^2\Omega}{32\pi^2}\log[\Sigma_o e^{-3/2}] + \int \frac{d^4x}{16\pi^2}\Big[\Sigma_o \log[\Sigma_o/e](\Sigma - \Sigma_o) + \frac{1}{2}(\Sigma - \Sigma_o)^2 \log \Sigma_o \\ &\quad -(\Sigma - \Sigma_o)\Pi(\Delta)(\Sigma - \Sigma_o) + \mathcal{O}(\Sigma - \Sigma_o)^3\Big]. \end{aligned} \tag{37}$$

This reduces to (32) for constant backgrounds ($\Sigma = \Sigma_o$). The $\mathcal{O}(\Sigma - \Sigma_o)^3$ and higher order terms can be obtained by the method of appendix C. But these higher order terms are scale-invariant. We will show in section 3.4 that the part of tr $\log[-\nabla^2 + \Sigma(x)]$ that is not scale-invariant, is restricted to the first three terms on the rhs of (37). Note that $\Delta = -\nabla^2/\Sigma_o$ and

$$\Pi(\Delta) = \sum_{n=1}^{\infty} \frac{(-\Delta)^n}{n(n+1)(n+2)} = \frac{\Delta(3\Delta + 2) - 2(\Delta + 1)^2 \log(1 + \Delta)}{4\Delta^2} = -\frac{\Delta}{6} + \frac{\Delta^2}{24} - \frac{\Delta^3}{60} + \cdots \tag{38}$$

Thus, the effective action at $N = \infty$ is

$$\begin{aligned} \Gamma_0(B, \Sigma) \;&=\; \frac{1}{2}\int d^4x \Big[(\nabla B)^2 + \sigma B^2 + W_0(\Sigma) + \frac{\hbar}{16\pi^2}\Big\{\frac{1}{2}\Sigma_o^2 \log[\Sigma_o e^{-3/2}] + \Sigma_o \log[\Sigma_o/e](\Sigma - \Sigma_o) \\ &\quad + \frac{1}{2}\log[\Sigma_o](\Sigma - \Sigma_o)^2 - (\Sigma - \Sigma_o)\Pi(\Delta)(\Sigma - \Sigma_o) + \mathcal{O}(\Sigma - \Sigma_o)^3\Big\}\Big]. \end{aligned} \tag{39}$$

In deriving (39) we did *not* assume Σ is slowly varying but rather that $\Sigma - \Sigma_o$ is small and that Σ_o is not a negative real number or zero.

3.3 Fixing the interaction at $N = \infty$ by requiring scale-invariance

$W_0(\sigma)$ must be chosen so that $\Gamma_0(B,\Sigma)$ in (39) is scale-free. The choice that does the job is

$$W_0(\sigma) = \lambda\sigma^2 - m^2\sigma - \frac{\hbar}{16\pi^2}\Big\{\frac{\sigma_o^2}{2}\log[\frac{\sigma_o e^{-\frac{3}{2}}}{M^2}] + \sigma_o\log[\frac{\sigma_o}{M^2 e}](\sigma-\sigma_o) + \frac{1}{2}\log[\frac{\sigma_o}{M^2}](\sigma-\sigma_o)^2\Big\} \quad (40)$$

with $m = 0$. For $m \neq 0$ we have a mass deformation. M sets the scale for logarithms, but cancels out in the $N = \infty$ effective action

$$\Gamma_0 = \int\frac{d^4x}{2}\Big[(\nabla B)^2 + \Sigma B^2 - m^2\Sigma + \lambda\Sigma^2 - \frac{\hbar}{16\pi^2}\Big\{(\Sigma-\Sigma_o)\Pi(\Delta)(\Sigma-\Sigma_o) + \mathcal{O}(\Sigma-\Sigma_o)^3\Big\}\Big]. \quad (41)$$

$\Gamma_0(B,\Sigma)$ is now free of divergences (for $\Sigma_o \notin \mathbf{R}^-$) and defines the large-$N$ effective action for background fields Σ whose deviation from a constant Σ_o is small. The higher order terms in $\varsigma = \Sigma - \Sigma_o$ are all finite, scale-free and calculable by the method of appendix C. The fact that $\Gamma_0(B,\Sigma)$ is not quadratic when regulators are removed, indicates that our theory is not trivial.

3.4 Cancelation of scale anomaly

We show that the effective action $\Gamma_0(B,\Sigma)$ in (41) is scale invariant for $m = 0$ ($m \neq 0$ is treated subsequently). Coordinates and fields are rescaled according to their canonical dimensions. M sets the scale for logarithms and explicitly introduces a scale into the theory. To encode this physical fact, M is *not* rescaled, the same is true of m. We define dilations as

$$D_a x^\mu = a^{-1}x^\mu, \quad D_a b = ab, \quad D_a\sigma = a^2\sigma, \quad D_a\lambda = \lambda, \quad D_a\nabla = a\nabla. \quad (42)$$

The generator of infinitesimal dilations δ_D is defined as

$$\begin{aligned} \delta_D f &= \lim_{\epsilon\to 0}\frac{D_{1+\epsilon}f - f}{\epsilon}, \\ \delta_D x^\mu = -x^\mu, \quad \delta_D b = b, \quad \delta_D\sigma = 2\sigma, &\quad \delta_D\lambda = 0, \quad \delta_D^0\nabla = \nabla, \quad \delta_D dx = -dx \end{aligned} \quad (43)$$

and may be represented as

$$\delta_D = -x^\mu\frac{\partial}{\partial x^\mu} + b(x)\frac{\partial}{\partial b(x)} + 2\sigma(x)\frac{\partial}{\partial\sigma(x)}. \quad (44)$$

We now show that $\Gamma_0(B,\Sigma)$ (22) is invariant under dilations for (not necessarily constant) backgrounds $B(x)$ and $\Sigma(x)$ if $W_0(\Sigma)$ is chosen according to (40) with $m = 0$. First,

$$D_a\int d^4x\{(\nabla B)^2 + \Sigma B^2\} = \int d^4x\{(\nabla B)^2 + \Sigma B^2\} \quad\Rightarrow\quad \delta_D\int d^4x\{(\nabla B)^2 + \Sigma B^2\} = 0. \quad (45)$$

$W_0(\Sigma)$ and $\mathrm{tr}\ \log[-\nabla^2 + \Sigma]$ are the only terms in (22) with non-trivial (in fact inhomogeneous) scale transformations. W_0 transforms as

$$\begin{aligned} D_a\int d^4x W_0(\Sigma) &= \int d^4x W_0(\Sigma) - \frac{\hbar}{16\pi^2}\int d^4x\Big[\frac{1}{2}\Sigma_o^2\log[a^2] + \Sigma_o\log[a^2]\varsigma + \frac{1}{2}\log[a^2]\varsigma^2\Big] \\ &= \int d^4x W_0(\Sigma) - \frac{\hbar\Sigma_o^2\Omega\log a}{8\pi^2}\Big[\frac{1}{2} + \frac{\langle\varsigma\rangle}{\Sigma_o} + \frac{\langle\varsigma^2\rangle}{2\Sigma_o^2}\Big] \end{aligned}$$

$$\Rightarrow \quad \delta_D \int d^4x W_0(\Sigma) \quad = \quad -\frac{\hbar\Omega\Sigma_o^2}{8\pi^2}\left[\frac{1}{2}+\frac{\langle\varsigma\rangle}{\Sigma_o}+\frac{\langle\varsigma^2\rangle}{2\Sigma_o^2}\right] \tag{46}$$

where $\varsigma = \Sigma - \Sigma_o$, Σ_o is a constant background and $\langle f\rangle = \frac{1}{\Omega}\int d^4x f$. On the other hand,

$$\begin{aligned} &\text{tr } \log[-\nabla^2+\Sigma] = -\zeta'(0) \quad \text{and} \quad \zeta(s) = \text{ tr } [-\nabla^2+\Sigma]^{-s} \\ &\Rightarrow \quad D_a\zeta(s) = a^{-2s}\zeta(s) \quad \text{and} \quad D_a\zeta'(s) = -2\zeta(s)a^{-2s}\log a + a^{-2s}\zeta'(s). \end{aligned} \tag{47}$$

Now set $s=0$ and use the result for $\zeta(0)$ calculated in Appendix C.4,

$$\begin{aligned} &D_a\zeta'(0) = \zeta'(0) - 2\zeta(0)\log a \quad \Rightarrow \quad \delta_D\zeta'(0) = -2\zeta(0) = -\frac{\Omega\Sigma_o^2}{8\pi^2}\left[\frac{1}{2}+\frac{\langle\varsigma\rangle}{\Sigma_o}+\frac{\langle\varsigma^2\rangle}{2\Sigma_o^2}\right], \\ &\Rightarrow \quad \delta_D\hbar \text{ tr } \log[-\nabla^2+\Sigma] \quad = \quad -\hbar\delta_D\zeta'(0) = \frac{\hbar\Omega\Sigma_o^2}{8\pi^2}\left[\frac{1}{2}+\frac{\langle\varsigma\rangle}{\Sigma_o}+\frac{\langle\varsigma^2\rangle}{2\Sigma_o^2}\right]. \end{aligned} \tag{48}$$

We see that the scale anomaly of tr $\log[-\nabla^2+\Sigma]$ exactly cancels that of $\int d^4x W_0(\Sigma)$. Therefore the large-N effective action (41) (for $m=0$) is invariant under scale transformations (42): $\delta_D\Gamma_0(B,\Sigma)=0$. However, Γ_0 with a mass term (41) is not scale-invariant. In fact,

$$\delta_D\Gamma_0(B,\Sigma) = -2\int d^4x\ m^2\ \Sigma(x) = -m\frac{\partial}{\partial m}\int d^4x\ m^2\ \Sigma(x). \tag{49}$$

Define $\beta_0^m = m$. Though Γ_0 is not scale-invariant, it satisfies a 'renormalization group equation'

$$(\delta_D + \beta_0^m\frac{\partial}{\partial m})\Gamma_0(B,\Sigma) = 0. \tag{50}$$

β_0^m is the $N=\infty$ 'beta function' of mass. In general, $\beta^m = \beta_0^m + \frac{1}{N}\beta_1^m + \frac{1}{N^2}\beta_2^m + \cdots$. We could also have β^λ at higher orders in $1/N$, but we want to ensure $\beta^\lambda = 0$ for at least one value of λ when $m=0$. Let us define a new operator δ_0, the large-N 'RG vector field':

$$\delta_0 = -x^\mu\frac{\partial}{\partial x^\mu} + b(x)\frac{\partial}{\partial b(x)} + 2\sigma(x)\frac{\partial}{\partial\sigma(x)} + \beta_0^m\frac{\partial}{\partial m}, \quad \text{where} \quad \beta_0^m = m. \tag{51}$$

Then $\delta_0\Gamma_0(B,\Sigma)=0$. The RG vector field may also receive corrections: $\delta = \delta_0 + \frac{\delta_1}{N} + \frac{\delta_2}{N^2} + \cdots$.

4 Small Oscillations around constant classical solutions

4.1 Constant extrema of $N=\infty$ effective action

Field configurations that extremize the $N=\infty$ effective action $\Gamma_0(B,\Sigma)$ (41) dominate the path integral over b and σ in the saddle point approximation in $1/N$. Extrema of Γ_0 must satisfy the large-N 'classical' equations of motion (also called gap equations elsewhere)

$$\begin{aligned} \frac{\delta\Gamma_0}{\delta B} &= (-\nabla^2+\Sigma)B = 0 \quad \text{and} \\ \frac{\delta\Gamma_0}{\delta\Sigma} &= \frac{1}{2}(B^2-m^2) + \left(\lambda - \frac{\hbar}{16\pi^2}\Pi(\Delta)\right)\Sigma + \mathcal{O}(\Sigma-\Sigma_o)^2 = 0. \end{aligned} \tag{52}$$

Unlike, say, the gap equations of the non-linear sigma model in the large-N limit, these equations are finite and do not require any renormalization, since the divergences in tr $\log[-\nabla^2+\Sigma]$ have been canceled by the choice of $W_0(\Sigma)$. The complication here is that the equation of motion for Σ has been obtained in a series in powers of $\Sigma-\Sigma_o$ for constant Σ_o. Let us begin by looking for constant extrema $(B=B_o, \Sigma=\Sigma_o)$ of Γ_0. The classical equations of motion become

$$\Sigma_o B_o = 0 \quad \text{and} \quad B_o^2 - m^2 + 2\lambda\Sigma_o = 0. \tag{53}$$

Assuming λ and m are non-zero, which is the generic situation, there are two types of extrema: (**S**) $B_o = 0, \Sigma_o = m^2/2\lambda$ where $O(N+1)$ symmetry is unbroken and (**B**) $B_o = \pm m$ and $\Sigma_o = 0$ where $O(N+1)$ symmetry is spontaneously broken to $O(N)$. The vev of the real scalar field b must be real. This justifies our choice for the sign of the $m^2\sigma$ term in $W_0(\sigma)$ in (40). In the broken phase, the vev $\Sigma_o = 0$. This is fine for constant backgrounds, since the effective potential is entire. But we cannot yet analyze the vicinity of the broken vacuum for non-constant backgrounds, since we found the effective action only in an expansion in powers of $\Sigma - \Sigma_o$ which involves inverse powers of Σ_o. Perhaps a cleverer method can be invented to study the phase where $O(N+1)$ symmetry is broken. There may also be phases where fields are not constant. Now, we study oscillations around the vacuum where $O(N+1)$ symmetry is unbroken.

4.2 Mass of long wavelength small oscillations in unbroken phase

To study oscillations around an extremum of the large-N effective potential, we expand the large-N effective action (41) to quadratic order around a constant background $B = B_o+\beta, \Sigma = \Sigma_o+\varsigma$:

$$\begin{aligned}\Gamma_0(B_o+\beta,\Sigma_o+\varsigma) \;=\; & \frac{1}{2}\int d^4x\Big[(\nabla\beta)^2 + \Big\{2\Sigma_o B_o\beta + (B_o^2 - m^2 + 2\lambda\Sigma_o)\varsigma\Big\} \\ & + \Big\{\Sigma_o\beta^2 + 2B_o\beta\varsigma + \varsigma\Big(\lambda - \frac{\hbar}{16\pi^2}\Pi(\Delta)\Big)\varsigma\Big\} + \cdots\Big].\end{aligned} \tag{54}$$

We have omitted an additive constant in Γ_0. Assuming that (B_o, Σ_o) is a constant solution of the classical equations of motion (53), the linear terms drop out and we get

$$\Gamma_0(B_o+\beta,\Sigma_o+\varsigma) = \frac{1}{2}\int d^4x\Big[(\nabla\beta)^2 - \frac{\hbar}{16\pi^2}\varsigma\Pi(\Delta)\varsigma + \Sigma_o\beta^2 + 2B_o\beta\varsigma + \lambda\varsigma^2 + \cdots\Big]. \tag{55}$$

Here $\Delta = -\nabla^2/\Sigma_o$ and $\Pi(\Delta)$ is given in (117). The lightest 'particles' of the theory are the longest wavelength oscillations around the classical vacua. To find the mass of the longest wavelength oscillations we may assume that the deviation $\varsigma = \Sigma - \Sigma_o$ is slowly varying in space. It suffices to keep the leading term in $\Pi(\Delta) = -\frac{\Delta}{6} + \frac{\Delta^2}{24} - \frac{\Delta^3}{60} + \cdots$. Thus

$$\Gamma_0(B_o+\beta,\Sigma_o+\varsigma) = \frac{1}{2}\int d^4x\Big[(\nabla\beta)^2 + \frac{\hbar}{96\pi^2\Sigma_o}\varsigma(-\nabla^2)\varsigma + \Sigma_o\beta^2 + 2B_o\varsigma\beta + \lambda\varsigma^2 + \cdots\Big]. \tag{56}$$

This should describe small amplitude oscillations since we also ignored higher powers of ς. In symmetric phase (**S**) where $O(N+1)$ is unbroken, the classical minimum is at $B_o = 0$, $\Sigma_o = m^2/2\lambda$. Γ_0 expanded around this minimum is

$$\Gamma_0(\beta,\varsigma) = \frac{1}{2}\int d^4x\Big[(\nabla\beta)^2 + \frac{\hbar\lambda}{48\pi^2 m^2}(\nabla\varsigma)^2 + \frac{m^2}{2\lambda}\beta^2 + \lambda\varsigma^2\Big]. \tag{57}$$

Recall that a field ϕ with Lagrangian $(\nabla\phi)^2 + m^2\phi^2$ has as its longest wavelength excitation, a particle of mass m. We deduce that the lightest particle-like excitation of the b field has a mass $M_b = \frac{m}{\sqrt{2\lambda}}$ and transforms in the fundamental representation of $O(N+1)$. Small oscillations of σ correspond to a particle of mass $M_\sigma = \frac{4\sqrt{3}\pi m}{\sqrt{\hbar}}$. This particle is an $O(N+1)$ singlet. There could also be heavier particles.

5 Leading $1/N$ corrections due to fluctuations of b and σ

At $N=\infty$, quantum fluctuations of b and σ could be ignored, but averaging over them at $\mathcal{O}(1/N)$ will lead to divergences and violations of scale-invariance in the effective action Γ. The technical difficulty in obtaining $\Gamma(B,\Sigma)=\Gamma_0(B,\Sigma)+\frac{1}{N}\Gamma_1(B,\Sigma)$ is that $\Gamma_0(B,\Sigma)$ (41), when expanded, involves arbitrarily high powers and derivatives of Σ,

$$S_0(b,\sigma)=\Gamma_0(b,\sigma)=\frac{1}{2}\int d^4x\Big[(\nabla b)^2+\sigma b^2-m^2\sigma+\lambda\sigma^2-\frac{1}{16\pi^2}\Big\{\sigma\Pi(\Delta)\sigma+\mathcal{O}(\sigma-\sigma_o)^3\Big\}\Big]. \quad (58)$$

$\Pi(-\nabla^2/\sigma_o)$ is defined in (117). The action of our model at this order is

$$S(b,\sigma)=S_0(b,\sigma)+\frac{1}{2N}\int d^4x W_1(\sigma)+\mathcal{O}(\frac{1}{N^2}). \quad (59)$$

We must pick $W_1(\Sigma)$ so that $\Gamma_1(B,\Sigma)$ is finite and scale-free for $m=0$. However, this is most likely not possible since $W_1(\Sigma)$ is independent of B and can at best cancel divergences in $\Gamma_1(0,\Sigma)$. The remaining divergences must be canceled via mass and coupling constant renormalization and anomalous dimensions. These will modify the generator of RG flow

$$\delta=\delta_0+\frac{1}{N}\delta_1+\frac{1}{N^2}\delta_2+\cdots; \qquad \delta_1=-\beta_1^m\frac{\partial}{\partial m}-\gamma_1^B B\frac{\partial}{\partial B}-\gamma_1^\Sigma\Sigma\frac{\partial}{\partial\Sigma}-\beta_1^\lambda\frac{\partial}{\partial\lambda} \quad (60)$$

such that Γ still satisfies the RGE $\delta\Gamma(B,\Sigma)=0$ at each order. In order to have a fixed point, which would make it natural to have small masses, we will impose $\beta_1^\lambda=0$ for at least one value of λ when $m=0$. Since $\delta_0\Gamma_0=0$ (50), the RGE for the effective action at order $1/N$ reads

$$\delta_0\Gamma_1+\delta_1\Gamma_0=0. \quad (61)$$

$\delta_0=-x^\mu\frac{\partial}{\partial x^\mu}+b\frac{\partial}{\partial b}+2\sigma\frac{\partial}{\partial\sigma}+m\frac{\partial}{\partial m}$ and Γ_0 (39) are known, so we must determine δ_1 and Γ_1.

5.1 Calculation of effective action at $\mathcal{O}(1/N)$

Using (20) and doing the gaussian integral at order $1/N$ we get the change in effective action

$$\Gamma_1(B,\Sigma)=\frac{1}{2}\Big[\text{tr }\log\Gamma_0''(B,\Sigma)+\int d^4x\ W_1(\Sigma)\Big]. \quad (62)$$

$\Gamma_0''(B,\Sigma)$ is the hessian of Γ_0 acting on the 2-component column vector $(b\ \ \sigma)$,

$$\Gamma_0''(B,\Sigma)=\begin{pmatrix}-\nabla^2+\Sigma & B\\ B & -\frac{\Pi(-\nabla^2/\Sigma_o)}{16\pi^2}+\lambda\end{pmatrix} \quad (63)$$

Γ_1 is independent of m even if Γ_0 includes a mass deformation.

5.1.1 $W_1(\Sigma)$ for slowly varying quantum fluctuations & general background $\Sigma(x)$

To find $W_1(\Sigma)$, we calculate Γ_1 for $B=0$ and choose $W_1(\Sigma)$ to make $\Gamma_1(0,\Sigma)$ finite and scale-free for arbitrary $\Sigma(x)$. (However, even for $B=0$, Σ acquires an anomalous dimension, see sec. 5.1.2.) For $B=0$, the hessian (63) is diagonal, so

$$\Gamma_1(0,\Sigma)=\frac{1}{2}\Big[\text{tr }\log[-\nabla^2+\Sigma]+\text{ tr }\log\Big[-\frac{\Pi(-\nabla^2/\Sigma_o)}{16\pi^2}+\lambda\Big]+\int W_1(\Sigma)d^4x\Big]. \quad (64)$$

Since $\Pi(-\nabla^2/\Sigma_o)$ is complicated (117), we make the further approximation that quantum fluctuations are slowly varying[10]. Then we may ignore higher derivatives in $\Pi(-\nabla^2/\Sigma_o)$ and get

$$\Gamma_1(0,\Sigma) = \frac{1}{2}\Big[\text{ tr }\log[-\nabla^2+\Sigma] + \text{ tr }\log[-\frac{\nabla^2}{96\pi^2\Sigma_o}+\lambda] + \int W_1(\Sigma) d^4x\Big]. \tag{65}$$

The 1st term is identical to what appeared in Γ_0 (22) and was calculated in an expansion around a constant background Σ_o in (37). The 2nd term can be calculated exactly in zeta function regularization. Let $\zeta(s) = \text{ tr }[-\nabla^2 + 96\pi^2\lambda\Sigma_o]^{-s}$, then using (31) we get,

$$\begin{aligned}
\text{tr }\log[-\frac{\nabla^2}{96\pi^2\Sigma_o}+\lambda] &= -\zeta(0)\log[96\pi^2\Sigma_o] - \zeta'(0) \\
\text{where } \quad \zeta(0) = \frac{(96\pi^2)^2\lambda^2\Sigma_o^2\Omega}{32\pi^2} \quad &\text{and} \quad \zeta'(0) = \frac{(96\pi^2)^2\lambda^2\Sigma_o^2\Omega}{32\pi^2}\log[e^{-3/2}96\pi^2\lambda\Sigma_o] \\
\Rightarrow \text{ tr }\log[-\frac{\nabla^2}{96\pi^2\Sigma_o}+\lambda] &= -\frac{(96\pi^2)^2\lambda^2\Sigma_o^2\Omega}{32\pi^2}\log[e^{-3/2}(96\pi^2)^2\lambda\Sigma_o^2].
\end{aligned} \tag{66}$$

$\lambda\Sigma_o$ cannot be a negative real number. So $\lambda = 0$ is singular within our approximations. Thus

$$\Gamma_1(0,\Sigma) = \frac{1}{2}\Big[\text{ tr }\log[-\nabla^2+\Sigma] - \frac{(96\pi^2)^2\lambda^2\Sigma_o^2\Omega}{32\pi^2}\log[e^{-3/2}(96\pi^2)^2\lambda\Sigma_o^2/M^4] + \int W_1(\Sigma)d^4x\Big]. \tag{67}$$

The 1st and 2nd terms violate scale-invariance and involve a dimensional parameter M as in section 3. So $W_1(\Sigma)$ is determined by the condition that it must cancel these scale anomalies. Aside from the 2nd term, we had the same condition for $W_0(\Sigma)$ in (39). So[11] as in (40),

$$\begin{aligned}
W_1(\Sigma) &= -\frac{1}{16\pi^2}\Big\{\frac{1}{2}\Sigma_o^2\log[\frac{\Sigma_o e^{-3/2}}{M^2}] + \Sigma_o\log[\frac{\Sigma_o}{M^2 e}](\Sigma-\Sigma_o) + \frac{1}{2}\log[\frac{\Sigma_o}{M^2}](\Sigma-\Sigma_o)^2 \\
&\quad -\frac{(96\pi^2)^2\lambda^2\Sigma_o^2}{2}\log[\frac{(96\pi^2)^2\lambda\Sigma_o^2}{e^{3/2}M^4}]\Big\}.
\end{aligned} \tag{68}$$

$W_0 + \frac{1}{N}W_1$ is the potential for which $\Gamma_0 + \frac{1}{N}\Gamma_1$ is finite and scale-free for backgrounds $\Sigma(x)$ and $B = 0$, after including slowly varying quantum fluctuations at $\mathcal{O}(1/N)$. For this choice,

$$\Gamma_1(0,\Sigma) = \frac{1}{2}\int d^4x\Big[-\frac{1}{16\pi^2}(\Sigma-\Sigma_o)\Pi(-\nabla^2/\Sigma_o)(\Sigma-\Sigma_o) + \mathcal{O}(\Sigma-\Sigma_o)^3\Big]. \tag{69}$$

5.1.2 RGE for slowly varying fluctuations and constant backgrounds

In section 5.1.1 we found $W_1(\Sigma(x))$ that ensures Γ_1 is finite and scale-free for $B = 0$. But there could be further divergences which lead to running couplings and anomalous dimensions. We determine β^m, γ^σ and γ^b, while enforcing $\beta^\lambda = 0$ for constant backgrounds B_o & Σ_o. We do this by deriving an RGE for the effective action. We assume quantum fluctuations of σ are slowly varying on the scale of the constant background Σ_o, so that

$$\Gamma_0''(B_o,\Sigma_o) = \begin{pmatrix} -\nabla^2+\Sigma_o & B_o \\ B_o & -\frac{\nabla^2}{96\pi^2\Sigma_o}+\lambda \end{pmatrix} = \begin{pmatrix} A & B_0\mathbf{1} \\ B_0\mathbf{1} & D \end{pmatrix}. \tag{70}$$

[10] Quantum fluctuations need not be slowly varying. We hope to relax this assumption in future work.

[11] We could add a finite term $\lambda_1\Sigma^2$ to W_1 without violating scale invariance. We did this for W_0, but do not for the other W_n, just as we do not add an arbitrary finite counter term $\lambda_n\phi^4$ at n-loop order of $\lambda\phi^4$ theory. In a sense, we allow the most general W_0 consistent with the symmetries but the remaining ones are the minimal choices that ensure cancelation of divergences and preservation of scale invariance.

Here $A = -\nabla^2 + \Sigma_o$, $D = -\nabla^2/\tilde{\Sigma}_o + \lambda$ and $\tilde{\Sigma}_o = 96\pi^2\Sigma_o$. Since $B_0\mathbf{1}$ commutes with D, $\det[\Gamma_0''] = \det[AD - B_o^2\mathbf{1}]$, i.e., the characteristic polynomial of AD. To calculate it, let

$$\zeta(s) = \text{ tr } [AD - B_o^2]^{-s} = \frac{\Omega}{(2\pi)^4}\int d\Omega_4 \int_0^\infty \frac{p^3 dp}{\Big(p^4/\tilde{\Sigma}_o + (\Sigma_o/\tilde{\Sigma}_o + \lambda)p^2 + (\lambda\Sigma_o - B_o^2)\Big)^s}. \quad (71)$$

The integral converges for $\Re s > 1$ and defines meromorphic $\zeta(s)$. Changing variables to $q = p^2$ and defining $2c = \Sigma_o + \lambda\tilde{\Sigma}_o$, $d = (\lambda\Sigma_o - B_o^2)\tilde{\Sigma}_o$, $\zeta(s)$ is expressed as

$$\zeta(s) = \frac{\Omega\tilde{\Sigma}_o^s}{16\pi^2}\int_0^\infty \frac{q\ dq}{(q^2 + 2cq + d)^s} = \frac{\Omega\tilde{\Sigma}_o^s}{32\pi^2}\Big[\frac{d^{1-s}}{s-1} - \frac{c^{2-2s}\ {}_2F_1(s-\frac{1}{2}, s, s+\frac{1}{2}; 1 - d/c^2)}{(s-\frac{1}{2})}\Big]. \quad (72)$$

$\zeta(s)$ is analytic at $s = 0$ and we define $\text{tr}\ \log[AD - B_o^2] = -\zeta'(0)$. Then from (62)

$$\Gamma_1(B_o, \Sigma_o) = -\frac{1}{2}\zeta'(0) + \frac{1}{2}\Omega W_1(\Sigma_o). \quad (73)$$

$\zeta'(0)$ is complicated, but to understand the RGE $\delta_0\Gamma_1 + \delta_1\Gamma_0 = 0$ (61), we only need

$$\zeta(0) = \frac{\Omega}{32\pi^2}(2c^2 - d) = \frac{\Omega}{64\pi^2}\Big[\Sigma_o^2 + \lambda^2\tilde{\Sigma}_o^2 + 2B_o^2\tilde{\Sigma}_o\Big] = \frac{\Omega}{64\pi^2}\Big[\{1 + (96\pi^2)^2\lambda^2\}\Sigma_o^2 + 192\pi^2 B_o^2\Sigma_o\Big]. \quad (74)$$

Recall from Sec. 3.4 that $\delta_0\zeta'(0) = -2\zeta(0)$. Therefore,

$$\delta_0\Gamma_1 = \delta_0\Big(-\frac{1}{2}\zeta'(0) + \frac{1}{2}\Omega W_1(\Sigma_o)\Big) = \zeta(0) + \frac{1}{2}\delta_0\ \Omega W_1(\Sigma_o). \quad (75)$$

Thus, the RGE (61) for the effective action becomes

$$\zeta(0) + \frac{1}{2}\delta_0\ \Omega W_1(\Sigma_o) = -\delta_1\ \Gamma_0. \quad (76)$$

From (35) we know that $\Gamma_0(B_o, \Sigma_o) = \frac{\Omega}{2}[\Sigma_o B_o^2 - m^2\Sigma_o + \lambda\Sigma_o^2]$. Let us parameterize the leading $1/N$ correction to the renormalization group vector field as

$$\delta_1 = -\beta^m\frac{\partial}{\partial m} - \gamma^b B_o\frac{\partial}{\partial B_o} - \gamma^\sigma\Sigma_o\frac{\partial}{\partial\Sigma_o}. \quad (77)$$

$\beta^m(\lambda, m)$ is the beta function[12] of m while $\gamma^{b,\sigma}(\lambda, m)$ are anomalous dimensions. We are explicitly imposing the condition $\beta^\lambda = 0$, i.e. that λ remains unrenormalized and we have a line of fixed points. We only need one fixed point for naturalness, but as the sequel shows, we find a line of them within our approximations. $W_1(\Sigma_o)$, β and γ must satisfy

$$\begin{aligned}\zeta(0) + \frac{1}{2}\delta_0\Omega W_1(\Sigma_o) \quad &= \quad \Big[\beta_m\frac{\partial}{\partial m} + \gamma^b B_o\frac{\partial}{\partial B_o} + \gamma^\sigma\Sigma_o\frac{\partial}{\partial\Sigma_o}\Big]\frac{\Omega}{2}\Big(\Sigma_o B_o^2 - m^2\Sigma_o + \lambda\Sigma_o^2\Big) \\ \Rightarrow\ \frac{1}{2}\delta_0\Omega W_1(\Sigma_o) \quad &+ \quad \frac{\Omega}{64\pi^2}\Big[\{1 + (96\pi^2)^2\lambda^2\}\Sigma_o^2 + 192\pi^2 B_o^2\Sigma_o\Big] \\ &= \quad \frac{\Omega}{2}\Big[(-2m\beta^m - m^2\gamma^\sigma)\Sigma_o + 2\lambda\gamma^\sigma\Sigma_o^2 + (2\gamma^b + \gamma^\sigma)\Sigma_o B_o^2\Big]. \end{aligned} \quad (78)$$

[12] β and γ may receive corrections at $\mathcal{O}(1/N^2)$ as part of δ_2, so β^m is short for β_1^m etc.

We already determined $W_1(\Sigma)$ in sec. 5.1.1, from it we calculate

$$\frac{1}{2}\delta_0 \Omega W_1(\Sigma_0) = -\frac{\Omega\Sigma_o^2}{32\pi^2} + \frac{(96\pi^2)^2\lambda^2\Omega\Sigma_o^2}{16\pi^2}. \tag{79}$$

Putting this in (78) we get

$$\left(720\pi^2\lambda^2 - \frac{1}{64\pi^2}\right)\Sigma_o^2 + 3B_o^2\Sigma_o = \frac{1}{2}\Big[(-2m\beta^m - m^2\gamma^\sigma)\Sigma_o + 2\lambda\gamma^\sigma\Sigma_o^2 + (2\gamma^b + \gamma^\sigma)\Sigma_o B_o^2\Big]. \tag{80}$$

This equation must hold for *all* constant backgrounds B_o and Σ_o. If our model is not renormalizable or does not have a line of fixed points (for slowly varying quantum fluctuations at $\mathcal{O}(1/N)$), there would be no choice of $\beta^m, \gamma^{b,\sigma}$ for which it is identically satisfied. Comparing coefficients of monomials in the fields, we get the relations

$$2m\beta^m + m^2\gamma^\sigma = 0, \quad \lambda\gamma^\sigma = 720\pi^2\lambda^2 - \frac{1}{64\pi^2}, \quad \text{and} \quad \gamma^b + \frac{1}{2}\gamma^\sigma = 3, \tag{81}$$

whose unique solution is[13] (notice that $\beta^m \propto m$ so that $m = 0$ is preserved by RG flow)

$$\beta^m = m\left(\frac{1}{128\pi^2\lambda} - 360\pi^2\lambda\right); \quad \gamma^\sigma = 720\pi^2\lambda - \frac{1}{64\pi^2\lambda}, \quad \gamma^b = 3 + \frac{1}{128\pi^2\lambda} - 360\pi^2\lambda. \tag{82}$$

The resulting RG vector field is

$$\begin{aligned}\delta \;=\; &\left\{1 + \frac{1}{N}\left(360\pi^2\lambda - \frac{1}{128\pi^2\lambda}\right)\right\} m\frac{\partial}{\partial m} \;+\; \left\{1 + \frac{1}{N}\left(360\pi^2\lambda - 3 - \frac{1}{128\pi^2\lambda}\right)\right\} B_o\frac{\partial}{\partial B_o} \\ &+ \left\{2 + \frac{1}{N}\left(\frac{1}{64\pi^2\lambda} - 720\pi^2\lambda\right)\right\}\Sigma_o\frac{\partial}{\partial\Sigma_o} \;+\; \mathcal{O}(1/N^2).\end{aligned} \tag{83}$$

For constant backgrounds and slowly varying quantum fluctuations of σ, we have a consistent solution of the RGE for which λ remains unrenormalized ($\beta^\lambda = 0$). Under these assumptions, our model is renormalizable at order $1/N$. It remains to see whether renormalizability and $\beta^\lambda = 0$ can be maintained (for at least one λ), for non-constant backgrounds and rapidly varying quantum fluctuations, perhaps via an expansion in inverse powers of Σ_o.

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A Examples of naturalness

By a naturalness explanation for an 'unreasonably small' quantity we mean that the model acquires an additional symmetry when that quantity vanishes [1]. In the absence of such a

[13] As noted earlier, $\lambda = 0$ is a singular limit within our approximations, this is reflected in $\beta^m, \gamma^\sigma, \gamma^b$.

symmetry, if the quantity is dimensionless, the 'reasonable' or natural value for it is of order 1, and if it has dimensions, its natural value is of the order of the Planck scale, though there may be another scale depending on the context. The symmetry used in a naturalness explanation may be either continuous or discrete. Sometimes, it is more convenient to refer to the conservation law that follows from the symmetry. The actual small value of the quantity (if non-zero) is usually related to explicit breaking of the symmetry and can often be treated perturbatively. We give a few examples of naturalness explanations from different branches of physics. It appears that this concept explains several small parameters both in experimentally tested theories and mathematical models. Indeed, except for the Higgs mass, there is a naturalness explanation for every small parameter in the standard model. This gives us some confidence to turn things around: if there is an unreasonably small parameter in nature or in a model, then there must be some symmetry, which if exact, would make that parameter vanish. In this way, naturalness can be a useful guide to model building.

(1) Small electron mass compared to Planck mass: If $m_e = 0$ QED would gain chiral symmetry. The same applies to muon and tau masses. There is a different chiral symmetry for each. This puts no constraint on the ratios of lepton masses.

(2) Small current quark masses: For $N_{flavors} > 1$ if current quark masses are set to zero, QCD gains a partial unbroken chiral symmetry $SU(N_f)_V$ (isospin for $N_f = 2$).

(3) Small coupling constants can be explained by the separate conservation laws for particles, gained by setting coupling constants to zero.

(4) Some near-degeneracies of energy levels in atomic spectroscopy can be explained as due to the presence of a symmetry. In hydrogen-like atoms, the difference in energy between levels with the same principal and angular momentum quantum numbers n, l but different magnetic quantum number m vanishes if we have spherical symmetry. The small energy difference can be due to a magnetic field whose direction explicitly breaks spherical symmetry.

(5) Naturalness in classical mechanics: The fact that planetary orbits are nearly closed and nearly lie on a plane, are related to the conservation of angular momentum and the Laplace-Runge-Lenz vector in the Kepler problem. Quantum mechanically, in hydrogen-like atoms, the 'accidental degeneracy' of energy levels with the same value of l is due to a hidden $SO(4)$ symmetry whose conserved quantities are angular momentum and Laplace-Runge-Lenz vectors.

(6) Some near-degeneracies in atomic energy levels can be explained as due to parity symmetry. The small splittings are due to parity violation in the weak interactions.

(7) Experimentally, the mass of a photon is less than 10^{-16} eV outside a superconductor. This is explained by the exact $U(1)$ gauge symmetry if the photon is massless.

(8) The near degeneracy in the proton-neutron masses (and pion masses) may be explained as a consequence of isospin symmetry. If u and d quarks were degenerate in mass, isospin would be an exact symmetry of the strong interactions and the neutron and proton would be degenerate in mass (as would the three pions). Isospin is explicitly broken by the quark mass difference as well as electromagnetic interactions, which explain the small $n-p$ and $\pi^+ - \pi^0$ mass differences.

(9) Pions are naturally light compared to say, ρ mesons due to chiral symmetry. They are the pseudo-goldstone bosons of spontaneously broken chiral symmetry. If the quarks were massless, chiral symmetry would be exact at the level of the lagrangian, and be spontaneously broken to $SU(N_f)_V$, and the pions would be massless goldstone bosons. But in fact, quarks are not massless, this explicit breaking of chiral symmetry gives the pions a small mass.

(10) Small neutrino masses: Chiral symmetry for each flavor is exact if neutrinos are massless.

(11) Parity is an exact symmetry of QCD in the absence of the topological θ term, which is parity odd. Thus, a small QCD theta angle is natural within the theory of strong interactions.

(12) The scalar field mass in a supersymmetric extension of the standard model can be naturally small since if it were zero, the theory would have unbroken global supersymmetry (when the super partner fermion is also massless, which would be natural due to chiral symmetry).

(13) The effective mass (inverse of correlation length) is very small in the neighborhood of a 2^{nd} order phase transition. A naturalness explanation is that at such a transition, when the effective mass vanishes, the system gains a new symmetry, scale-invariance. However, one might argue that a naturalness explanation is not needed here, since the temperature must be fine-tuned in order to have such a phase transition.

(14) Some linear combinations of correlations in large-N multi-matrix models vanish because of the presence of hidden non-anomalous symmetries [21].

(15) It has been suggested [22] that a discrete symmetry that relates real-valued space-time coordinates to pure-imaginary ones could ensure a naturally small cosmological constant.

(16) The amplitudes for gluon scattering either with the same helicity or with only one gluon with a different helicity from the others, vanishes in classical Yang-Mills theory. A naturalness explanation for this has been suggested, using an effective tree-level supersymmetry [23]. This is an example of naturalness in classical field theory.

(17) GIM mechanism and suppression of flavor-changing neutral currents (FCNC) [17]: The small quantity here is the $\Delta s = 1$ strangeness changing neutral current compared to the $\Delta s = 0$ neutral current. The GIM mechanism eliminates $\Delta s = 1$ FCNC at tree level by introducing a new quark doublet participating in the weak interactions, consisting of charm and the Cabibbo rotated strange quark. This is not a naturalness explanation, but could be turned into one by specifying a symmetry which ensures absence of FCNC.

B The quantum effective action

To be self-contained, we collect a few facts about the effective action (see eg. [19]). The generating series for (possibly disconnected) correlations of a scalar field ϕ is

$$Z(J) = \int [D\phi] e^{-\frac{1}{\hbar}[S(\phi)+J\cdot\phi]} \quad \text{where} \quad J\cdot\phi = \int d^4x \; J(x)\phi(x) \tag{84}$$

The generating series of connected correlations is $W(J) = -\hbar \log Z(J)$. The effective action $\Gamma(\Phi)$ is the generating series of proper vertices (1-particle irreducible diagrams). It is the Legendre transform of $W(J)$: $\Gamma(\Phi) = \text{ext}_J[W(J) - J\cdot\Phi]$. The solution of the *classical* theory (tree diagrams) defined by Γ is equivalent to a solution of the *quantum* theory of the original action S. An implicit integral representation for $\Gamma(\Phi)$ is

$$e^{-\Gamma(\Phi)/\hbar} = \int [D\phi] e^{-\frac{1}{\hbar}[S(\Phi+\phi)-\phi\cdot\frac{\delta\Gamma(\Phi)}{\delta\Phi}]}. \tag{85}$$

Here Φ is the background field and ϕ is the fluctuating field. Φ need not solve the classical equations of motion, i.e. $S'(\Phi)$ need not be zero. To obtain this integral representation we start from the relation $W(J) = \text{ext}_\Phi[\Gamma(\Phi) + J \cdot \Phi]$. The extremum occurs at $J = -\frac{\delta\Gamma(\Phi)}{\delta\Phi}$. With J and Φ related this way, the effective action is

$$\Gamma(\Phi) = W(J) - J \cdot \Phi = -\hbar \log Z(J) - J \cdot \Phi. \tag{86}$$

In other words, $e^{-\frac{1}{\hbar}\Gamma(\Phi)} = Z(J) e^{\frac{1}{\hbar} J \cdot \Phi}$. Inserting the path integral for $Z(J)$,

$$e^{-\frac{1}{\hbar}\Gamma(\Phi)} = \int [D\chi] e^{-\frac{1}{\hbar}[S(\chi) + J\cdot\chi - J\cdot\Phi]}. \tag{87}$$

Changing variables of integration to $\phi = \chi - \Phi$ and using $J = J(\Phi) = -\frac{\delta\Gamma(\Phi)}{\delta\Phi}$, we get the implicit formula (85) for the effective action. By expanding $S(\Phi + \phi)$ in powers of ϕ and doing the gaussian integrals, we get an asymptotic series for $\Gamma(\Phi)$ in powers of $\hbar$ (the same formula holds with $\hbar$ replaced by $1/N$)

$$\Gamma(\Phi) = S(\Phi) + \frac{\hbar}{2} \log\det S''(\Phi) + \mathcal{O}(\hbar^2). \tag{88}$$

B.1 Large-N effective potential via dimensional regularization

We wish to calculate $\text{tr}\ \log[-\nabla^2 + \Sigma_o] = \Omega \int [d^4p] \log[p^2 + \Sigma_o]$ appearing in the large-N effective action (22) via dimensional regularization. Analytically continuing to n Euclidean dimensions and differentiating, we get a convergent integral if $n < 2$

$$T_n = \int [d^np] \log[p^2 + \Sigma_o] \quad \Rightarrow \quad \frac{\partial T_n}{\partial \Sigma_o} = \int \frac{[d^np]}{(p^2 + \Sigma_o)} = (4\pi)^{-n/2}\ \frac{\Gamma(1 - n/2)}{\Sigma_o^{1-n/2}}. \tag{89}$$

Here $[d^np] = \frac{d^np}{(2\pi)^n}$. Expanding in a Laurent series around $n = 4$ dimensions,

$$\frac{\partial T_n}{\partial \Sigma_o} = \frac{\Sigma_o}{8\pi^2(n-4)} + \frac{\Sigma(\gamma - 1 + \log[\Sigma_o/4\pi])}{16\pi^2} + \mathcal{O}(n-4). \tag{90}$$

Now $\hbar\ \text{tr}\ \log[-\nabla^2 + \Sigma_o] = \hbar\Omega T_n$. So integrating with respect to Σ_o,

$$\hbar\ \text{tr}\ \log[-\nabla^2 + \Sigma_o] = \frac{\hbar\Omega\Sigma_o^2}{16\pi^2(n-4)} + \frac{\hbar\Omega\Sigma_o^2}{32\pi^2}(\gamma - \frac{3}{2} - \log 4\pi) + \frac{\hbar\Omega\Sigma_o^2 \log\Sigma_o}{32\pi^2} + c\hbar\Omega + \mathcal{O}(n-4). \tag{91}$$

Integration constant c is independent of Σ_o and plays no role since it only contributes an additive constant to the effective potential. $\gamma = .577$ is Euler's constant. We have a pole part, finite part and terms that vanish as $n \to 4$. Notice that the finite part that transforms inhomogeneously under scale transformations $\frac{\hbar\Omega\Sigma_o^2 \log\Sigma_o}{32\pi^2}$ is the same as in cutoff or zeta-function regularization. The choice of W_0 that makes the effective action finite and scale-free in the limit $n \to 4$ is

$$W_0(\Sigma_o, n) = -\frac{\hbar\Sigma_o^2}{16\pi^2(n-4)} - \frac{\hbar\Sigma_o^2\log\Sigma_o}{32\pi^2} + \lambda\Sigma_o^2. \tag{92}$$

The finite and scale-free effective action for constant backgrounds is thus

$$\Gamma_0(B_o, \Sigma_o) = \frac{\Omega}{2}\left[\left(\lambda + \frac{\hbar(\gamma - 3/2 - \log 4\pi)}{32\pi^2}\right) + \Sigma_o^2 B_o\right]. \tag{93}$$

We get a line of fixed points parameterized by λ, whose definition is scheme dependent

$$\lambda_{\text{zeta-fn}} = \lambda_{\text{dim-reg}} + \frac{\hbar(\gamma - 3/2 - \log 4\pi)}{32\pi^2}. \tag{94}$$

C Expansion of $\text{tr}\ \log[-\nabla^2+\sigma]$ in powers and derivatives of σ

C.1 Zeta function in terms of the heat kernel

Let $A = -\nabla^2 + \sigma(x)$ and $\zeta_A(s) = \ \text{tr}\ A^{-s}$. Then $\text{tr}\ \log A = \zeta'(0)$. We get an integral representation for $\zeta_A(s)$ by making a change of variable $t \mapsto At$ in the formula for $\Gamma(s)$:

$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt\ e^{-tA}\ t^{s-1}. \tag{95}$$

Now define the evolution operator $\hat{h}_t = e^{-tA}$ which satisfies a generalized heat equation

$$\frac{d\hat{h}_t}{dt} = -A\hat{h}_t = (\nabla^2 - \sigma)\ \hat{h}_t, \qquad \lim_{t\to 0^+} \hat{h}_t = 1 \tag{96}$$

It is convenient to work with the heat kernel $\hat{h}_t\ \psi(x) = \int\ dy\ h_t(x,y)\ \psi(y)$ which satisfies

$$\frac{\partial h_t(x,y)}{\partial t} = \left[\nabla^2 - \sigma(x)\right]\ h_t(x,y) \quad \text{and} \quad \lim_{t\to 0^+} h_t(x,y) = \delta(x-y). \tag{97}$$

Then $\zeta_A(s)$ is the Mellin transform of the trace of the heat kernel:

$$\zeta_A(s) = \ \text{tr}\ A^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1}\ \ \text{tr}\ e^{-tA} = \frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1}\ \int d^4x\ h_t(x,x). \tag{98}$$

To find $h_t(x,x)$ we need to solve (97). We find $h_t(x,y)$ and take $x \to y$ in the end. For $\sigma = 0$, $h_t(x,y)$ satisfies the diffusion equation and the solution is

$$h_t^o(x,y) = \frac{1}{(4\pi t)^2} e^{-\frac{(x-y)^2}{4t}}. \tag{99}$$

For constant complex $\sigma = \sigma_o$ (97) is a PDE with constant coefficients whose solution is

$$h_t^o(x,y) = \frac{1}{(4\pi t)^2} e^{-t\sigma_o} e^{-\frac{(x-y)^2}{4t}}. \tag{100}$$

C.2 Short time expansion for heat kernel

For non-constant σ we get an expansion for the heat kernel in derivatives and powers of σ for small t. We assume that the non-analytic part of the heat kernel is already captured by the exact solution (100). We let $\sigma = \sigma_o + \varsigma(x)$ and make the ansatz

$$h_t(x,y) = h_t^o(x,y)\ \sum_{n=0}^\infty a_n(x,y) t^n = \frac{e^{-\sigma_o t} e^{-(x-y)^2/4t}}{(4\pi t)^2}\ \sum_{n=0}^\infty a_n(x,y) t^n. \tag{101}$$

The average value of ς need not vanish. However, we assume that $\nabla\varsigma(x) \to 0$ as $|x| \to \infty$ so that the average value of derivatives of ς and its powers vanish

$$\int dx\ (\nabla^2)^p \varsigma^q(x) = 0, \quad p,q = 1,2,3\ldots \tag{102}$$

For (101) to satisfy the initial condition (97), we need $a_0 = 1$. If σ is a constant, $a_i = \delta_{0,i}$. In (101), we could absorb $e^{-\sigma_o t}$ into the infinite series since it is analytic in t, but that would amount to throwing away an exact result, so let us not do it. Moreover, $e^{-\sigma_o t}$ makes the Mellin transform (98) of the heat kernel convergent for $\Re\sigma_o > 0$, which is necessary to recover $\zeta_A(s)$. The coefficients $a_n(x,y)$ are to be obtained by putting (101) into the heat equation (97) and comparing coefficients of common powers of t. We need the expressions

$$\begin{aligned}\frac{\partial h_t}{\partial t} &= \frac{\partial h_t^o}{\partial t}\sum_0^\infty a_n\, t^n \;+\; h_t^o \sum_1^\infty n\, a_n\, t^{n-1} \qquad \text{and} \\ \nabla^2 h_t &= \nabla^2\, h_t^o \sum_0^\infty a_n\, t^n + 2\nabla_i h_t^o \sum_0^\infty t^n \nabla_i\, a_n + h_t^o \sum_0^\infty t^n \nabla^2 a_n. \end{aligned} \tag{103}$$

We put these into the generalized heat equation (97) and get

$$\begin{aligned}\frac{\partial h_t^o}{\partial t}\sum_0^\infty a_n\, t^n \;+\; h_t^o \sum_1^\infty n\, a_n\, t^{n-1} &= \nabla^2\, h_t^o \sum_0^\infty a_n\, t^n + 2\nabla_i h_t^o \sum_0^\infty t^n \nabla_i\, a_n \\ &\quad + h_t^o \sum_0^\infty t^n \nabla^2 a_n - (\sigma_o + \varsigma) h_t^o \sum_0^\infty a_n t^n. \end{aligned} \tag{104}$$

Using the fact that $\partial_t h^o = (\nabla^2 - \sigma_o) h^o$ this simplifies to

$$\sum_0^\infty (n+1)\, a_{n+1}\, t^n = 2\frac{\nabla_i h_t^o}{h_t^o} \sum_0^\infty t^n \nabla_i\, a_n + \sum_0^\infty t^n \nabla^2 a_n - \varsigma \sum_0^\infty a_n t^n. \tag{105}$$

Now

$$\frac{\nabla_i h^o}{h^o} = \nabla_i \log h^o = \nabla_i\Big[- (x-y)^2/4t\Big] = -\frac{(x-y)_i}{2t}. \tag{106}$$

So the generalized heat equation becomes

$$\sum_0^\infty (n+1)\, a_{n+1}\, t^n = -(x-y)_i \sum_0^\infty t^n \nabla_i\, a_{n+1} + \sum_0^\infty t^n \nabla^2 a_n - \varsigma \sum_0^\infty a_n t^n. \tag{107}$$

Comparing coefficients of t^n determines a_{n+1} given a_n with the initial condition $a_0 = 1$

$$\Big\{(x-y)_i + n + 1\Big\} a_{n+1}(x,y) = (\nabla^2 - \varsigma) a_n(x,y). \tag{108}$$

Now only $a_n(x,x)$ appear in $\zeta(s)$, so we specialize to

$$a_{n+1}(x,x) = \frac{1}{(n+1)} (\nabla^2 - \varsigma) a_n(x,x). \tag{109}$$

The first few $a_n(x,x)$ are

$$\begin{aligned} a_1 &= -\varsigma(x); \qquad a_2 = \frac{1}{2}(\nabla^2 - \varsigma) a_1 = \frac{1}{2}(\varsigma^2 - \nabla^2 \varsigma) \\ a_3 &= \frac{1}{3}(\nabla^2 - \varsigma) a_2 = \frac{1}{3!}(\nabla^2 - \varsigma)(\varsigma^2 - \nabla^2\varsigma) = \frac{1}{3!}(\varsigma \nabla^2 \varsigma - \varsigma^3 + \nabla^2 \varsigma^2 - (\nabla^2)^2 \varsigma) \end{aligned}$$

$$
\begin{aligned}
a_4 &= \frac{1}{4}(\nabla^2-\varsigma)a_3 = \frac{1}{4!}(\nabla^2-\varsigma)(\varsigma\nabla^2\varsigma - \varsigma^3 + \nabla^2\varsigma^2 - (\nabla^2)^2\varsigma)\\
&= \frac{1}{4!}(\nabla^2(\varsigma\nabla^2\varsigma) - \nabla^2\varsigma^3 + \nabla^4\varsigma^2 - \nabla^6\varsigma - \varsigma^2\nabla^2\varsigma + \varsigma^4 - \varsigma\nabla^2\varsigma^2 + \varsigma\nabla^4\varsigma). \qquad (110)
\end{aligned}
$$

To summarize, the heat kernel expansion is $h_t(x,x) = \frac{e^{-\sigma_o t}}{(4\pi t)^2}\sum_0^\infty a_n t^n$. If we drop cubic and higher powers of ς, then the a_n are

$$
\begin{aligned}
a_0 &= 1; \quad a_1 = -\varsigma; \qquad a_2 = \frac{1}{2}(\varsigma^2 - \nabla^2\varsigma); \quad a_3 = \frac{1}{3!}(\varsigma\nabla^2\varsigma + \nabla^2\varsigma^2 - (\nabla^2)^2\varsigma) + \mathcal{O}(\varsigma^3);\\
a_4 &= \frac{1}{4!}(\nabla^2(\varsigma\nabla^2\varsigma) + \nabla^4\varsigma^2 - \nabla^6\varsigma + \varsigma\nabla^4\varsigma) + \mathcal{O}(\varsigma^3) \quad \text{etc.} \qquad (111)
\end{aligned}
$$

For $\zeta_A(s)$ we need $\langle a_n\rangle = \int d^4x a_n(x,x)/\int d^4x$. Assuming $\varsigma \to$ constant as $|x| \to \infty$ and $\nabla\varsigma \to 0$ at ∞ we get (up to terms involving cubic and higher powers of ς),

$$
\langle a_0\rangle = 1, \quad \langle a_1\rangle = -\langle\varsigma\rangle, \quad \langle a_2\rangle = \frac{1}{2!}\langle\varsigma^2\rangle, \quad \langle a_3\rangle = \frac{1}{3!}\langle\varsigma\nabla^2\varsigma\rangle + \mathcal{O}(\varsigma^3) \quad \text{etc.} \qquad (112)
$$

More generally, $\langle a_n\rangle = \frac{1}{n!}\langle\varsigma(\nabla^2)^{n-2}\varsigma\rangle + \mathcal{O}(\varsigma^3)$ for $n = 3,4,5,\ldots$.

C.3 Derivative expansion for $\mathrm{tr}\ \log[-\nabla^2+\sigma]$

We use (98) and the expansion (101) to get an expansion for $\zeta_A(s)$ in derivatives of $\sigma = \sigma_o + \varsigma$

$$
\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty dt \int d^4x\ t^{s-1}\ h_t(x,x) = \frac{1}{\Gamma(s)}\int d^4x \int_0^\infty dt\ t^{s-1}\frac{e^{-\sigma_o t}}{(4\pi t)^2}\sum_0^\infty a_n(x,x)t^n\\
\frac{\zeta(s)}{\Omega} &= \frac{1}{16\pi^2\Gamma(s)}\sum_0^\infty \langle a_n\rangle \int_0^\infty dt\ t^{s+n-3}e^{-\sigma_o t} \qquad (113)
\end{aligned}
$$

where $\Omega = \int d^4x$. The integral over t is a Gamma function,

$$
\begin{aligned}
\frac{\zeta(s)}{\Omega} &= \frac{1}{16\pi^2\Gamma(s)}\sum_n \langle a_n\rangle\sigma_o^{2-n-s}\Gamma(s+n-2) = \frac{\sigma_o^{2-s}}{16\pi^2}\sum_n \frac{\langle a_n\rangle}{\sigma_o^n}\frac{\Gamma(s+n-2)}{\Gamma(s)}\\
&= \frac{\sigma_o^{2-s}}{16\pi^2}\left[\frac{\langle a_0\rangle}{(s-1)(s-2)} + \frac{\langle a_1\rangle}{(s-1)\sigma_o} + \frac{\langle a_2\rangle}{\sigma_o^2} + \frac{s\langle a_3\rangle}{\sigma_o^3} + \frac{s(s+1)\langle a_4\rangle}{\sigma_o^4}\cdots\right]. \qquad (114)
\end{aligned}
$$

Differentiating and setting $s = 0$ we get

$$
\begin{aligned}
\frac{\zeta'(0)}{\Omega} &= -\frac{\sigma_o^2\log\sigma_o}{16\pi^2}\left[\frac{\langle a_0\rangle}{2} - \frac{\langle a_1\rangle}{\sigma_o} + \frac{\langle a_2\rangle}{\sigma_o^2}\right] + \frac{\sigma_o^2}{16\pi^2}\left[\frac{3\langle a_0\rangle}{4} - \frac{\langle a_1\rangle}{\sigma_o} + \sum_{n=3}^\infty \frac{(n-3)!\langle a_n\rangle}{\sigma_o^n}\right]\\
&= -\frac{\sigma_o^2}{16\pi^2}\left[\frac{\langle a_0\rangle}{2}\log[\sigma_o e^{-3/2}] + \frac{\langle a_1\rangle}{\sigma_o}(1-\log\sigma_o) + \frac{\langle a_2\rangle}{\sigma_o^2}\log\sigma_o - \sum_{n=3}^\infty \frac{(n-3)!\langle a_n\rangle}{\sigma_o^n}\right] \qquad (115)
\end{aligned}
$$

Inserting expressions for $\langle a_n\rangle$ from (112), we get a formula for $\mathrm{tr}\ \log[-\nabla^2+\sigma] = -\zeta'(0)$.

$$
\begin{aligned}
\mathrm{tr}\ \log[-\nabla^2+\sigma] &= \frac{\sigma_o^2\Omega}{16\pi^2}\left[\frac{1}{2}\log[\frac{\sigma_o}{e^{3/2}}] + \frac{\langle\varsigma\rangle}{\sigma_o}\log[\frac{\sigma_o}{e}] + \frac{\langle\varsigma^2\rangle}{2\sigma_o}\log\sigma_o\right.\\
&\left.-\sum_{n=3}^\infty \frac{\langle\varsigma\nabla^{2n-4}\varsigma\rangle}{n(n-1)(n-2)\sigma_o^n}\right] + \mathcal{O}(\varsigma^3)
\end{aligned}
$$

$$= \frac{\sigma_o^2\Omega}{32\pi^2}\log[\frac{\sigma_o}{e^{3/2}}] + \int d^4x\Big[\frac{\sigma_o\log[\frac{\sigma_o}{e}]}{16\pi^2}\varsigma + \frac{\log\sigma_o}{32\pi^2}\varsigma^2 - \frac{1}{16\pi^2}\sum_{n=3}^{\infty}\frac{\varsigma(\nabla^2)^{n-2}\varsigma}{n(n-1)(n-2)(\sigma_o)^{n-2}} + \mathcal{O}(\varsigma^3)\Big]. \quad (116)$$

The sum over powers of ∇^2 can be performed. Let $\Delta = -\frac{\nabla^2}{\sigma_o}$ and

$$\Pi(\Delta) = \sum_{n=1}^{\infty}\frac{(-\Delta)^n}{n(n+1)(n+2)} = \frac{\Delta(3\Delta+2) - 2(\Delta+1)^2\log(1+\Delta)}{4\Delta^2}. \quad (117)$$

$\Pi(\Delta)$ is analytic at $\Delta = 0$, $\Pi(\Delta) = -\frac{\Delta}{6} + \frac{\Delta^2}{24} - \frac{\Delta^3}{60} + \cdots$. For large [14] Δ,

$$\Pi(\Delta) \to -\frac{1}{2}\log\Delta + \frac{3}{4} - \frac{\log\Delta}{\Delta} + \mathcal{O}(\Delta^{-2}). \quad (118)$$

The final result is

$$\text{tr }\log[-\nabla^2+\sigma] = \frac{\sigma_o^2\Omega}{32\pi^2}\log\left[\frac{\sigma_o}{e^{3/2}}\right] + \int\frac{d^4x}{16\pi^2}\left[\sigma_o\log\left[\frac{\sigma_o}{e}\right]\varsigma + \frac{\log\sigma_o}{2}\varsigma^2 - \varsigma\Pi(\Delta)\varsigma + \mathcal{O}(\varsigma^3)\right]. \quad (119)$$

Here $\sigma(x) = \sigma_o + \varsigma(x)$, Ω is the volume of space-time and $\Pi(\Delta)$ is defined above. This formula is valid for small deviations of σ from a constant background σ_o. The term proportional to ς vanishes if σ_o is the average value of σ. σ need not be slowly varying. We assumed that ς approaches a constant as $|x| \to \infty$ and that $\nabla\varsigma \to 0$ as $|x| \to \infty$.

Remark: If σ is slowly varying, we ignore terms with more than two derivatives to get

$$\text{tr }\log[-\nabla^2+\sigma] = \frac{\sigma_o^2\Omega}{32\pi^2}\log[\frac{\sigma_o}{e^{3/2}}] + \int\frac{d^4x}{16\pi^2}\left[\sigma_o\log[\frac{\sigma_o}{e}]\varsigma + \frac{\log\sigma_o}{2}\varsigma^2 - \frac{\varsigma\nabla^2\varsigma}{6\sigma_o} + \mathcal{O}(\varsigma^3)\right] \quad (120)$$

where $\sigma = \sigma_o + \varsigma$ and σ_o is a constant.

C.4 Scale anomaly $\zeta(0)$ for general backgrounds

Though we only got an asymptotic series for $\zeta'(0)$ around a constant background, we can get an exact closed-form expression for its scale anomaly. Under a scale transformation $\sigma \mapsto a^2\sigma$,

$$\begin{array}{rcl} \zeta(s) \mapsto a^{-2s}\zeta(s) & \Rightarrow & \zeta'(s) \mapsto -2\zeta(s)a^{-2s}\log a + a^{-2s}\zeta'(s) \\ \zeta'(0) & \mapsto & \zeta'(0) - 2\zeta(0)\log a \end{array} \quad (121)$$

Now we use our result to find the scale anomaly $\zeta(0)$:

$$\frac{\zeta(s)}{\Omega} = \frac{(\sigma_o)^{2-s}}{16\pi^2}\Big[\frac{\langle a_0\rangle}{(s-1)(s-2)} + \frac{\langle a_1\rangle}{(s-1)\sigma_o} + \frac{\langle a_2\rangle}{\sigma_o^2} + \frac{s\langle a_3\rangle}{\sigma_o^3} + \frac{s(s+1)\langle a_4\rangle}{\sigma_o^4}\cdots\Big]. \quad (122)$$

Since all higher order terms are proportional to s, only a_0, a_1 and a_2 contribute to $\zeta(0)$:

$$\zeta(0) = \frac{\Omega\sigma_o^2}{16\pi^2}\Big[\frac{\langle a_0\rangle}{2} - \frac{\langle a_1\rangle}{\sigma_o} + \frac{\langle a_2\rangle}{\sigma_o^2}\Big] = \frac{\Omega\sigma_o^2}{16\pi^2}\Big[\frac{1}{2} + \frac{\langle\varsigma\rangle}{\sigma_o} + \frac{\langle\varsigma^2\rangle}{2\sigma_o^2}\Big]. \quad (123)$$

[14] $\Delta = -\frac{\nabla^2}{\sigma_o}$. Note that $-\nabla^2$ is a positive operator and σ_o is not a negative real number.

D Original potential $V(\phi)$ in zeta function regularization

The simplest way to describe the interactions of our model in the large-N limit is via the finite and scale-invariant effective action $\Gamma_0(B, \Sigma)$ of eqn. (41). Physical intuition and approximation methods can be applied to Γ. The original potential $V(\phi^2/N)$ is *not* the effective potential, that honor goes to $\Gamma_0(B, \Sigma)$ for constant backgrounds. The minima of $V(\phi^2/N)$ have no physical significance. What is more, $V(\phi^2/N)$ is *not* scale-free, it depends on a scale parameter M. But, M is not a parameter of the theory, it is canceled by scale 'anomalies' from quantum fluctuations. Moreover, $W(\sigma)$ is divergent, so $V(\eta)$ is not strictly defined independent of a regularization scheme. Despite all these warnings, many physicists wish to know what $V(\eta)$ is, so we find the $V(\eta)$ that corresponds to $W(\sigma)$ obtained in zeta function regularization. In the large-N limit, we find that for a constant background field σ, $V(\eta)$ grows as $\eta^2/\log\eta$ for large $\eta = \phi^2/N$. We haven't yet determined its behavior for small η.

Recall that $e^{-NV(\eta(x))}$ is the inverse Laplace transform of $e^{-NW(\sigma(x))}$ for each x:

$$\int_{\mathcal{C}} \frac{d\sigma}{2\pi i} e^{-(N/2\hbar)(W(\sigma)+\sigma\eta)} = e^{-(N/2\hbar)V(\eta)}. \tag{124}$$

We found $W(\sigma)$ only at $N = \infty$, so it makes sense to invert the Laplace transform for large-N. We do this here for constant σ, for which we have found[15] $W(\sigma)$ exactly in (34):

$$W(\sigma) + \eta\sigma = \eta\sigma - m^2\sigma - \frac{\hbar\sigma^2}{32\pi^2}\log(\tilde{\lambda}\sigma) \quad \text{where} \quad \tilde{\lambda} = e^{-[32\pi^2\lambda\hbar^{-1}+3/2]}. \tag{125}$$

The branch cut of $W(\sigma)$ implies $\mathcal{C}$ runs from $-i\infty$ to $i\infty$ avoiding the negative real σ axis. Putting $\sigma = u + iv$ for $u, v \in \mathbf{R}$, the real and imaginary parts of $W(\sigma) + \eta\sigma = \varphi + i\psi$ are ($\hbar = 1$)

$$\begin{aligned} \varphi &= (\eta - m^2)u - \frac{(u^2 - v^2)\log\tilde{\lambda}\sqrt{u^2+v^2}}{32\pi^2} + \frac{uv\arctan(v/u)}{16\pi^2} \\ \psi &= (\eta - m^2)v - \frac{(u^2 - v^2)\arctan(v/u)}{32\pi^2} - \frac{uv\log(\tilde{\lambda}\sqrt{u^2+v^2})}{16\pi^2}. \end{aligned} \tag{126}$$

We wee that $\Re(W(\sigma) + \eta\sigma) \to \infty$ as $v \to \pm\infty$ for any $u \geq 0$. So the integrand vanishes along the lines $u \pm i\infty$ for any u. Thus, the end points of $\mathcal{C}$ can be moved from $\pm i\infty$ to $\pm i\infty + u_\pm$ for any real $u_\pm$ without altering the integral. Simply put, it does not matter along which longitude $\mathcal{C}$ leaves the south pole or along which longitude it approaches the north pole.

The general strategy for estimating such integrals is as follows [20]. $W(\sigma) + \eta\sigma$ is in general complex on $\mathcal{C}$. Its imaginary part ψ will lead to a highly oscillatory integral as $N \to \infty$ and make it difficult to estimate. The trick is to use analyticity of $W(\sigma) + \eta\sigma$ to deform the contour to a (union of) contour(s) along which $\Im(W(\sigma) + \eta\sigma)$ is constant and other contours where the integrand vanishes. Such contours are called constant phase contours and coincide with the steepest contours, those along which the absolute value of the integrand changes fastest. If $\mathcal{C}$ is such a contour (assumed to be a single one for simplicity), then

$$\int_{\mathcal{C}} \frac{d\sigma}{2\pi i} e^{-(N/2\hbar)(W(\sigma)+\sigma\eta)} = e^{-\frac{Ni}{2\hbar}\Im(W(\sigma)+\sigma\eta)} \int_{\mathcal{C}} \frac{d\sigma}{2\pi i} e^{-(N/2\hbar)\Re(W(\sigma)+\sigma\eta)}. \tag{127}$$

[15] We should have a mass scale M to set the scale for the logarithms. We set $M = 1$ in this section.

Now we have eliminated the oscillating phase and for large-N, the asymptotic behavior is determined by the local minima of $\varphi = \Re(W(\sigma) + \sigma\eta))$ along $\mathcal{C}$. Since φ diverges at the end points of $\mathcal{C}$, local minima must occur at interior points of $\mathcal{C}$. Moreover, there must be an odd number $1, 3, 5, \cdots$ of such local minima along $\mathcal{C}$. At any one, the directional derivatives of both φ and ψ vanish in the direction tangent to the curve. Since $\varphi + i\psi$ is analytic, it follows that these local minima of φ are saddle points, i.e. $\partial_\sigma(W(\sigma) + \sigma\eta) = 0$, where two or more steepest curves intersect. Not all saddle points of $W(\sigma) + \sigma\eta$ need lie on $\mathcal{C}$ and those that don't will not contribute to the asymptotic behavior of the integral.

Suppose $\sigma = \sigma_s(\eta)$ is the only saddle point along the constant phase contour $\mathcal{C}$. The integrand attains a local maximum at σ_s along $\mathcal{C}$ and decays exponentially in either direction away from σ_s. The contour can be approximated by a straight line tangent to $\mathcal{C}$ at σ_s and of length ϵ on either side. $\varphi(\sigma) = \Re(W + \eta\sigma)$ is approximated by its quadratic Taylor polynomial around σ_s, whose linear term vanishes. Now we let $\epsilon \to \infty$. $\varphi(\sigma_s)$ gives the leading contribution while the quadratic term in its Taylor series gives a gaussian integral proportional to $\frac{1}{\sqrt{N}}$. So

$$\begin{aligned} e^{-\frac{N}{2}V(\eta)} &= e^{-\frac{iN}{2}\psi(\sigma_s)} e^{-\frac{N}{2}\varphi(\sigma_s)} \frac{1}{2\pi i} \mathcal{O}(\frac{1}{\sqrt{N}}), \\ \Rightarrow \quad V(\eta) &= \varphi(\sigma_s) + i\psi(\sigma_s) + \mathcal{O}\left(\frac{\log N}{N}\right). \end{aligned} \tag{128}$$

If there is more than one saddle point on $\mathcal{C}$, we add up their contributions in this formula for $V(\eta)$. Moreover, if the saddle point σ_s is on the real axis, then $\psi(\sigma_s) = 0$ does not contribute.

So our job is to find a convenient constant phase contour and identify the saddle points on it. In practice, we look for saddle points and then a suitable contour. The saddle point condition for $W(\sigma) + \sigma\eta$ for given η, m^2 and λ is

$$\frac{16\pi^2}{\hbar}(\eta - m^2) = \sigma \log(\tilde{\lambda}\sigma\sqrt{e}). \tag{129}$$

Taking imaginary and real parts, it is a pair of transcendental equations (for $\tilde{\lambda} = 1$ and $\hbar = 1$)

$$\begin{aligned} \frac{v}{2} + \frac{v}{2}\log(u^2 + v^2) &= -u \arctan(v/u) \quad \text{and} \\ 16\pi^2(\eta - m^2) &= \frac{u}{2} + \frac{u}{2}\log(u^2 + v^2) - v \arctan(v/u). \end{aligned} \tag{130}$$

The fundamental domain for the arctangent is taken as $-\pi < \arctan < \pi$. We must solve for $\sigma = u + iv$ assuming it is off the negative real axis. Any $u > 0$ for $v = 0$ satisfies the first condition (i.e. saddle points can lie on the positive real σ axis), but there are other possibilities. The imaginary part of the saddle point condition is also satisfied along a curve (found numerically) in the $u-v$ plane that encircles the origin and is symmetric under reflections about either axis[16] and lies within the rectangle[17] $|u| \leq e^{-3/2}, |v| \leq \frac{1}{\sqrt{e}}$. However, the second saddle point condition is satisfied on this curve only for a limited range of values of $\eta - m^2$, namely $m_c^2 \geq \eta - m^2 \geq -\frac{1}{32\pi\sqrt{e}}$ for $-e^{-3/2} \leq u \leq 0$ and $-\frac{1}{32\pi\sqrt{e}} \leq \eta - m^2 \leq -m_c^2$ for $0 \leq u \leq e^{-3/2}$ where $m_c^2 = \frac{\hbar e^{-3/2}}{16\pi^2\tilde{\lambda}}$. For $\eta - m^2$ in this range, saddle points could occur on the above mentioned curve as well as the positive real σ axis, making their analysis more involved.

[16] This follows from the even and oddness of the condition in u and v respectively.

[17] The limiting values are obtained by solving the first saddle point condition for small v and u respectively.

For now, we set aside the behavior of $V(\eta)$ for small[18] η, i.e. $\eta - m^2 \leq m_c^2$. For $\eta - m^2 \geq m_c^2$ the only possible saddle points are located on the positive real σ axis ($v = 0$). In this case, the positions of the saddle points are given by the solutions to equation (129) where $\sigma = u$ is real. Since we assumed $\eta - m^2 > m_c^2$, the LHS is in particular positive, and there is a unique solution σ_s which can be found recursively

$$\sigma_s = \frac{\tilde{\eta}}{\log(\tilde{\lambda}\sqrt{e}\sigma_s)} = \frac{\tilde{\eta}}{\log\left(\frac{\tilde{\lambda}\sqrt{e}\tilde{\eta}}{\log(\tilde{\lambda}\sqrt{e}\sigma_s)}\right)} = \frac{\tilde{\eta}}{\log(\tilde{\lambda}\sqrt{e}\tilde{\eta}) - \log\log(\tilde{\lambda}\sqrt{e}\sigma_s)} = \cdots \tag{131}$$

where $\tilde{\eta} = 16\pi^2(\eta - m^2)/\hbar$. Thus, for sufficiently large $\eta - m^2$, there is only a single saddle point σ_s. Moreover, we have checked numerically that there is a constant phase (actually zero phase, since $\psi(\sigma_s) = 0$) contour that starts at the south pole, passes through σ_s and goes to the north pole. For large $\eta - m^2$, the leading approximation for the position of the saddle point is (with $\bar{\lambda} = 16\pi^2\sqrt{e}\tilde{\lambda}/\hbar$)

$$\sigma_s \to \frac{16\pi^2(\eta - m^2)}{\hbar\log[\bar{\lambda}(\eta - m^2)]} \qquad \text{as} \qquad \eta - m^2 \to \infty. \tag{132}$$

The original potential is given by $V(\eta) = W(\sigma_s(\eta)) + \eta\sigma_s(\eta) + \mathcal{O}(\log N/N)$. Using the saddle point equation (129), we simplify this to

$$V(\eta) = W(\sigma_s) + \eta\sigma_s = \frac{\hbar\sigma_s^2}{64\pi^2} + \frac{1}{2}\sigma_s(\eta - m^2). \tag{133}$$

For large $\eta - m^2$ we get[19]

$$V(\eta) \to \frac{8\pi^2}{\hbar}\frac{(\eta - m^2)^2}{\log[\bar{\lambda}(\eta - m^2)]}\left[1 + \frac{1}{2\log[\bar{\lambda}(\eta - m^2)]}\right] \quad \text{as} \quad \eta - m^2 \to \infty \tag{134}$$

Recalling that $\eta = \frac{\phi^2}{N}$, we see that up to a multiplicative factor, $V(\phi^2/N) \sim \frac{(\phi^4/N^2)}{\log(\phi^2/N)}$ for large ϕ^2/N and fixed m. Thus, the original potential grows logarithmically slower that a quartic potential in the large-N limit. It would be interesting to find the behavior of $V(\eta)$ for small η. This requires a careful study of saddle points and constant phase contours for small η.

References

[1] G. 't Hooft, "Naturalness, Chiral Symmetry, And Spontaneous Chiral Symmetry Breaking," 'Recent developments in gauge theories'. Ed. by G. 't Hooft et al., (Plenum Press, New York NY 1980).

[2] S. G. Rajeev, "Non-trivial fixed point in four dimensional scalar field theory and the Higgs mass," arXiv:hep-th/9607100.

[3] M. Luscher and P. Weisz, "scaling laws and trivality bounds in the lattice phi**4 theory. 1,2,3" Nucl. Phys. B **290**, 25 (1987); **295**, 65 (1988); **318**, 705 (1989).

[4] J. Kuti, L. Lin and Y. Shen, "Upper Bound On The Higgs Mass In The Standard Model," Phys. Rev. Lett. **61**, 678 (1988).

[18] Recall that we are working in units where the scale parameter $M = 1$.

[19] Recall that $\bar{\lambda} = 16\pi^2\sqrt{e}\tilde{\lambda}/\hbar$ and $\tilde{\lambda} = e^{-(32\pi^2\lambda/\hbar + 3/2)}$

[5] C. M. Bender and H. F. Jones, "A New Nonperturbative Calculation: Renormalization And The Triviality Of Lambda Phi**4 In Four-Dimensions Field Theory," Phys. Rev. D **38** (1988) 2526.

[6] R. Fernandez, J. Frohlich and A. D. Sokal, "Random walks, critical phenomena, and triviality in quantum field theory," Berlin, Springer, Texts and monographs in physics (1992).

[7] K. G. Wilson and J. B. Kogut, "The Renormalization group and the epsilon expansion," Phys. Rept. **12**, 75 (1974).

[8] B. W. Lee, C. Quigg and H. B. Thacker, "Weak Interactions At Very High-Energies: The Role Of The Higgs Boson Mass," Phys. Rev. D **16**, 1519 (1977).

[9] R. F. Dashen and H. Neuberger, "How To Get An Upper Bound On The Higgs Mass," Phys. Rev. Lett. **50**, 1897 (1983).

[10] H. P. Nilles, "Supersymmetry, Supergravity And Particle Physics," Phys. Rept. **110**, 1 (1984).

[11] E. Farhi and L. Susskind, "Technicolor," Phys. Rept. **74**, 277 (1981).

[12] N. Arkani-Hamed, A. G. Cohen and H. Georgi, "Electroweak symmetry breaking from dimensional deconstruction," Phys. Lett. B **513**, 232 (2001) [arXiv:hep-ph/0105239].

[13] S. R. Coleman and E. Weinberg, "Radiative Corrections As The Origin Of Spontaneous Symmetry Breaking," Phys. Rev. D **7**, 1888 (1973).

[14] W. A. Bardeen, "On naturalness in the standard model," FERMILAB-CONF-95-391-T *Presented at the 1995 Ontake Summer Institute, Ontake Mountain, Japan, Aug 27 - Sep 2, 1995*

[15] A. A. Slavnov, "Higgs mechanism as a collective effect due to extra dimension," Theor. Math. Phys. **148**, 1159 (2006) [Teor. Mat. Fiz. **148**, 339 (2006)] [arXiv:hep-th/0604052].

[16] K. Halpern and K. Huang, "Non-Trivial Directions for Scalar Fields," Phys. Rev. D **53**, 3252 (1996) [arXiv:hep-th/9510240]. H. Gies, "Flow equation for Halpern-Huang directions of scalar O(N) models," Phys. Rev. D **63**, 065011 (2001) [arXiv:hep-th/0009041].

[17] B. Richter, "Theory in particle physics: Theological speculation versus practical knowledge", Reference Frame, Physics Today, October 2006, page 8.

[18] I. R. Klebanov and A. M. Polyakov, "AdS dual of the critical O(N) vector model," Phys. Lett. B **550**, 213 (2002) [arXiv:hep-th/0210114]. H. J. Schnitzer, "Gauged vector models and higher-spin representations in AdS(5)," Nucl. Phys. B **695**, 283 (2004) [arXiv:hep-th/0310210].

[19] V. P. Nair, "Quantum field theory: A modern perspective," Springer, New York (2005).

[20] C. M. Bender and S. A. Orszag, "Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory", Springer (2005).

[21] L. Akant and G. S. Krishnaswami, "Non-anomalous 'Ward' identities to supplement large-N multi-matrix loop equations for correlations," arXiv:hep-th/0611350.

[22] G. 't Hooft and S. Nobbenhuis, "Invariance under complex transformations, and its relevance to the cosmological constant problem," Class. Quant. Grav. **23**, 3819 (2006) [arXiv:gr-qc/0602076].

[23] L. J. Dixon, "Calculating scattering amplitudes efficiently," arXiv:hep-ph/9601359.