We wish to apply the results of Lecture.2. To this end, we introduce some relevant basic notions of scattering.

**Scattering**

The importance of scattering can be realized by the fact that the discovery of nucleus was made via scattering of alpha particles by thin gold foil. The high energy scattering of electrons by proton revealed that the proton is not a point-particle, but has structures. In fact to find the features of interaction between two particles, scattering by them is one of the recommended method. We shall first consider 'classical scattering'. In here the notion of trajectory is acceptable. Let us consider a beam of particles of known intensity $I$ (particles per unit area per second, area being perpendicular to the beam direction) going. Let us place a detector in the passage. Then the counting rate $N$ (counts per second) of the detector is $Na$ where $a$ is the aperture of the detector. Thus, in this case,

$$a = \frac{N}{I}. \quad (1)$$

Now consider the case in which the beam is scattered by a target. After scattering, the scattered particles are recorded by the detector. First, the number of particles arriving the counter will depend on the scattering angle $\theta$ (angle between the incident direction and the scattered direction) and energy of the incident beam. Second, the role of the aperture in (1) will be replaced by an 'effective area', called the "cross section" because the detector records those particles that originate in a narrow cone. The differential count will be, in view of (1),

$$d\sigma(\theta, E) = \frac{dN(\theta, E)}{I}. \quad (2)$$

Clearly the value of $dN(\theta, E)$ varies with the size of the counter or the solid angle $d\Omega$ that the counter aperture subnets at the target. A measure of the scattering independent of the size of the counter is the ratio

$$\frac{d\sigma(\theta, E)}{d\Omega} = \frac{1}{I} \frac{dN(\theta, E)}{d\Omega}. \quad (3)$$
The quantity \( \frac{d\sigma}{d\Omega} \) is called the **differential cross section**. Let us consider a repulsive scattering and introduce 'impact parameter' \( s \), the perpendicular distance of the target from the incident direction. Then the effective area can be shown to be \( d\sigma = s \, ds \, d\phi \). Using \( d\Omega = \sin \theta \, d\theta \, d\phi \), we have

\[
\frac{d\sigma}{d\Omega} = \frac{s \, ds}{\sin \theta \, d\theta}.
\] (4)

As an example, consider the scattering of particles of charge \( Z'e \) by the Coulomb field of a nucleus of charge \( Ze \). This is a repulsive scattering. The orbit of the incident particle will be easily seen to be a hyperbola. In this case (see Goldstein, Classical Mechanics),

\[
s = \frac{ZZ'e^2}{2E} \cot \frac{\theta}{2},
\] (5)

where \( E \) is the energy of the incident particle.\(^1\) Substituting (5) in (4), we obtain the differential cross section for repulsive Coulomb scattering as

\[
\frac{d\sigma}{d\Omega} = \left( \frac{ZZ'e^2}{E} \right)^2 \frac{\cosec^4 \theta/2}{16},
\] (6)

which is the famous Rutherford’s formula. There are two remarks to be made. First, a statistical feature is there as the particles arrive at random and enough counts are to be recorded. Second, the notion of trajectory has been used.

Before we embark the quantum theory of scattering, we wish to introduce an important transformation. Scattering is a two-body process. We have come across a two-body issue in Hydrogen atom where we introduced Centre of Mass (CM) and relative coordinates. There the CM motion was a plane

\(^1\)The orbit equation for hyperbola is \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) where the potential is \( k/r \), the eccentricity is \( \epsilon = \sqrt{1 + \frac{2E\ell^2}{mk^2}} \), the mass of the planet (particle) is \( m \) and \( \ell \) is the orbital angular momentum which is \( mvs \). For asymptotes set \( r = \infty \) and for Coulomb potential \( k = ZZ'e^2 \). The orbit angle \( \phi \) is the angle between the asymptote and the periapsis and the orbit is symmetric about the direction of the periapsis. The angle between the outgoing asymptote and incoming asymptote extended is \( \theta \), the scattering angle and these are related by \( 2\phi + \theta = \pi \). Then (5) follows.
wave. We consider a coordinate system in which the CM is at rest. The theory is generally in this frame. The experiments (conducted in laboratories) are carried out when one of the two particles is initially at rest and is in Laboratory Frame. We must know how to transform from one to another.

Let a particle of mass \( m \) and velocity \( \vec{v} \) approach a target of mass \( M \) at rest. Then, the CM will move with a velocity \( \vec{v} \) so that by momentum conservation, \( (m + M)\vec{v} = m\vec{v} \) from which \( \vec{v} = \frac{m}{m+M} \vec{v} \). We have introduced 'reduced mass' as \( \mu = \frac{mM}{m+M} \). In order to bring the CM to rest, we give \( -\vec{v} \) to all. Then, for \( m \), the velocity will be \( \vec{v} - \vec{v} \) which is \( \frac{m}{m+M} \vec{v} \) so that its momentum will be \( \mu\vec{v} \). Now for \( M \), the velocity will be \( \vec{0} - \vec{v} = -\vec{v} \) and its momentum will be \( -M \vec{v} = -\mu\vec{v} \). So in the CM frame the momentum of \( m \) and \( M \) are equal and opposite. The total kinetic energy in the Lab.frame is \( \frac{1}{2}mv^2 \) while in the CM frame \( \frac{1}{2}\mu v^2 \) which is \( \frac{1}{2}\mu v^2 \). So we have: \( E_L = \frac{1}{2}mv^2 \) and \( E_{CM} = \frac{1}{2}\mu v^2 \) from which it follows \( E_L = \frac{m+M}{M}E_{CM} \).

Now, the momentum of \( m \) in CM frame being \( \mu\vec{v} \), it follows, the velocity of \( m \) in the CM frame is \( \vec{v}_{CM} = \frac{M}{m+M} \vec{v} \). Similarly, the momentum of \( M \) in the CM frame being \( -\mu\vec{v} \), it follows the velocity of \( M \) in the CM frame is \( \vec{V}_{CM} = -\frac{m}{m+M} \vec{v} \). You see \( \vec{V}_{CM} \) and \( \vec{v}_{CM} \) are opposite. We can rewrite these as

\[
\vec{v}_{CM} = \vec{v}_L - \vec{v} ; \quad \vec{v}_L = \vec{v}_{CM} + \vec{v}.
\]

\[
\vec{V}_{CM} = -\vec{v} ; \quad \vec{V}_L = \vec{V}_{CM} + \vec{v} (= 0).
\]

The angle between \( \vec{v}_L \) and \( \vec{v} \) will be \( \theta_L \) and the angle between \( \vec{v}_{CM} \) and \( \vec{v} \) extended will be \( \theta_{CM} \). Draw a vector diagram. Then

\[
tan\theta_L = \frac{\vec{v}_{CM}sin\theta_{CM}}{\vec{v} + \vec{v}_{CM}cos\theta_{CM}} = \frac{\vec{v}}{\vec{v}_{CM}} + cos\theta_{CM},
\]

\[
= \frac{sin\theta_{CM}}{\vec{v}_{CM}} = \frac{m}{M} + cos\theta_{CM}.
\]

If \( m << M \), we see \( \theta_L = \theta_{CM} \) and if \( m = M \), we get \( \theta_L = \frac{1}{2} \theta_{CM} \) pertinent to neutron-neutron, neutron-proton, proton-proton scattering at low energies.
We have related the scattering angles in the two frames. How to transform the differential cross sections? A key point here is: the actual counting rate does not depend on the use of coordinates! From (3) and noting the intensity of the incident beam is not going to change, we have
\[
\left(\frac{d\sigma}{d\Omega}\right)_{CM} \sin\theta_{CM} d\theta_{CM} d\phi = \left(\frac{d\sigma}{d\Omega}\right)_{L} \sin\theta_{L} d\theta_{L} d\phi,
\]
\[
\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \left(\frac{d\sigma}{d\Omega}\right)_{L} \frac{dcos\theta_{L}}{dcos\theta_{CM}}.
\] (9)

In the above note \(\phi\) does not change. From (8), using \(1 + \tan^2 \theta = \sec^2 \theta\), we have
\[
cos\theta_{L} = \frac{\frac{m}{M} + \cos\theta_{CM}}{\sqrt{\left(\frac{m^2}{M^2} + 1 + 2\frac{m}{M} \cos\theta_{CM}\right)}}, \text{ and so}
\]
\[
\frac{d}{dcos\theta_{CM}} \cos\theta_{L} = \frac{1 + \frac{m}{M} \cos\theta_{CM}}{(1 + \frac{m^2}{M^2} + 2\frac{m}{M} \cos\theta_{CM})^{\frac{3}{2}}}
\] (10)

From (9) and (10), we see that
\[
\left(\frac{d\sigma}{d\Omega}\right)_{L} = \frac{(1 + \frac{m^2}{M^2} + 2\frac{m}{M} \cos\theta_{CM})^{\frac{3}{2}}}{1 + \frac{m}{M} \cos\theta_{CM}} \left(\frac{d\sigma}{d\Omega}\right)_{CM}.
\] (11)

The above expression can be considered for scattering of \(m\) by \(M\) producing \(m'\) and \(M'\) particles. If an amount of energy \(Q\) is converted from internal energy kinetic energy of the emerging particles, and the particle \(m'\) is observed, then the ratio \(m/M\) should be replaced by \(\left(\frac{m'm}{MM'} \frac{E}{E+Q}\right)^{\frac{3}{2}}\) where \(E = mMv^2/(2(m + M))\). However, the above derivation is true for non-relativistic scattering.