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Parabolic Bundles on Algebraic Surfaces II - Irreducibility of the Moduli Space

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To Professor Ramanan on his 70th birthday

Abstract. In this paper we prove irreducibility of the moduli space of parabolic rank 2 bundles over an algebraic surface for $c_2 \gg 0$ and with an irreducible parabolic divisor D of X. This gives parabolic analogues of theorems of O'Grady and Gieseker-Li.

1. Introduction

Let X be a smooth projective surface over the ground field $\mathbb{C}$ of complex numbers and let D be a smooth irreducible divisor. Let H be a very ample line bundle on X which we fix throughout. We study bundles with $c_1 = 0$ in this paper.

We denote by $M_{k,d}^{\alpha}$ the *moduli space of parabolic H–stable parabolic bundles of rank* 2 with parabolic structure on D together with rational weights $\alpha := (\alpha_1, \alpha_2)$ (see (2.4) and (2.5) for the definition of the invariant d) and where k stands for the second Chern class c_2 of a vector bundle.

In [**3**], the Donaldson-Uhlenbeck compactification $\overline{M_{k,d}^{\alpha}}$ of the moduli space $M_{k,d}^{\alpha}$ was constructed as a projective variety by realizing it as the closure of $M_{k,d}^{\alpha}$ in a certain projective scheme $\mathfrak{M}_{k,d}^{\alpha}$ endowed with the reduced scheme structure; it was also shown to be non-empty for large k. There are also obvious bounds on the invariant d for quasi-parabolic structures to exist.

Let $\mathfrak{M}_H(2, c_1, c_2)$ (resp. $\mathfrak{M}_H(2, c_1, c_2)^s$) denote the moduli space of H-semistable (resp. stable) torsion free sheaves of rank 2 whose Chern classes are c_1 and c_2 respectively.

Since the topological type of the bundles is fixed for the problem as also is the ample polarization H, we will have the following convenient notations:

$$M_{\alpha}^{s} := M_{k,d}^{\alpha};\ M_{\alpha} := \overline{M_{k,d}^{\alpha}}$$

and

$$\mathfrak{M}^s := \mathfrak{M}_H(2, 0, c_2)^s.$$

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We carry the weight tuple α as a part of the notation since this parameter will be varied in the arguments and the moduli spaces will be compared for differing weights.

We say a moduli space as above is *asymptotically irreducible* if it is irreducible for $c_2 \gg 0$, i.e the second Chern class of the bundle underlying the parabolic bundle is *large*. In particular we do not quantify c_2 when we address the question of asymptotic irreducibility.

In this paper we prove asymptotic irreducibility of the moduli space M_α when obvious bounds are imposed on d for the existence of quasi-parabolic structures. These moduli spaces for rank 2 have been studied from a differential geometric standpoint in [**12**] where $k = c_2$ stands for the "instanton number".

Our theorem generalizes the theorem of Gieseker-Li and O'Grady ([**6**] and [**16**]) to the parabolic case. The parabolic case has been of independent interest; for example, in [**14**] Maruyama has shown links between the parabolic moduli spaces for special parabolic weights and the moduli space of instantons. Maruyama uses these links to prove irreducibility of some of these spaces.

The assumptions on the parabolic divisor, rank and full flag quasi-parabolic structure can be relaxed; one could take the parabolic divisor to be a divisor with simple normal crossings and the bundles to be of arbitrary rank and any quasi-parabolic type. We have made the special choices to make the paper more readable. The choice of rational weights is the natural one and real weights are really an artifice and do not occur in the classical setting. In any case this is not a serious issue as far as the question of irreducibility of the moduli space is concerned since the "yoga of parabolic weights" allows us to deduce geometric statements for moduli spaces with real weights from those with nearby rational weights.

The assumption on large second Chern class is what makes the statement an *asymptotic one*; the result is shown only for large c_2.

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2. Preliminaries

Our basic tool is the Seshadri-Biswas correspondence between the category of parabolic bundles on X and the category of Γ–bundles on a suitable Kawamata cover. This strategy has been employed in several papers. Most of the material written in this section is taken from §2 of [**3**] and the reader will find details of the Seshadri-Biswas correspondence in this reference. However in this note we are only interested in the rank 2 case, and we will give definitions in the rank 2 case alone and lay stress on those points which are relevant to our purpose.

2.1. The category of bundles with parabolic structures. Let X be a smooth projective surface over the ground field $\mathbb{C}$ and let D be an irreducible smooth divisor in X. Let H be a very ample line bundle on X.

DEFINITION 2.2. Let E be a rank 2 torsion-free sheaf on X. A parabolic structure (with respect to D) on E is a filtration (quasi–parabolic structure)

$$E_* : E = \mathcal{F}^1(E) \supset \mathcal{F}^2(E) \supset \mathcal{F}^3(E) = E \otimes \mathcal{O}_X(-D) \tag{2.1}$$

together with a system of weights

$$0 \le \alpha_1 < \alpha_2 < 1$$

where α_i is the weight associated with $\mathcal{F}^i(E)$.

(See [**12**, Section 8] where the weights are given in $[-\frac{1}{2}, \frac{1}{2})$ following the *balanced convention.*)

We will use the notation E_* to denote a parabolic sheaf and by E (without the subscript "$*$") when it is without its parabolic luggage. The notation E_* therefore carries the data of the weight tuple α as well. A parabolic sheaf E_* is called a parabolic bundle if the underlying sheaf E is a vector bundle.

2.3. Some assumptions. The class of parabolic vector bundles that are dealt with in the present work satisfy certain constraints which will be explained now.

(2.2) *All parabolic weights are rational numbers.*

(2.3) $\mathcal{F}^1(E)/\mathcal{F}^2(E)$ *is torsion-free as a sheaf on* D.

We need to impose these in order to have the *Seshadri–Biswas* correspondence (cf. [**3**, Remark 2.3] for details). *Henceforth, all parabolic vector bundles will be assumed to have the constraints* (2.2) *and* (2.3).

Also note that the filtration (2.1) is equivalent to a filtration on $E\mid_D$ given by

$$E\mid_D = \mathcal{F}^1_D(E) \supset \mathcal{F}^2_D(E) \supset \mathcal{F}^3_D(E) = 0. \tag{2.4}$$

To see this, simply define

$$\mathcal{F}^i(E) = \ker\left(E \to \frac{E\mid_D}{\mathcal{F}^i_D(E)}\right).$$

In the notation $M^{\alpha}_{k,d}$ in the introduction, the numerical invariant d is given by

$$d = c_1(\mathcal{F}^2_D(E)).D. \tag{2.5}$$

The slope of a rank 2 parabolic sheaf E_* is defined as

$$\mu_\alpha(E_*) = \frac{[c_1(E) + (\alpha_1 + \alpha_2)D] \cdot H}{2}. \tag{2.6}$$

Let $\mathrm{PVect}(X, D)$ denote the category whose objects are rank 2 parabolic vector bundles over X with parabolic structure over the divisor D satisfying (2.2) and (2.3), and whose morphisms are homomorphisms of parabolic vector bundles (see [**3**] for more detail).

For an integer $N \ge 2$, let $\mathrm{PVect}(X, D, N) \subseteq \mathrm{PVect}(X, D)$ denote the subcategory consisting of all parabolic vector bundles all of whose parabolic weights are multiples of $1/N$.

Let E_* be a rank 2 parabolic bundle on X with parabolic weight (α_1, α_2). Let L be a line subbundle of E, the underlying bundle of the parabolic bundle E_*. The parabolic weights on E_* induces a parabolic weight on L denoted by α_L; α_L equals

α_2 if $L \subset \mathcal{F}_D^2(E)$, and is α_1 otherwise. Denote this parabolic line bundle with this induced structure by L_*. The slope of this parabolic line bundle L_* is given by

$$\mu_{\alpha_L}(L_*) \;=\; (c_1(L) + \alpha_L D) \cdot H. \tag{2.7}$$

It is not hard to check that for the purposes of stability it suffices to worry about parabolic line subbundles L_* of a rank 2 parabolic bundle E_* which are obtained from a line subbundle L of E with a weight α_L defined as above. The parabolic bundle E_* is α-stable (resp. α-semi–stable) if

$$\mu_{\alpha_L}(L_*) \;<\; \mu_\alpha(E_*)\,(\text{resp. } \leq) \tag{2.8}$$

for all parabolic line subbundles L_* of E_*.

2.4. The Kawamata covering lemma. Let $D \subset X$ be an irreducible divisor. Take any $E_* \in \mathrm{PVect}(X, D)$ such that all the parabolic weights of E_* are multiples of $1/N$, i.e. $E_* \in \mathrm{PVect}(X, D, N)$. The "covering lemma" of Y. Kawamata ([**11**, Theorem 1.1.1], [**10**, Theorem 17]) says that there is a connected smooth projective variety Y over $\mathbb{C}$ and a Galois covering

$$p : \quad Y \quad \longrightarrow \quad X \tag{2.9}$$

such that the reduced divisor $\tilde{D} := (p^*D)_{\text{red}}$ is a normal crossing divisor of Y and furthermore the pull-back p^*D equals $kN\tilde{D}$, for some positive integer k. Let Γ denote the Galois group for the covering map p (2.9).

2.5. The category of Γ–bundles. Let $\Gamma \subsetneq \mathrm{Aut}(Y)$ be a finite subgroup of the group of automorphisms of a connected smooth projective variety $Y/\mathbb{C}$ and $\tilde{H}$ be a fixed polarization on Y.

A Γ–vector bundle V on Y is a vector bundle V together with a collection of isomorphisms of vector bundles

$$\bar{g} : V \quad \longrightarrow \quad (g^{-1})^* V$$

indexed by $g \in \Gamma$ and satisfying the condition that $\overline{gh} \;=\; \bar{g} \circ \bar{h}$ for any $g, h \in \Gamma$ (see §2, [**3**] for more detail).

A Γ*–homomorphism* between two Γ–vector bundles is a homomorphism between the two underlying vector bundles which commutes with the Γ–actions. Let $\mathrm{Vect}_\Gamma(Y)$ denote the category of Γ–vector bundles on Y with the morphisms being Γ–homomorphisms.

Having fixed the parabolic divisor and the Kawamata cover together with the ramification indices, one has the concept of *local type of a* Γ*–bundle* which is described in [**3**, 2.4.1] (see [**17**] for the terminology). This is needed in order to set up the correspondence between Γ–bundles and parabolic bundles with specified parabolic datum on X.

Let $\mathrm{Vect}_\Gamma^D(Y, N)$ denote the subcategory of $\mathrm{Vect}_\Gamma(Y)$ consisting of all rank 2 Γ–vector bundles V over Y of *fixed local type* (see [**3**, 2.4.1] for details).

A Γ–vector bundle V is called Γ–stable (resp. Γ–semistable) iff for all Γ–invariant line subbundles $L \subset V$ the following holds

$$c_1(L) \cdot \tilde{H} \;<\; (\text{resp. } \leq)\, \frac{c_1(V) \cdot \tilde{H}}{2}. \tag{2.10}$$

Note that the above definition of Γ–stability is strictly weaker than the usual definition of stability; in particular the notion of Γ– stability does not imply the stability of the underlying Γ–bundle. In contrast, the notion of Γ–semistability is equivalent to the usual notion of semistability of the underlying Γ–bundle because of the uniqueness of the *Harder-Narasimhan filtration.*

2.6. Parabolic bundles and Γ–bundles. In [**4**] a categorical correspondence between the objects of $\mathrm{PVect}(X, D, N)$ and the objects of $\mathrm{Vect}^D_\Gamma(Y, N)$ has been constructed, induced by the "invariant direct image" functor p^Γ_*. The details of this identification is also given in [**2**, Section 2].

Let $\tilde{H}$ denote the pullback $p^*(H)$. Then the above correspondence between *parabolic bundles* on X and Γ–bundles on Y identifies the Γ–semistable (resp. Γ–stable) objects with parabolic semistable (resp. parabolic stable) objects as well. The invariant direct image functor p^Γ_* giving this equivalence of categories is moreover a "tensor functor" which sends the usual dual of a Γ–vector bundle to the "parabolic dual" of the corresponding parabolic vector bundle .

2.7. Γ–derived functors Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category with enough injectives. Let Γ be a finite group. Let $\mathcal{C}^\Gamma$ be the category whose objects are pairs of the form $(A, \rho : \Gamma \to \mathrm{Aut}_{\mathcal{C}}(A))$ where $A \in \mathcal{C}$. A morphism between pairs $(A, \rho : \Gamma \to \mathrm{Aut}_{\mathcal{C}}(A))$, $(B, \rho' : \Gamma \to \mathrm{Aut}_{\mathcal{C}}(B))$ is defined as a Γ–equivariant morphism in $\mathcal{C}$, i.e. the diagram

$$\begin{CD} A @>{\rho(\gamma)}>> A \\ @V{f}VV @V{f}VV \\ B @>{\rho'(\gamma)}>> B \end{CD} \tag{2.11}$$

is required to commute for all $\gamma \in \Gamma$.

Since the ground field is assumed to be of characteristic 0, for any object $(A, \rho) \in \mathcal{C}^\Gamma$, we have a subobject $A^\Gamma_\rho \subset A$ defined as follows. Given $\gamma \in \Gamma$ and $A \in \mathcal{C}$, we can define the γ-invariant subobject A^γ of A to be the *kernel* of the composite map:

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{\mathrm{id}\oplus(-\rho(\gamma))} A$$

where Δ is the diagonal morphism. We define the Γ-invariant subobject A^Γ_ρ of A to be the *intersection* of the A^γ's in A, $\gamma \in \Gamma$, i.e.

$$A^\Gamma_\rho := \bigcap_{\gamma\in\Gamma} \mathrm{Ker}((\mathrm{id} \oplus (-\rho(\gamma))) \circ \Delta). \tag{2.12}$$

Note that the induced action of Γ on A^Γ_ρ is trivial. Therefore we will just write A^Γ instead of A^Γ_ρ.

Let F be a covariant left exact functor from $\mathcal{C}$ to an abelian category $\mathcal{B}$. Any k-linear additive functor $F : \mathcal{C} \to \mathcal{D}$ extends uniquely to a functor $\tilde{F} : \mathcal{C}^\Gamma \to \mathcal{D}^\Gamma$ since any action of Γ on an object A extends to an action of Γ on $F(A)$. Let F^Γ be the invariant functor which sends A to $F(A)^\Gamma$. It is a subfunctor of F. We have the following useful observation.

Lemma 2.8. *F^Γ is a direct summand of F. Consequently the right derived functors R^iF^Γ are direct summands of R^iF.*

Proof. The fact that F^Γ is a direct summand of F follows immediately from the assumption on the characteristic of the ground field. □

Now we return to the case in which we are interested. Let Y be a Kawamata cover of X with a finite group Γ acting on Y such that $Y/\Gamma = X$. Note that by our previous notations if $\mathcal{C}$ denotes the category $\mathrm{Vect}(Y)$ of vector bundles on Y then $\mathcal{C}^\Gamma$ is the category $\mathrm{Vect}^\Gamma(Y)$ of Γ-vector bundles on Y . The global section functor $\mathrm{Hom}(Y, -)$ gives rise to a left exact functor from $\mathcal{C}^\Gamma$ to category of k-linear spaces. We denote the i-th right derived functor of this by $\mathrm{Ext}^i(Y, -)$. Let $\mathrm{Ext}^i_\Gamma(Y, -)$ be the right derived functor of the global invariant section functor $\mathrm{Hom}_\Gamma(Y, -)$.

By Lemma 2.8 we have the following proposition.

PROPOSITION 2.9. $\mathrm{Ext}^i_\Gamma(Y, -)$ *is a direct summand of* $\mathrm{Ext}^i(Y, -)$. *Hence*

$$\mathrm{ext}^i_\Gamma(Y, F) \leq \mathrm{ext}^i(Y, F)$$

for all $F \in C^\Gamma$, *where* "ext" *denotes the dimension of the vector space* "Ext".

Let us consider the category $\mathcal{C}_\bullet$ of filtered $\mathcal{O}_Y$-modules whose objects are denoted by $F_\bullet$, i.e. sheaves F with a filtration of subsheaves

$$F_\bullet : 0 = F_0 \subset F_1 \cdots \subset F_n = F. \tag{2.13}$$

Let $\mathcal{C}^{\Gamma,-}_\bullet$ be the category whose objects are given by Γ-filtered sheaves of $\mathcal{O}_Y$ modules (as in (2.13)). For any two objects $F_\bullet$, $G_\bullet$ in $\mathcal{C}^{\Gamma,-}_\bullet$, morphisms in $\mathcal{C}^{\Gamma,-}_\bullet$ are defined as:

$$\mathrm{Hom}_{\Gamma,-}(F_\bullet, G_\bullet) = \{\phi : F \to G : \ \phi(F_i) \subset G_i \ \text{ for all } 0 \leq i \leq n\}.$$

Let $\mathcal{C}^{\Gamma,+}_\bullet$ be the category whose objects are the same as in $\mathcal{C}^{\Gamma,-}_\bullet$, and morphisms between two objects $F_\bullet$, $G_\bullet$ are defined as

$$\mathrm{Hom}_{\Gamma,+}(F_\bullet, G_\bullet) = \mathrm{Hom}_{\mathcal{C}}(F, G)/\mathrm{Hom}_{\Gamma,-}(F_\bullet, G_\bullet).$$

Both these categories $\mathcal{C}^{\Gamma,\pm}_\bullet$ are abelian categories with enough injectives and $\mathrm{Hom}_{\Gamma,\pm}(F_\bullet, -)$ are both left exact covariant functors. Let $\mathrm{Ext}^i_{\Gamma,\pm}(F_\bullet, -)$ be the right derived functors of $\mathrm{Hom}_{\Gamma,\pm}(F_\bullet, -)$.

We have a long exact sequence (cf. [**9**, page 49])

$$\cdots \longrightarrow \mathrm{Ext}^i_{\Gamma,-}(F_\bullet, G_\bullet) \longrightarrow \mathrm{Ext}^i_\Gamma(F_\bullet, G_\bullet) \longrightarrow \mathrm{Ext}^i_{\Gamma,+}(F_\bullet, G_\bullet) \longrightarrow \cdots \tag{2.14}$$

3. R^{ss}_α is irreducible for small α

3.1. A description of R^{ss}_α. We briefly recall the construction of semistable sheaves over X. For details see ([**9**, Chapter 4]). Let $\mathcal{H} = \mathcal{O}_X(-m)^p$ for some m and p. Let

$$Q := \mathrm{Quot}(\mathcal{H}, P) \tag{3.1}$$

be the Quot scheme which parametrizes quotients of $\mathcal{H}$ with fixed Hilbert polynomial P given by

$$P(n) := n^2H^2 + n(c_1 \cdot H - K_X \cdot H) + \frac{c_1^2 - c_1 \cdot K_X}{2} - c_2 + 2\chi(\mathcal{O}_X), \tag{3.2}$$

where K_X is the canonical line bundle and the c_i are the Chern classes of the sheaves which we wish to parametrize.

For fixed Chern classes c_1 and c_2, it is known that rank 2 semistable sheaves F with $c_i(F) = c_i$ can be realized as quotients of a fixed $\mathcal{H} = \mathcal{O}_X(-m)^p$ for suitably chosen m and p. Let $\mathfrak{R}^{ss} \subset Q$ (resp. $\mathfrak{R}^s \subset Q$) consist of points $[\mathcal{H} \to F] \in Q$ such that the quotients F are semi-stable (resp. stable) torsion-free sheaves and the quotient map $\mathcal{H} \to F$ induces an isomorphism $\mathbb{C}^p = H^0(X, \mathcal{H}(m)) \cong H^0(X, F(m))$. Let $R^s \subset \mathfrak{R}^s$ (resp $R^{ss} \subset \mathfrak{R}^{ss}$) denote the open subschemes parametrizing *locally free sheaves*.

In [**13**] the Donaldson-Uhlenbeck moduli space has been constructed as the closure of the moduli space M^s of the moduli space of stable *locally free sheaves* in the scheme $\mathfrak{M}^{ss}$ together with the reduced scheme structure. The scheme $\mathfrak{M}^{ss}$ is realized as the image of a $PGL(m)$-invariant mapping

$$\pi : \mathfrak{R}^{ss} \to \mathfrak{M}^{ss}$$

Furthermore, the scheme $\mathfrak{M}^{ss}$ is projective. Note that $\mathfrak{M}^{ss}$ is not a GIT quotient but it maps $\mathfrak{R}^s$ to an open subset $\mathfrak{M}^s \subset \mathfrak{M}^{ss}$ and $\pi\ |_{\mathfrak{R}^s}$ is the GIT quotient $\mathfrak{R}^s//PGL(m)$. The closure M^{DU} of M^s in $\mathfrak{M}^{ss}$ with the *reduced scheme structure* is the precise algebro-geometric analogue of the differential geometric construction due to Donaldson. The key property of the moduli space M^{DU} is that the boundary of M^s is describable in terms of *locally free* polystable sheaves with lower c_2 and certain zero cycles.

LEMMA 3.2. *$\mathfrak{R}^{ss}$ is irreducible for large c_2, for a fixed c_1.*

PROOF. Observe that $\mathfrak{R}^s$ is a dense open subset in $\mathfrak{R}^{ss}$ for large c_2 ([**9**, Theorem 9.1.2, page 200]). So irreducibility of $\mathfrak{R}^{ss}$ is equivalent to the irreducibility of $\mathfrak{R}^s$. Now $\mathfrak{M}^s$ is a geometric quotient of $\mathfrak{R}^s$ for the action of $PGL(m)$. By [**9**, Theorem 9.4.3, page 203] the scheme $\mathfrak{M}^s$ is irreducible for large c_2. Since the quotient map $f : \mathfrak{R}^s \to \mathfrak{M}^s$ is an open map with both base and fibre being irreducible, it follows that $\mathfrak{R}^s$ is irreducible (see Lemma 3.3 below) . □

LEMMA 3.3. *If $f : X \to Y$ is a morphism of schemes such that f is an open surjective morphism and each closed fibre is irreducible, then*

$$Y \text{ irreducible} \implies X \text{ irreducible} .$$

PROOF. Let U and V be two nonempty open sets in X. Since f is an open surjective map and Y is irreducible, $f(U)$ and $f(V)$ are open nonempty subsets of Y such that $f(U) \cap f(V) \neq \emptyset$. Let $y \in f(U) \cap f(V)$ be a closed point and $x_1 \in U$ and $x_2 \in V$ such that $y = f(x_1) = f(x_2)$. Clearly, $U \cap f^{-1}(y)$ and $V \cap f^{-1}(y)$ are two nonempty open subsets of $f^{-1}(y)$. Since $f^{-1}(y)$ is irreducible it implies that $U \cap V \supset U \cap V \cap f^{-1}(y) \neq \emptyset$. Hence Y is irreducible. □

3.4. The small weight case. Since our final aim in this paper is to show that the Donaldson-Uhlenbeck spaces constructed in [**3**] are asymptotically irreducible, we will assume for the rest of the paper that the sheaves that we consider in the Quot scheme are *locally free*. Recall that the scheme R^{ss} (resp. R^s) parametrizes semistable (resp. stable) *locally free quotients. We will stick to these assumptions and notations in the paper from here onwards.*

We now consider bundles equipped with parabolic structures. The weight $\alpha := (\alpha_1, \alpha_2)$ is called *small* if it satisfies the condition

$$(\alpha_2 - \alpha_1) D \cdot H < \frac{1}{2}. \tag{3.3}$$

Now a key observation, which is easy to check, is the following.

LEMMA 3.5. *For small weights α* (3.3)*, for any $E \in R^s$ and any quasi-parabolic structure* (2.4)*, the parabolic bundles E_* is α-stable and conversely any α-stable parabolic bundle E_* has the property that its underlying bundle E is* semistable.

Let $\tilde{T}$ be the total family parametrizing quasi-parabolic structures on rank 2 bundles. The notation $\tilde{T}$ is loose since it hides the topological and parabolic data of its underlying objects. However, we observe that $\tilde{T}$ is independent of the parabolic weights α, β etc.

Since we do not need to make modifications in the topological datum to prove *asymptotic irreducibility* we do not carry it as a part of the notation. Again since the rank of the bundle is 2 there is not much in terms of the quasi-parabolic structure except the degree of the subbundle when restricted to the parabolic divisor D. This will figure in the discussion that follows. It will be mentioned whenever needed and should cause no confusion.

Let $R^{\alpha}_{k,d}$ be the total family for H–stable parabolic bundles with weight α. For a formal definition of $R^{\alpha}_{k,d}$ we direct the reader to [**15**]. We simplify the notation and have

$$R^s_\alpha := R^{\alpha}_{k,d}.$$

By the definition of $\tilde{T}$ we have the obvious morphism, namely *forget* : $\tilde{T} \to R$ which "forgets" the quasi-parabolic structure. Note however that under this map the image of R^{ss}_β or R^s_β need not be contained in R^{ss}; similarly, the inverse image of R^s can fall outside R^s_β for an arbitrary weight β.

In our simple setting of a flag which is only one-step on a rank 2 bundle, when the weights are small (3.3), the morphism *forget* : $R^s_\alpha \to R^{ss}$ is well-defined and the inverse image of R^s is contained in R^s_α (this is a consequence of Lemma 3.5). Let T^s and T^{ss} denote the inverse images of R^s and R^{ss} in $\tilde{T}$. In other words, T^{ss} is the total family of *quasi-parabolic structures on semistable bundles*.

The upshot is that, if the weight α is small we have *open* inclusions:

$$T^s \subset R^s_\alpha \subset T^{ss}. \tag{3.4}$$

We now describe the space T^{ss} of quasi-parabolic structures on rank 2 semistable bundles. Let $\mathcal{F}$ be the universal sheaf on $X \times R^{ss}$ and let $\mathcal{L}$ be the Poincaré line bundle over $D \times \mathrm{Pic}^l(D)$. We have a diagram of various projections:

$$\begin{array}{ccccc} & & D \times R^{ss} \times \mathrm{Pic}^l(D) & & \\ & \swarrow^{p_1} & \downarrow^{p_2} & \searrow^{p_3} & \\ D \times R^{ss} & & R^{ss} \times \mathrm{Pic}^l(D) & & D \times \mathrm{Pic}^l(D). \end{array} \tag{3.5}$$

Let

$$\mathcal{W} := p_{2*}(\mathcal{H}om(p_1^*(\mathcal{F}\mid_{D \times R^{ss}}), p_3^*(\mathcal{L}))),$$

where we have assumed that $l = \deg(\mathcal{L}_t)$ is sufficiently large so that

$$p_{2*}(\mathcal{H}om(p_1^*(\mathcal{F}\mid_{D \times R^{ss}}), p_3^*(\mathcal{L})))$$

is locally free ([**8**, page 288]). Let

$$Z = \underline{Spec}\; Sym(\mathcal{W}^*)$$

be the underlying geometric vector bundle. This scheme parametrizes all morphisms from $(F\mid_D) \to L$ for $F \in R^{ss}$ and $L \in \mathrm{Pic}^l(D)$. Let

$$Z_{\mathrm{sur}} \subset Z$$

be the open subscheme which parametrizes the *surjective morphisms*. It is not hard to show that by choosing $l \gg 0$ we can have a non-empty set of surjective morphisms (see for example [**1**, Theorem 2, page 426]). By taking kernels of these morphisms which give line subbundles, we get the quasi-parabolic structures. Thus, after choosing l suitably, we see that $Z_{\text{sur}} \subset Z$ is a *non-empty* open subset. By the definition of T^{ss} (for suitable l) it is immediate that there is an isomorphism $Z_{\text{sur}} \xrightarrow{\sim} T^{ss}$.

By Lemma 3.2 the scheme R^{ss} is asymptotically irreducible, being open in $\mathfrak{R}^{ss}$; furthermore, $\text{Pic}^l(D)$ is also irreducible for any integer l. Hence Z and therefore Z_{sur} is asymptotically irreducible. This implies that T^{ss} is *asymptotically irreducible*. It follows by (3.4), that R^s_α is *asymptotically irreducible*.

Observe that bounds on l in turn give bounds on $d = c_1(\mathcal{F}^2_D(F)).D$, where the quasi-parabolic structure $\mathcal{F}^2_D(F) \subset F\mid_D$ is obtained as the kernel to $F\mid_D \to L$. We isolate this key result in the following proposition.

Proposition 3.6. *For small α, R^s_α is asymptotically irreducible for suitable $d = c_1(\mathcal{F}^2_D(E)).D$.*

Remark 3.7. The bounds on l, which in turn give bounds on d, ensure that, irrespective of the weight α, the bundles E have enough quasi-parabolic structures on the given divisor $D \subset X$.

4. The density of $\mathcal{M}^s_\alpha$ in $\mathcal{M}^{ss}_\alpha$

Let $\mathcal{M}^s_\alpha = \mathcal{M}^\alpha_{k,d}$ be the moduli stack of α-stable bundles and $\mathcal{M}^{ss}_\alpha$ the moduli stack of α-semistable bundles on X with topological and parabolic datum as in §2. The aim of this section is to prove that the open substack $\mathcal{M}^s_\alpha$ in the moduli stack $\mathcal{M}^{ss}_\alpha$ is dense for any α. We handle the problem by converting it to the equivariant Γ-bundle setting. The general set-up is as in §2 and we use the same notation. Let Y be a Kawamata cover of X. The advantage in doing this is that the technical complications arising in handling obstruction theory in the parabolic setting is considerably simplified when we make this shift.

Let $\mathcal{H}$ be a Γ-sheaf over Y and P_1, P_2 are two fixed polynomials. Let $\text{Drap}_\Gamma(\mathcal{H}, P_1, P_2)$ denote the "generalized flag scheme" which parametrizes Γ–subsheaves of $\mathcal{H}$

$$\mathcal{H}_* := 0 \subset \mathcal{H}_3 \subset \mathcal{H}_2 \subset \mathcal{H}_1 = \mathcal{H}$$

such that the Hilbert polynomial of $\mathcal{H}_{i-1}/\mathcal{H}_i$ is P_{i-1}. These can be defined as Γ-fixed points of the usual Drap scheme (cf. [**3**, page 15], [**9**, Appendix 2.A, page 48]).

Lemma 4.1. *The dimension of* $\text{Drap}_\Gamma(\mathcal{H}, P_1, \cdots, P_k)$ *at the point $\mathcal{H}_*$ satisfies the following inequality*

$$\text{ext}^0_{\Gamma,+}(\mathcal{H},\mathcal{H}) \geq \dim_{\mathcal{H}_*}(\text{Drap}_\Gamma(\mathcal{H}, P_1, \cdots, P_k)) \geq \text{ext}^0_{\Gamma,+}(\mathcal{H},\mathcal{H}) - \text{ext}^1_{\Gamma,+}(\mathcal{H},\mathcal{H}).$$

Proof. The proof of this lemma is a routine equivariant generalization of the one given in [**9**, Proposition 2.A.12, page 54] and we omit the details. □

4.2. Parabolic Chern classes Let E_* be a rank 2 parabolic vector bundle over X with underlying bundle E. The parabolic Chern classes are defined as (see [**3**, Lemma 6.1])

$$\text{par}(c_1)(E_*) = c_1(E) + (\alpha_1 + \alpha_2) \cdot D, \tag{4.1}$$

$$\text{par}(c_2)(E_*) = c_2(E) + (\alpha_1 + \alpha_2)c_1(E) \cdot D + \alpha_1\alpha_2 D^2. \tag{4.2}$$

Observe that when $c_1(E) = 0$ (which is our base assumption), both $\mathrm{par}(c_1)$ and $\mathrm{par}(c_2)$ differ from the usual c_1 and c_2 by terms which involve only the parabolic divisor.

Let V be a Γ–bundle on Y. We have (see [**4**, Equation 3.11])

$$c_1(V) \;=\; p^*(\mathrm{par}(c_1)(\{p_*^\Gamma(V)\}_*)), \tag{4.3}$$

where $\{p_*^\Gamma(V)\}_*$ is the invariant direct image of V with the canonical parabolic structure coming from the Seshadri-Biswas correspondence.

For a Γ-bundle F of rank 2 of *local type* $\tau(\alpha)$ (see [**3**, Definition 2.12] for the definition) we have the equation in the second Chern classes of the underlying bundle:

$$c_2(p_*^\Gamma(\mathcal{H}om(F,F))) = c_2\left[\{p_*^\Gamma(F)\}_*^\vee \hat{\otimes} \{p_*^\Gamma(F)\}_*\right] = 4\; c_2((p_*^\Gamma(F))), \tag{4.4}$$

where the last equality follows by a splitting principle argument as in [**3**, Lemma 6.1] and the assumption that $c_1((p_*^\Gamma(F)) = 0$.

Here and elsewhere, "$\vee$" denotes the parabolic dual and $\hat{\otimes}$ denotes the parabolic tensor product. By the naturality of parabolic Chern classes we have

$$\mathrm{par}(c_i)(E_*^\vee) = (-1)^i \mathrm{par}(c_i)(E_*). \tag{4.5}$$

When we work with a Kawamata cover as in our case, then we have the following relation between the Γ–cohomology and the usual cohomology on $Y/\Gamma = X$:

$$H^i_\Gamma(Y,\mathcal{F}) = H^i(X, p_*^\Gamma(\mathcal{F})), \quad \forall i. \tag{4.6}$$

DEFINITION 4.3. For Γ-bundles F and G on Y, define

$$\chi_\Gamma(F,G) := \sum_i (-1)^i \mathrm{ext}^i_\Gamma(F,G).$$

Let V be a Γ-bundle of rank r on Y. Define the Γ-discriminant of V as:

$$\Delta_\Gamma(V) :=\; 2r\; c_2(p_*^\Gamma(V)) - (r-1)\; c_1(p_*^\Gamma(V))^2.$$

4.4. Γ-total families. Let R^{ss}_Γ (resp. R^s_Γ) parametrize Γ–semistable (resp. stable) bundles of type $\tau(\alpha)$ and fixed topological datum (c_1, c_2) over Y. In §3 and §4 of [**3**] we give the construction of R^{ss}_Γ which parametrizes Γ-torsion–free sheaves. We recall that there is an action of Γ on a suitable Quot scheme of quotients on the Kawamata cover Y of X. The scheme R_Γ is the subscheme of Γ–fixed points in the Quot scheme which consists of torsion–free sheaves and R^{ss}_Γ is an open subscheme of R_Γ. We stick to locally free sheaves in this work since we work with the Donaldson-Uhlenbeck compactifications.

4.5. Cohomological computations. The following lemmas play a key role in proving that R^s_Γ is dense in R^{ss}_Γ.

LEMMA 4.6. *Let F be a Γ–vector bundle of rank 2 on Y of some type $\tau(\alpha)$, such that*

$$c_1(p_*^\Gamma(F)) \;= 0\,. \tag{4.7}$$

Then

$$\chi_\Gamma(F,F) = -\Delta_\Gamma(F) + 4\chi(\mathcal{O}_X).$$

PROOF. We have $\chi_{\Gamma}(F,F) = \chi_{\Gamma}(Y, \mathcal{H}om(F,F))$
$= \chi(X, p_*^{\Gamma}(\mathcal{H}om(F,F)))$ (because of (4.6))
$= -c_2(p_*^{\Gamma}(\mathcal{H}om(F,F))) + 4\chi(\mathcal{O}_X)$ (by Hirzebruch-Riemann-Roch and (4.7))
$= -4\ c_2(p_*^{\Gamma}(F)) + 4\chi(\mathcal{O}_X)$ (by (4.4))
$= -\Delta_{\Gamma}(F) + 4\chi(\mathcal{O}_X)$ (by Definition 4.3). □

Let $F \in R_{\Gamma}^{ss} - R_{\Gamma}^{s}$ be a rank 2 strictly Γ-semistable bundle on Y such that $c_1(p_*^{\Gamma}(F)) = 0$. Let

$$0 \to F_1 \to F \to F_2 \to 0$$

be the Γ-Jordan-Hölder filtration of F. Observe that the F_2 is a torsion–free Γ–sheaves of rank 1, while F_1 is locally free.

Let $p_*^{\Gamma}(F) = E_*$ and $p_*^{\Gamma}(F_i) = E_{i,*}$, $i = 1,2$. Then

$$0 \to E_{1,*} \to E_* \to E_{2,*} \to 0$$

is the parabolic Jordan-Hölder filtration of E_* on X. Note that $E_{2,*}$ is a parabolic torsion-free sheaf of rank 1. Further, $c_1(E) = c_1(p_*^{\Gamma}(F)) = 0$. We remark that if the parabolic line bundle $E_{1,*}$ has weight α_1, then its parabolic dual $E_{1,*}^{\vee}$ has weight $1-\alpha_1$ (cf. [**12**, Section 8] where the weight will be simply $-\alpha_i$ in the balanced convention).

LEMMA 4.7. *Let $E_{i,*}$ be as above with weights α_i on X. Then*

$$\chi(X, E_{2,*}\hat{\otimes}E_{1,*}^{\vee}) = \chi(X, E_2 \otimes E_1^*)$$

E_1^* being the usual dual of E_1.

PROOF. By the Hirzebruch-Riemann-Roch theorem (K being the canonical divisor on X), we see that

$$\chi(X, E_{2,*}\hat{\otimes}E_{1,*}^{\vee}) = \frac{c_1(E_{2,*}\hat{\otimes}E_{1,*}^{\vee})^2}{2} - \frac{c_1(E_{2,*}\hat{\otimes}E_{1,*}^{\vee})\cdot K}{2} + \chi(\mathcal{O}_X).$$

We write $c_1(E_{2,*}\hat{\otimes}E_{1,*}^{\vee})$ for the Chern class of the underlying bundle (and not its parabolic Chern class) since it is notationally inconvenient to shed the parabolic luggage on the tensor product $E_{2,*}\hat{\otimes}E_{1,*}^{\vee}$, the reason being that the underlying sheaf of $E_{2,*}\hat{\otimes}E_{1,*}^{\vee}$ is *not* $E_2 \otimes E_1^*$.

Observe that
$c_1(E_{2,*}\hat{\otimes}E_{1,*}^{\vee}) = \mathrm{par}(c_1)(E_{2,*}\hat{\otimes}E_{1,*}^{\vee}) - (\alpha_2 - \alpha_1)D$
$= \mathrm{par}(c_1)(E_{2,*}) + \mathrm{par}(c_1)(E_{1,*}^{\vee}) - (\alpha_2 + 1 - \alpha_1)D$
$= [\mathrm{par}(c_1)(E_{2,*}) - \alpha_2 D] + [\mathrm{par}(c_1)(E_{1,*}^{\vee}) - (1-\alpha_1)D]$
$= c_1(E_2) + c_1({E_1}^*) = c_1(E_2 \otimes E_1^*)$
and the result follows. □

Let

$$\xi_{21} = c_1(E_2) - c_1(E_1).$$

Observe that, by the definition of Δ_{Γ}, we have

$$\Delta_{\Gamma}(F) = 4c_2(E) = 4c_1(E_1)\cdot c_1(E_2). \tag{4.8}$$

Now

$$\begin{aligned}[c_1(E_2) - c_1(E_1)]^2 &= [c_1(E_2) + c_1(E_1)]^2 - 4[c_1(E_2)\cdot c_1(E_1)] \\ &= -4[c_1(E_2)\cdot c_1(E_1)]\end{aligned} \tag{4.9}$$

since $c_1(E_2) + c_1(E_1) = c_1(E) = 0$.

Hence by (4.8) and (4.9)

$$\xi_{21}^2 = [c_1(E_2) - c_1(E_1)]^2 = -\Delta_{\Gamma}(F). \tag{4.10}$$

The Γ-Euler characteristic has the following description for the Γ-line bundles F_i.

$$\chi_\Gamma(F_1,F_2) = \chi(X, p_*^\Gamma(\mathcal{H}om(F_1,F_2))) = \chi(X, p_*^\Gamma(F_2\otimes F_1^*)) \tag{4.11}$$
$$= \chi(X, E_{2,*}\hat{\otimes}E_{1,*}^\vee) = \frac{\xi_{21}^2}{2} - \frac{\xi_{21}.K}{2} + \chi(\mathcal{O}_X)$$

by the proof of Lemma 4.7. Hence

$$\chi_\Gamma(F_1,F_2) = \frac{\xi_{21}^2}{2} - \frac{\xi_{21}.K}{2} + \chi(\mathcal{O}_X). \tag{4.12}$$

LEMMA 4.8. *Let $F \in R_\Gamma^{ss} - R_\Gamma^s$ be as above such that $c_1(p_*^\Gamma(F)) = 0$. Then*

$$\mathrm{ext}^1_{\Gamma,-}(F,F) \leq \frac{3}{4}\Delta_\Gamma(F) + B$$

where B is an irrelevant number not involving the Chern classes of the bundles.

PROOF. By a Γ-equivariant version of the spectral sequence in [**9**, 2.A.4] we have

$\mathrm{ext}^1_{\Gamma,-}(F,F) \leq \mathrm{ext}^1_\Gamma(F_1,F_1) + \mathrm{ext}^1_\Gamma(F_2,F_1) + \mathrm{ext}^1_\Gamma(F_2,F_2)$
$= \{\mathrm{ext}^0_\Gamma(F_1,F_1) + \mathrm{ext}^2_\Gamma(F_1,F_1) - \chi_\Gamma(F_1,F_1)\} + \{\mathrm{ext}^0_\Gamma(F_2,F_1) + \mathrm{ext}^2_\Gamma(F_2,F_1) -$
$\chi_\Gamma(F_2,F_1)\} + \{\mathrm{ext}^0_\Gamma(F_2,F_2) + \mathrm{ext}^2_\Gamma(F_2,F_2) - \chi_\Gamma(F_2,F_2)\}$
$\leq B_1 - \{\chi_\Gamma(F_1,F_1) + \chi_\Gamma(F_2,F_1) + \chi_\Gamma(F_2,F_2)\}$ [1]
$= B_1 + \chi_\Gamma(F_1,F_2) - \chi_\Gamma(F,F)$
$= B_1 + \frac{\xi_{21}^2}{2} - \frac{\xi_{21}.K}{2} + \Delta_\Gamma(F) - 3\chi(\mathcal{O}_X)$ (by Lemma 4.6 and (4.12))
$= \frac{3}{4}\Delta_\Gamma(F) + \frac{\xi_{21}^2}{4} - \frac{\xi_{21}.K}{2} + B_1 - 3\chi(\mathcal{O}_X)$ (by (4.10))
$= \frac{3}{4}\Delta_\Gamma(F) + \left[\frac{\xi_{21}}{2} - \frac{K}{2}\right]^2 + B_1 - 3\chi(\mathcal{O}_X) - \frac{K^2}{4}$
$\leq \frac{3}{4}\Delta_\Gamma(F) + B.$

The last inequality with the irrelevant number B comes by the following reasoning. By the Hodge index theorem,

$$\left[\frac{\xi_{21}}{2} - \frac{K}{2}\right]^2 \leq \frac{\left(\left[\frac{\xi_{21}}{2} - \frac{K}{2}\right]\cdot H\right)^2}{H^2}.$$

Further, by the parabolic semistability of E_*, since $E_{i,*}$ are its parabolic Jordan-Hölder terms we have

$$\mathrm{par}(c_1)(E_{1,*})\cdot H = \frac{\mathrm{par}(c_1)(E_*)\cdot H}{2} = \mathrm{par}(c_1)(E_{2,*})\cdot H.$$

Hence, $(c_1(E_2) - c_1(E_1))\cdot H = \xi_{21}\cdot H = (\alpha_1 - \alpha_2)D\cdot H$ is an irrelevant number. The remaining terms in $(\left[\frac{\xi_{21}}{2} - \frac{K}{2}\right]\cdot H)$ are clearly irrelevant. □

4.9. The density of R_Γ^s in R_Γ^{ss}. In the rest of this section we conclude the density of R_Γ^s in R_Γ^{ss}.

LEMMA 4.10. *There is an irrelevant number B depending on the rank, X, H, the parabolic datum α_i and D, such that*

$$\dim(R_\Gamma^{ss} - R_\Gamma^s) \leq \mathrm{end}_\Gamma(\mathcal{H}) + \frac{3}{4}\Delta_\Gamma(F) + B$$

where $F \in R_\Gamma^{ss} - R_\Gamma^s$

[1] Since F_i are rank 1 Γ-torsion–free sheaves, the dimensions $\mathrm{ext}^0_\Gamma(F_i,F_j)$ and $\mathrm{ext}^2_\Gamma(F_i,F_j)$ are bounded $\forall i,j$ and the irrelevant number B_1 is to take care of these terms.

PROOF. Let $\{\rho : \mathcal{H} \to F\} \in R_\Gamma^{ss} - R_\Gamma^s$ with Γ–Jordan-Hölder filtration

$$0 = F_0 \subset F_1 \subset F_2 = F \tag{4.13}$$

such that F_1 and F_2/F_1 are rank 1 torsion–free sheaves with the same $\mu = \mu(F)$ and F_1 is locally free. The filtration 4.13 induces a Γ-filtration on $\mathcal{H}$

$$0 \subset \mathcal{H}_0 \subset \mathcal{H}_1 \subset \mathcal{H}_2 = \mathcal{H} \tag{4.14}$$

such that $F_1 = \mathcal{H}_1/\mathcal{H}_0$ and $F_2 = \mathcal{H}_2/\mathcal{H}_0$.

Let $P_* := (P_1, P_2)$ be the Hilbert polynomials of $\mathcal{H}_2/\mathcal{H}_1$ and $\mathcal{H}_1/\mathcal{H}_0$. Since we fix the topological type of these quotients, we get a bounded family of sheaves with fixed $\mu = \mu(F)$ giving only finitely many choices of P_*. Let Z be the finite union of $\mathrm{Drap}_\Gamma(\mathcal{H}, P_1, P_2)$. There is a morphism $f : Z \to Q_\Gamma$ (the Γ–fixed points of the Quot scheme on Y) sending $\mathcal{H}_0 \subset \mathcal{H}_1 \subset \mathcal{H}$ to $\mathcal{H}_0 \subset \mathcal{H}$. It is clear that $R_\Gamma^{ss} - R_\Gamma^s \subset f(Z)$ (since every strictly semistable object has a Jordan-Hölder filtration).

We have by Lemma 4.1

$$\dim(R_\Gamma^{ss} - R_\Gamma^s) \leq \dim Z \leq \mathrm{ext}^0_{\Gamma,+}(\mathcal{H}, \mathcal{H}). \tag{4.15}$$

The definition of $\mathrm{Ext}_\pm$ gives an exact sequence

$$0 \to \mathrm{Ext}^0_{\Gamma,-}(\mathcal{H}, \mathcal{H}) \to \mathrm{Ext}^0_\Gamma(\mathcal{H}, \mathcal{H}) \to \mathrm{Ext}^0_{\Gamma,+}(\mathcal{H}, \mathcal{H}) \to \mathrm{Ext}^1_{\Gamma,-}(\mathcal{H}, \mathcal{H}).$$

Hence

$$\begin{aligned} \mathrm{ext}^0_{\Gamma,+}(\mathcal{H}, \mathcal{H}) &\leq \mathrm{end}_\Gamma(\mathcal{H}) - \mathrm{ext}^0_{\Gamma,-}(\mathcal{H}, \mathcal{H}) + \mathrm{ext}^1_{\Gamma,-}(\mathcal{H}, \mathcal{H}) \\ &\leq \mathrm{end}_\Gamma(\mathcal{H}) - 1 + \mathrm{ext}^1_{\Gamma,-}(\mathcal{F}, \mathcal{F}). \end{aligned}$$

The last inequality follows from the fact that a filtration of $\mathcal{F}$ canonically induces a filtration on $\mathcal{H}$, and we also have $\mathrm{ext}^1_{\Gamma,-}(\mathcal{F}, \mathcal{F}) = \mathrm{ext}^1_{\Gamma,-}(\mathcal{H}, \mathcal{H})$. Hence,

$$\mathrm{ext}^0_{\Gamma,+}(\mathcal{H}, \mathcal{H}) \leq \mathrm{end}_\Gamma(\mathcal{H}) + \frac{3}{4}\Delta_\Gamma(F) + B \quad \text{(by Lemma 4.8).}$$

□

PROPOSITION 4.11. *For any* $\{\rho : \mathcal{H} \to F\} \in R_\Gamma^{ss}$,

$$\mathit{dim}_\rho(R_\Gamma^{ss}) \geq \mathit{end}_\Gamma(\mathcal{H}) + \Delta_\Gamma(F) - 4\chi(\mathcal{O}_X)$$

PROOF. We follow the proof of O'Grady (see [**16**] and [**9**, Theorem. 4.5.8, page. 104]. Let K be the kernel of the morphism ρ. Applying $\mathrm{Hom}_\Gamma(-, F)$ to

$$0 \longrightarrow K \longrightarrow \mathcal{H} \longrightarrow F \longrightarrow 0,$$

we get

$$0 \longrightarrow \mathrm{End}_\Gamma(F) \longrightarrow \mathrm{Hom}_\Gamma(\mathcal{H}, F) \longrightarrow \mathrm{Hom}_\Gamma(K, F) \longrightarrow \mathrm{Ext}^1_\Gamma(F, F) \longrightarrow 0$$

Suppose that a positive integer m has been already chosen for which F is m–regular. Therefore, as we have seen earlier, $H^0(\mathcal{H}(m)) = H^0(F(m))$. Thus we have

$$\mathrm{Hom}_\Gamma(\mathcal{H}, F) = \mathrm{Hom}_\Gamma(H^0(\mathcal{H}(m)), H^0(F(m))) = \mathrm{Hom}_\Gamma(\mathcal{H}, \mathcal{H})$$

and we have the following equality of dimensions:

$$\mathrm{hom}_\Gamma(K, F) = \{\mathrm{end}_\Gamma(\mathcal{H}) - \mathrm{end}_\Gamma(F) + \mathrm{ext}^1_\Gamma(F, F)\}.$$

Using this computation we get the following inequality of dimensions:

$$\begin{aligned} &\dim_\rho(R_\Gamma^{ss}) \geq \mathrm{hom}_\Gamma(K, F) - \mathrm{ext}^2_\Gamma(F, F) \\ &= \mathrm{end}_\Gamma(H) - \mathrm{end}_\Gamma(F) + \mathrm{ext}^1_\Gamma(F, F) - \mathrm{ext}^2_\Gamma(F, F) = \mathrm{end}_\Gamma(\mathcal{H}) - \chi_\Gamma(F, F) \\ &= \mathrm{end}_\Gamma(\mathcal{H}) + \Delta_\Gamma(F) - 4\chi(\mathcal{O}_X) \quad \text{(by Lemma 4.6).} \end{aligned}$$

□

Proposition 4.12. *Let B be as in Lemma 4.10. If $\Delta_\Gamma(F) > 4(B + 4\chi(\mathcal{O}_X))$, then R^s_Γ is dense in R^{ss}_Γ.*

Proof. We have the following inequalities:
$\dim(R^{ss}_\Gamma - R^s_\Gamma) \le \{\mathrm{end}_\Gamma(\mathcal{H}) + \frac{3}{4}\Delta_\Gamma(F) + B\}$
$< \{\mathrm{end}_\Gamma(\mathcal{H}) + \Delta_\Gamma(F) - 4\chi(\mathcal{O}_X)\} \le \min\{\dim(X_i) : X_i \text{ a component of } R^{ss}_\Gamma\}$.

Note that the above inequalities show that the dimension of any component is at least $(\mathrm{end}_\Gamma(\mathcal{H}) + \Delta_\Gamma(F) - 4\chi(\mathcal{O}_X))$. Hence R^s_Γ intersects all components of R^{ss}_Γ. Hence R^s_Γ is dense in R^{ss}_Γ. □

Corollary 4.13. *If the second Chern class c_2 of the underlying bundles is* large*, then $\mathcal{M}^s_\alpha$ is dense in $\mathcal{M}^{ss}_\alpha$ for any weight α.*

Proof. By the *Seshadri–Biswas* correspondence we see that the moduli stack $\mathcal{M}^{ss}_\Gamma$ of Γ–semistable bundles (resp. $\mathcal{M}^s_\Gamma$ of Γ–stable bundles) is isomorphic to $\mathcal{M}^{ss}_\alpha$ (resp. $\mathcal{M}^s_\alpha$). Hence by Proposition 4.12, since $c_2 \gg 0$ it follows that R^s_Γ is dense in R^{ss}_Γ. Since R^{ss}_Γ (resp. R^s_Γ) is the *atlas* of the Artin stack $\mathcal{M}^{ss}_\Gamma$ (resp. $\mathcal{M}^s_\Gamma$) we have a surjective morphism $R^{ss}_\Gamma \to \mathcal{M}^{ss}_\Gamma$ and the result follows. □

5. Variation of $\mathcal{M}^{ss}_\alpha$ and the Main Theorem

Let $\mathcal{M}^{ss}_\alpha$ (resp. $\mathcal{M}^s_\alpha$) be the moduli stack of α-semistable (resp. α-stable) bundles with first Chern class 0. Here we tacitly assume that d is chosen as in Proposition 3.6 (see Remark 3.7). Now we study these moduli stacks as we vary weights.

Let $N_0 = D \cdot H$,

$$W = \{(\alpha_1, \alpha_2) : 0 < \alpha_1 < \alpha_2 < 1\}$$

and

$$\delta W := \left\{(\alpha_1, \alpha_2) : 0 < \alpha_1 < \alpha_2 < 1 \quad \text{such that } \mid \alpha_1 - \alpha_2 \mid = \frac{k}{2N_0},\ 1 \le k \le 2N_0\right\}.$$

Let $W^\circ = W - \delta W$ A connected component of W° is called a chamber. Observe that, if α is a weight within a chamber then $\mathcal{M}^{ss}_\alpha = \mathcal{M}^s_\alpha$. Moreover if α and β are in same chamber then $\mathcal{M}^{ss}_\alpha = \mathcal{M}^{ss}_\beta$ and also $\mathcal{M}^s_\alpha = \mathcal{M}^s_\beta$.

We have the following lemma:

Lemma 5.1. *If α is in a chamber and ω is on an adjacent wall, then any ω-stable bundle is α-stable and any α-stable bundle is ω-semistable.*

Proof. For $0 \le t \le 1$, let α_t denote the vector $t\alpha + (1-t)\omega$ in W. Then, α_t is also in the chamber for $t \ne 0$. Suppose that E_* is ω-stable and suppose that E_* is not α-stable. Then, there exists a subbundle E'_* of E_* such that $\mu_\alpha(E'_*) \ge \mu_\alpha(E_*)$. The continuous function $t \mapsto \mu_{\alpha_t}(E'_*) - \mu_{\alpha_t}(E_*)$ assumes a negative value at $t = 0$ and is non-negative at $t = 1$ and hence takes the value 0 for some $0 < t_0 \le 1$. But then E_* is strictly semistable with respect to the weight α_{t_0} contradicting the fact that α_{t_0} is within the chamber.

Similarly, if E_* is α-semistable (therefore in fact α-stable) and E_* is not ω-semistable, then there exists a subbundle E'_* of E_* such that $\mu_\omega(E'_*) > \mu_\omega(E_*)$. Thus, $\mu_{\alpha_t}(E_*) - \mu_{\alpha_t}(E'_*)$ is negative at $t = 0$ and non-negative at $t = 1$; this would imply that with respect to some α_{t_0} within the chamber, E_* is strictly semistable, again a contradiction. □

We have the following:

COROLLARY 5.2. *If α is in a chamber and ω is in a adjacent wall then we have the following inclusions:*

$$\mathcal{M}^s_\omega \subset \mathcal{M}^s_\alpha = \mathcal{M}^{ss}_\alpha \subset \mathcal{M}^{ss}_\omega.$$

THEOREM 5.3. *M_β is asymptotically irreducible for all β.*

PROOF. Recall that the scheme R^s_α is an *atlas* for the Artin stack $\mathcal{M}^s_\alpha$ and we have a canonical surjective morphism $R^s_\alpha \to \mathcal{M}^s_\alpha$. Hence by Proposition 3.6, $\mathcal{M}^s_\alpha$ is *asymptotically irreducible* for small α.

For any ω in an adjacent wall, by Corollary 5.2 we see that $\mathcal{M}^s_\omega$ is asymptotically irreducible being an open substack of $\mathcal{M}^s_\alpha$. Now by Corollary 4.13 it follows that $\mathcal{M}^{ss}_\omega$ is asymptotically irreducible.

Now taking β in any chamber with ω in an adjacent wall and different from the "small" chamber, we see again by Corollary 4.13 that $\mathcal{M}^{ss}_\beta$ is asymptotically irreducible. We proceed similarly to reach all weights in W using the connectedness of W and finiteness of the number of walls; since $\mathcal{M}^s_\beta$ surjects onto M^s_β it follows that M^s_β is asymptotically irreducible.

Now recall that M_β is by definition the closure of M^s_β (with the reduced scheme structure) in a certain $\mathfrak{M}^\beta_{k,d}$. This implies that M_β is also asymptotically irreducible and the theorem follows. □

REMARK 5.4. The subtle point is that even though we finally need to prove that M^s_α is asymptotically irreducible, we are forced to go to the semistable bundles since we need to go over various weight chambers.

REMARK 5.5. Observe that the arguments in this paper automatically give as a consequence the *generic smoothness* and *asymptotic non-emptiness* of the moduli space M_α. In specific situations, one can use the techniques of this paper to conclude rationality of certain parabolic moduli.

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