RESEARCH STATEMENT

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1. Introduction

My research interests lie in areas of representation theory, vertex operator algebras and algebraic combinatorics—more precisely, involving the fascinating interplay between (integer) partition identities and representation theory of affine Kac–Moody Lie algebras, using vertex-operator-theoretic techniques. The highlight of my doctoral research (under the supervision of Prof. Robert Wilson at Rutgers University) was the discovery of three new conjectured partition identities, arising from the level 4 standard modules for the affine Kac–Moody Lie algebra $A_2^{(2)}$. These identities affords interesting new features not seen in previously known examples of this type.

2. History and Background

Historically, the discovery of vertex operator constructions of representations of affine Kac–Moody Lie algebras was motivated by a conjectured interplay between classical partition identities and standard modules for affine Kac–Moody Lie algebras.

2.1. **Rogers–Ramanujan identities.** The first famous example of such interplay arises from the Rogers–Ramanujan identities, which may be stated as follows:

(1) *The number of partitions of a nonnegative integer n in which the difference between any two successive part is at least 2 is the same as the number of partitions of n into parts congruent to 1 or 4 modulo 5.*
(2) *The number of partitions of a nonnegative integer n in which the difference between any two successive part is at least 2 and such that the smallest part is at least 2 is the same as the number of partitions of n into parts congruent to 2 or 3 modulo 5.*

For each of these identities, the left-hand side counts the number of partitions satisfying certain "difference conditions" and "initial conditions," while the right-hand side counts the number of partitions satisfying certain "congruence conditions." In generating-function form, for each of these identities, the LHS can be naturally expressed as an infinite sum, and the RHS as an infinite product. Consequently, they are often informally referred to as the "sum-side" and the "product-side," respectively.

2.2. **Lepowsky–Wilson's approach.** A connection between these identities and their generalizations, and standard modules for the affine Lie algebra $A_1^{(1)}$ $(= \widehat{\mathfrak{sl}(2)})$ was discovered and further elucidated in the early works of Lepowsky–Milne [LM78], Lepowsky–Wilson [LW78, LW81, LW82, LW84, LW85], Lepowsky–Primc [LP85] and Meurman–Primc [MP87]. The work of [LM78] led to the conjecture that the Rogers–Ramanujan identities "take place" in the level 3 standard modules for $A_1^{(1)}$.

In Lepowsky–Wilson's vertex operator theoretic interpretation and proof of the Rogers–Ramanujan identities [LW81, LW82, LW84], they showed that the product-sides (in generating-function form) of these identities are precisely the graded dimensions (sometimes called "characters") of certain infinite-dimensional spaces, namely the "vacuum spaces" with respect to the principal Heisenberg sublalgebra, of the level 3 standard $\mathrm{A}_1^{(1)}$-modules. They constructed bases for these modules—inventing certain vertex-operator-theoretic structures which they called the Z-algebras—using monomials, acting on a highest weight vector, in certain new operators (Z-operators) whose indices reflected the difference and initial conditions, explaining the sum-sides.

They extended their work to all the standard modules for $\mathrm{A}_1^{(1)}$ in [LW82, LW84, LW85], giving a vertex-algebraic interpretation of a family of Rogers–Ramanujan-type identities, discovered by Gordon–Andrews–Bressoud. The linear independence of the relevant monomials for standard $\mathrm{A}_1^{(1)}$-modules of level greater than 3 was not proved in [LW82, LW84, LW85]. This problem was solved by Meurman–Primc [MP87], providing a vertex-algebraic proof of Gordan–Andrews–Bressoud identities beyond the case of Rogers–Ramanujan identities.

The concept of Z-algebra is universal, in that it treated all affine Lie algebras at all levels. In this way, one gets very general partition identities. However, the hard part is to *explicitly* construct a concrete basis reflecting the sum-side information.

2.3. **Capparelli's identities.** In his Ph.D. thesis [Cap88], S. Capparelli proposed a pair of combinatorial identities based on the level 3 standard modules for the affine Lie algebra $\mathrm{A}_2^{(2)}$. He also demonstrated that the construction of the level 2 standard modules for $\mathrm{A}_2^{(2)}$ in this way gives rise to another vertex-operator-theoretic interpretation of the classical Rogers–Ramanujan identities (see also [Cap92, Cap93]). It was believed that once a few low level cases for standard $\mathrm{A}_2^{(2)}$-modules had been successfully analyzed, a general construction for all levels would emerge. However, the cases for $\mathrm{A}_2^{(2)}$ turned out to be much harder and subtler than those for $\mathrm{A}_1^{(1)}$ which had been extensively studied. One of Capparelli's identities, arising from the level 3 standard $\mathrm{A}_2^{(2)}$-modules, may be stated as follows:

> *The number of partitions of a nonnegative integer n into parts different from* 1 *and such that the difference of two successive parts is at least* 2*, and is exactly* 2 *or* 3 *only if their sum is a multiple of* 3*, is the same as the number of partitions of n into parts congruent to* $\pm 2, \pm 3$ *modulo* 12.

A q-series proof of this identity was given by G. Andrews [And94], proving Capparelli's conjecture. Capparelli also provided a direct vertex-operator-theoretic proof of his identities by proving the linear independence of his spanning sets in [Cap96]. Another vertex-operator-theoretic proof of Capparelli's identities was independently given by Tamba–Xie [TX95].

3. Research Summary

3.1. **New partition identities based on level 4 standard $\mathrm{A}_2^{(2)}$-modules.** In [Nan14a], I gave combinatorial interpretations of the graded dimensions of the three inequivalent level 4 standard modules for the affine Kac–Moody Lie algebra $\mathrm{A}_2^{(2)}$, and based on it, proposed a new set of partition identities. The level 4 case involves much more subtlety and complexity compared to the level 3 case, showing even more surprising results.

3.1.1. *The statements.* A *partition* of a nonnegative integer n may be described as a non-increasing sequence $(m_1, \ldots, m_s)$, $s \geq 0$, of positive integers such that $\sum_{i=1}^{s} m_i = n$. For each $1 \leq i \leq s$, the number m_i is referred to as a *part* of the partition; and s is the *length.* A nonempty partition $(m_1, \ldots, m_s)$ is said to satisfy the *difference condition* $[d_1, \ldots, d_{s-1}]$ if $m_i - m_{i+1} = d_i$ for all $1 \leq i < s$.

The new partition identities conjectured in [Nan14a] may be stated as follows:

(1) *The number of partitions of a nonnegative integer n into parts different from 1 and such that there is no sub-partition satisfying the difference conditions $[1]$, $[0,0]$, $[0,2]$, $[2,0]$ or $[0,3]$, and such that there is no sub-partition with an odd sum of parts satisfying the difference conditions $[3,0]$, $[0,4]$, $[4,0]$ or $[3,2^*,3,0]$ (where 2^* indicates zero or more occurrence of 2), is the same as the number of partitions of n into parts congruent to $\pm 2, \pm 3$ or ± 4 modulo 14.*

(2) *The number of partitions of a nonnegative integer n such that 1, 2 and 3 may occur at most once as a part, and such that there is no sub-partition satisfying the difference conditions $[1]$, $[0,0]$, $[0,2]$, $[2,0]$ or $[0,3]$, and such that there is no sub-partition with an odd sum of parts satisfying the difference conditions $[3,0]$, $[0,4]$, $[4,0]$ or $[3,2^*,3,0]$ (where 2^* indicates zero or more occurrence of 2), is the same as the number of partitions of n into parts congruent to $\pm 1, \pm 4$ or ± 6 modulo 14.*

(3) *The number of partitions of a nonnegative integer n into parts different from $1$and 3, such that 2 may occur at most once as a part, and such that there is no sub-partition satisfying the difference condition $[3,2^*]$ (where 2^* denotes zero or more occurrence of 2) ending with a 2, and such that there is no sub-partition satisfying the difference conditions $[1]$, $[0,0]$, $[0,2]$, $[2,0]$ or $[0,3]$, and such that there is no sub-partition with an odd sum of parts satisfying the difference conditions $[3,0]$, $[0,4]$, $[4,0]$ or $[3,2^*,3,0]$, is the same as the number of partitions of n into parts congruent to $\pm 2, \pm 5$ or ± 6 modulo 14.*

Each of the above statements corresponds to computing the graded dimension of a level 4 standard $\mathrm{A}_2^{(2)}$-module in two ways—from the principal specialization of the Weyl–Kac character formula given by the numerator formula in [LM78, Lep78], describing the congruence conditions on the product-side, and an explicit construction of a graded basis for the module, describing the difference and initial conditions on the sum-side. The "spanning set" results in [Nan14a] prove that for each nonnegative integer n, the LHS (the sum-side) is more than or equal to the RHS (the product-side) in each of the above statements. A computer verification shows that the equality holds for at least $n \leq 170, n = 180, 190$ and 200.

3.1.2. *A brief overview.* A level 4 standard module V for $\mathrm{A}_2^{(2)}$ may be viewed as embedded in $U^{\otimes 4}$, where U is the (unique) basic module (i.e., level 1 standard module) for $\mathrm{A}_2^{(2)}$. The starting point in [Nan14a] is a spanning set consisting of elements of the form $\alpha(-\lambda)X(-\mu)v_0$, where $v_0 \in U^{\otimes 4}$ is a highest weight vector for V (depending on the particular level 4 module), and $\alpha(\bullet)$, $X(\bullet)$ are families of operators in End $U^{\otimes 4}$ parametrized by certain partitions λ and μ. To find a "tighter" spanning set matching the graded dimension as closely as possible, we need relations among these elements.

There are two types of relations among the above elements. Some relations come from relations among various monomials in the operators $\alpha(\bullet)$, $X(\bullet)$ viewed inside End $U^{\otimes 4}$. They explain the difference conditions on the sum-sides. (Note that the difference conditions are the same in all the three partition identities above). The resulting sub-partitions, satisfying these difference conditions in the sum-sides of the above three partition identities, are called "forbidden" sub-partitions.

The other relations are specific to the particular level 4 module V. They are valid among monomials in the operators $\alpha(\bullet)$, $X(\bullet)$ applied to a highest weight vector $v_0 \in U^{\otimes 4}$, depending on the particular level 4 module V. These relations explain the initial conditions on the sum-sides for the three partition identities stated above.

3.1.3. *The "surprise".* The most surprising and curious result in [Nan14a] is the discovery of the forbidden sub-partitions of arbitrary lengths (described by the difference conditions $[3, 2^*, 3, 0]$). In all previously known analogous situations arising from representations of affine Kac–Moody Lie algebras, the forbidden sub-partitions could be described by difference conditions of bounded length. For example, in Rogers–Ramanujan identities and Capparelli's identities, the forbidden sub-partitions are of the form $(k+1, k)$ with difference condition $[1]$.

The proof of the relations in [Nan14a] corresponding to the forbidden sub-partitions of arbitrary lengths is intricate, requiring to keep a lot of careful details in the calculations, but at the same time, it shows some elegant "recursive" and "periodic" properties.

3.1.4. *Experimental and computational approach.* It is interesting to note how experimental and computational methods played a large role in [Nan14a]. The discovery of the unexpected, "exceptional" forbidden sub-partitions was facilitated by experimental methods using Maple (a computer algebra system) programs.

Based on relations that we already knew at that time, we made educated guesses about possible forbidden sub-partitions that we hadn't discovered yet. We could then test our hypotheses by checking against the graded dimensions of the standard module under investigation. This way, wrong guesses could be eliminated, and we could focus on possible forbidden sub-partitions for which corresponding relations in the spanning set are yet to be found. These experimental tools provided us with valuable insights and intuitions that led us to the mathematically rigorous proofs of the corresponding relations.

Maple programs were also used (but not as an experimental tool) to carry out computations that were too involved and complicated to do manually. These computations were essential in the proofs for both the difference and initial conditions.

Finally, C programs were used to check the validity of the conjectured partition identities in [Nan14a]. The validity of these identities were verified for $n \leq 170$ and for $n = 180, 190$ and 200—giving us strong evidence in support of the conjectures.

All the programs that were used in [Nan14a] for computations and verification are published on my github page [Nan14b].

3.2. **Classification of hyperbolic Dynkin diagrams.** In another collaborative project [Car+10], we presented classification of hyperbolic Dynkin diagrams, root lengths and Weyl group orbits. We also gave a symmetrizability criterion for a Dynkin diagram, or equivalently, a generalized Cartan matrix.

4. Research Agenda

I intend to continue on my current line of research to explore further connections between combinatorial identities and representations of affine Kac–Moody Lie algebras, and to develop a better understanding of them. Since the partition identities (stated in §3.1.1) and the corresponding analyses of the relations in [Nan14a] are first of its kind, I hope that this will lead to discovery of similar (or, perhaps, even more complicated) partition identities from other standard modules for various affine Kac–Moody Lie algebras. I would like to apply my experimental methods as outlined in §3.1.4 to standard modules of higher levels for the algebra $\mathrm{A}_2^{(2)}$ in order to gain deeper insights and intuitions about the general case.

I would also like to apply my tools to standard modules for other affine Kac–Moody Lie algebras. In recent developments [KR14], using experimental methods, Kanade–Russell discovered 6 new conjectured partition identities of Rogers–Ramanujan type. The product-sides of three of these identities appear to come from level 3 standard modules for $D_4^{(3)}$. In collaboration with S. Kanade, I would like to see if we can apply our methods to find possible vertex-operator-theoretic interpretations and proofs of these identities.

Another of my research goals is to prove the linear independence of the spanning sets for the level 4 standard $A_2^{(2)}$-modules and thereby proving my conjectures in [Nan14a].

Last but not least, I am always eager to learn, and open to collaborating opportunities in related areas of research.

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