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Tannakian Krull-Schmidt reduction

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Abstract. In this continuation of [BBN] we show that the theory of Tannakian categories gives a very transparent construction of the reduction of structure group, obtained in [BBN], of principal bundles. In fact, a more general construction which is valid for all algebraically closed fields is given. □

Let k be an algebraically closed field of arbitrary characteristic. Let M be a quasi-projective variety defined over k such that $M \cong \overline{M} \backslash S$, where $\overline{M}$ is a projective variety and S a subscheme with $\operatorname{codim}(S) \geqq 2$. Let G be a linear algebraic group defined over k. The category of finite dimensional left G-modules will be denoted by $\operatorname{Rep}(G)$.

Let E_G be a principal G-bundle over M. Let $\operatorname{Aut}(E_G)$ denote the reduced algebraic group defined by the space of all (global) automorphisms of E_G. Fix a reduced abelian algebraic subgroup

$$\mathcal{T} \subset \operatorname{Aut}(E_G). \tag{1}$$

Take any $V \in \operatorname{Rep}(G)$. Let $E_V = E_G(V) = (E_G \times V)/G$ be the vector bundle associated to E_G for the G-module V. Any automorphism of E_G gives an automorphism of the associated vector bundle E_V. Therefore, we have a homomorphism of algebraic groups

$$\rho : \mathcal{T} \to \operatorname{Aut}(E_V), \tag{2}$$

where $\mathcal{T}$ is the subgroup in (1). Since $\mathcal{T}$ is abelian, we have a direct sum decomposition

$$E_V = \bigoplus_{i=1}^{n} E_V^{\lambda_i}, \tag{3}$$

where λ_i, $i \in [1, n]$, are the distinct characters of $\mathcal{T}$ given by the distinct eigen-values of $\rho(\mathcal{T})$ (the homomorphism ρ is defined in (2)), and $E_V^{\lambda_i}$ is the generalized eigen-bundle for the eigen-character λ_i. Since M does not admit any nonconstant regular function, the characteristic polynomial for any endomorphism of E_V remains unchanged over M. Using this the decomposition in (3) is obtained.

Let $W \in \operatorname{Rep}(G)$ be a G-module, and let

$$\eta : V \to W \tag{4}$$

be a homomorphism of G-modules. This homomorphism η induces a homomorphism of associated vector bundles

$$\hat{\eta} : E_V \to E_W := (E_G \times W)/G. \tag{5}$$

Let

$$E_W = \bigoplus_{i=1}^{m} E_W^{\lambda_i'} \tag{6}$$

be the decomposition constructed as in (3). It is straight-forward to check that for each $i \in [1, n]$ in (3), one of the following two holds:

(a) $\lambda_i = \lambda_j'$ for some $j \in [1, m]$, and furthermore, $\hat{\eta}(E_V^{\lambda_i})$ is a subbundle of $E_W^{\lambda_j'}$;

(b) $\hat{\eta}(E_V^{\lambda_i}) = 0$.

Consider the category $\mathscr{E}(\mathscr{T})$ of all vector bundles over M of the form $E_V^{\lambda_i}$ (see (3)) with V running over $\mathrm{Rep}(G)$. So $\mathscr{E}(\mathscr{T})$ consists of eigen-bundles for all vector bundles associated to E_G. For objects $E_V^{\lambda_i}$ and $E_W^{\lambda_j'}$ in $\mathscr{E}(\mathscr{T})$ as in (3) and (6) respectively, a morphism from $E_V^{\lambda_i}$ to $E_W^{\lambda_j'}$ is a homomorphism of vector bundles induced by $\hat{\eta}$ (see (5)) for some η as in (4). The category $\mathscr{E}(\mathscr{T})$ is equipped with direct sum and tensor product operations in an obvious way. It is easy to see that $\mathscr{E}(\mathscr{T})$ is an abelian category.

Now fix a k-rational point $x_0 \in M$. For any fiber bundle E over M, its fiber over x_0 will be denoted by E_{x_0}. We have a fiber functor defined on $\mathscr{E}(\mathscr{T})$ that sends any vector bundle E in $\mathscr{E}(\mathscr{T})$ to the k-vector space E_{x_0}.

It is easy to check that the above defined category $\mathscr{E}(\mathscr{T})$ equipped with the above fiber functor is a Tannakian category. Therefore, it defines an affine group scheme defined over k [No], Section 2.1, Theorem 1.1, [Sa], Theorem 1. Let $\mathscr{G}(\mathscr{T})$ denote the affine group scheme defined over k given by the Tannakian category $\mathscr{E}(\mathscr{T})$. We will see that $\mathscr{G}(\mathscr{T})$ is isomorphic to a subgroup scheme of G.

For any affine algebraic group scheme H defined over k, there is a bijective correspondence between the principal H-bundles over M and the functors that sends H-modules to vector bundles over M satisfying certain conditions; the functor corresponding a principal H-bundle E_H sends any H-module V to the vector bundle $E_H(V)$ associated to E_H for V (see [No], Section 2.2 for the details). There is a tautological principal $\mathscr{G}(\mathscr{T})$-bundle over M that sends any E in $\mathscr{E}(\mathscr{T})$ to the underlying vector bundle E.

The above constructions are summarized in the following proposition:

Proposition 0.1. *Let G be a linear algebraic group defined over an algebraically closed field k. Let M be a quasi-projective variety defined over k which is the complement of a subscheme of codimension at least two in a projective variety. Let E_G be a principal G-bundle over M and $\mathscr{T}$ a reduced abelian algebraic subgroup of the reduced algebraic group defined by the automorphisms of E_G. Then there is an associated affine algebraic group scheme $\mathscr{G}(\mathscr{T})$ defined over k and a principal $\mathscr{G}(\mathscr{T})$-bundle $E_{\mathscr{G}(\mathscr{T})}$ over M.*

Let $\mathrm{Ad}(E_G) = E_G(G) := (E_G \times G)/G$ be the adjoint group scheme over M. The fibers of $\mathrm{Ad}(E_G)$ are groups isomorphic to G. The group $\mathrm{Ad}(E_G)_{x_0}$ has the following Tannakian description:

Let $\mathscr{E}$ denote the category of all vector bundles associated to E_G. A morphism between two associated vector bundles E_V and E_W, associated to $V, W \in \mathrm{Rep}(G)$, is a vector bundle homomorphism given by a G-module homomorphism between V and W. The fiber functor sends an associated vector bundle E_V to its fiber $(E_V)_{x_0}$.

Note that $\mathscr{E}$ is identified with the category $\mathrm{Rep}(G)$. However, the fiber functor on the category $\mathrm{Rep}(G)$ defining the group G sends a G-module V to the underlying vector space V. So G and $\mathrm{Ad}(E_G)_{x_0}$ differ in the fiber functor, although their underlying tensor categories coincide.

Comparing the above description of $\mathrm{Ad}(E_G)_{x_0}$ with that of the group scheme $\mathscr{G}(\mathscr{T})$ in Proposition 0.1 it follows immediately that there is a tautological homomorphism

$$\tau : \mathscr{G}(\mathscr{T}) \to \mathrm{Ad}(E_G)_{x_0}. \tag{7}$$

From a criterion, given in [DM], p. 139, Proposition 2.21, for a closed immersion it follows immediately that the homomorphism τ in (7) is a closed immersion.

Using τ, the principal $\mathscr{G}(\mathscr{T})$-bundle $E_{\mathscr{G}(\mathscr{T})}$ in Proposition 0.1 gives a principal $\mathrm{Ad}(E_G)_{x_0}$-bundle over M. This principal $\mathrm{Ad}(E_G)_{x_0}$-bundle will be denoted by $E_{\mathrm{Ad}(E_G)_{x_0}}$.

Fix a k-rational point $z_0 \in (E_G)_{x_0}$. Using z_0 we have an isomorphism

$$f_{z_0} : G \to \mathrm{Ad}(E_G)_{x_0}$$

defined by $g \mapsto [(z_0, g)]$ (recall that $\mathrm{Ad}(E_G)$ is a quotient of $E_G \times G$). It is easy to see that the principal G-bundle $E_{\mathrm{Ad}(E_G)_{x_0}}(f_{z_0}^{-1})$ over M, given by the principal $\mathrm{Ad}(E_G)_{x_0}$-bundle $E_{\mathrm{Ad}(E_G)_{x_0}}$ using the homomorphism $f_{z_0}^{-1}$, is the principal G-bundle E_G. Indeed, this follows from the fact that for any $V \in \mathrm{Rep}(G)$, the associated vector bundle E_V is the vector bundle associated to $E_{\mathrm{Ad}(E_G)_{x_0}}(f_{z_0}^{-1})$ for the G-module V. Therefore, the principal $\mathscr{G}(\mathscr{T})$-bundle $E_{\mathscr{G}(\mathscr{T})}$ is a reduction of structure group of E_G to the subgroup scheme $\mathscr{G}(\mathscr{T}) \subset \mathrm{Ad}(E_G)_{x_0} = G$.

If the base point z_0 in $(E_G)_{x_0}$ is replaced by $z_0 h$, where $h \in G$, then the subgroup scheme $\mathscr{G}(\mathscr{T}) \subset G$ gets replaced by $h^{-1}\mathscr{G}(\mathscr{T})h$, and the reduction of structure group

$$E_{\mathscr{G}(\mathscr{T})} \subset E_G \tag{8}$$

gets replaced by $E_{\mathscr{G}(\mathscr{T})}h$.

Set the subgroup $\mathscr{T}$ in (1) to be a maximal torus of $\mathrm{Aut}(E_G)$. Since any two maximal tori are conjugate by some element of $\mathrm{Aut}(E_G)$, the corresponding reduction of structure group in (8) is independent of the choice of the maximal torus up to an automorphism of E_G. In other words, any two such reductions differ by an automorphism of E_G. When the characteristic of k is zero and G is reductive, this coincides with the reduction of structure group constructed in [BBN], Theorem 3.2.

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