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Parabolic bundles on algebraic surfaces I – The Donaldson–Uhlenbeck compactification

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Abstract. The aim of this paper is to construct the parabolic version of the Donaldson–Uhlenbeck compactification for the moduli space of parabolic stable bundles on an algebraic surface with parabolic structures along a divisor with normal crossing singularities. We prove the non-emptiness of the moduli space of parabolic stable bundles of rank 2.

Keywords.

1. Introduction

Let X be a smooth projective variety defined over the field $\mathbb{C}$ of complex numbers. Moduli spaces of sheaves with parabolic structures were defined and constructed in great generality by Maruyama and Yokogawa [24]. Their work generalises the earlier construction of Mehta and Seshadri [25] when $\dim(X) = 1$. When $\dim(X) = 2$, i.e. X is a smooth projective surface and if D is an effective divisor on X then one finds from the work of Kronheimer and Mrowka (cf. [17] and [18]) that the underlying geometry and topology of the moduli space of parabolic bundles of rank two and trivial determinant have very interesting applications arising out of a generalization of Donaldson polynomials defined from these moduli spaces. These moduli spaces and their compactifications were studied in the papers of Kronheimer and Mrowka but primarily from the differential geometric standpoint. In particular, the Kobayashi–Hitchin correspondence was conjectured in these papers and this has since been proven by a number of people in growing order of generality (cf. [5], [23], [30]).

The purpose of this paper and its sequel [1] is to initiate a comprehensive study of the geometry of the moduli space of μ-stable parabolic bundles of arbitrary rank on *smooth projective surfaces* with parabolic structures on a reduced divisor D with *normal crossing singularities*. More precisely, in this paper we construct the analogue of the Donaldson–Uhlenbeck compactification of the moduli space of μ-stable parabolic bundles of arbitrary rank and also prove the existence of μ-stable parabolic bundles when certain topological invariants are allowed to be arbitrarily large. We summarise our results in the following theorem. For notations see (4.20):

Theorem 1.1.

(1) *There exists a natural compactification of the moduli space* $M^{\boldsymbol{\alpha}}_{k,\mathbf{j},\mathbf{r}}(r,\mathcal{P},\kappa)$ *of* μ-*stable parabolic bundles with fixed determinant* $\mathcal{P}$ *and with fixed topological and parabolic datum. Furthermore, the compactification can be set-theoretically described as follows*:

$$\overline{M^{\boldsymbol{\alpha}}_{k,\mathbf{j},\mathbf{r}}(r,\mathcal{P},\kappa)} \subset \coprod_{l\geq 0} M^{\boldsymbol{\alpha}\text{-poly}}_{k',\mathbf{j}',\mathbf{r}}(r,\mathcal{P},\kappa-l)\times S^l(X), \tag{1.1}$$

where by, $M^{\boldsymbol{\alpha}\text{-poly}}_{k,\mathbf{j},\mathbf{r}}(r,\mathcal{P},\kappa)$, *we mean the set of isomorphism classes of polystable parabolic bundles with parabolic datum given by* $(\boldsymbol{\alpha},\mathbf{l},\mathbf{r},\mathbf{j})$, *fixed determinant* $\mathcal{P}$ *and with topological datum given by k and* κ.

(2) *The moduli space of* μ-*stable parabolic bundles of rank* 2 *is non-empty, when the invariants k and* $\mathbf{j}$ *are made sufficiently large and the weights satisfy some natural bounds* (*see Theorem* 5.1)

This paper can therefore be seen as completing the algebro-geometric analogue of the Kobayashi–Hitchin correspondence for parabolic bundles on surfaces. We compare the moduli spaces that we construct with that of Kronheimer–Mrowka when we restrict ourselves to the rank two case.

The main strategy used for the construction is to use the categorical correspondence of the category of Γ-bundles of fixed type τ on a certain Kawamata cover of the surface X with the category of parabolic bundles on X with fixed parabolic datum (see §1 for definitions and terminology). The Kawamata cover Y is non-canonical and is therefore employed only as a stepping stone for the construction. Although non-canonical, the moduli problem gets defined more naturally on Y and one takes recourse to the ideas of Li and Le Potier, as well as the earlier work of Donaldson to give an algebraic–geometric construction of the Donaldson–Uhlenbeck compactification of the moduli space of μ-stable Γ-bundles on Y. Then by using the correspondence one can interpret the compactification in a canonical manner as a compactification of the moduli space of parabolic bundles over the surface X with given parabolic datum, thereby removing the non-canonical nature of the construction. We believe that this moduli space can be realised, as in the usual setting, as a generalized blow-down of the Maruyama–Yokogawa moduli space. Unlike our moduli space, the Maruyama–Yokogawa space is a GIT construction using Gieseker type stability for parabolic sheaves.

We then go on to show that the moduli space of μ-stable parabolic bundles is non-empty for large topological invariants. The proof is a generalization of the classical Cayley–Bacharach construction to the setting of orbifold bundles. Our proof of *non-emptiness* and existence of components with smooth points gives the same results for the Maruyama–Yokogawa space as well in the case when X is a surface. To the best of our knowledge the non-emptiness of these moduli spaces have not been shown hitherto. In the sequel [1] we also show the asymptotic irreducibility and asymptotic normality of these spaces.

The moduli spaces are defined when some natural topological invariants of the underlying objects are kept fixed. We also relate the topological invariants that occur in [17], [18] with natural invariants for parabolic bundles namely parabolic Chern classes as defined in [7]. One observes that the concept of an *action* (as defined in [17]) of a parabolic bundle is precisely the *second parabolic Chern class*. Moreover, when we examine the Donaldson–Uhlenbeck compactification for these moduli spaces, as observed by Kronheimer and Mrowka, the falling of the instanton numbers is not perceived very precisely but what is seen to drop in the boundary is the *second parabolic Chern class* or equivalently the *action*. Indeed, this is exactly the phenomenon in the usual Donaldson–Uhlenbeck compactification of stable $SU(2)$-bundles on surfaces. For applications involving Donaldson invariants arising from moduli of parabolic bundles should yield topological invariants for the pair (D, X) together with the imbedding $D \hookrightarrow X$, we refer the reader to [17].

2. Preliminaries

2.1 *The category of bundles with parabolic structures*

We rely heavily on the correspondence between the category of parabolic bundles on X and the category of Γ-bundles on a suitable Kawamata cover. This strategy has been employed in many papers (for example [6]) but since we need its intricate properties, most of which are scattered in a few papers of Biswas and Seshadri, we recall them briefly. We stress only on those points which are relevant to our purpose.

Let D be an effective divisor on X. For a coherent sheaf E on X the image of $E \bigotimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$ in E will be denoted by $E(-D)$. The following definition of parabolic sheaf was introduced in [24].

DEFINITION 2.3

Let E be a torsion-free $\mathcal{O}_X$-coherent sheaf on X. A *quasi-parabolic* structure on E over D is a filtration by $\mathcal{O}_X$-coherent subsheaves

$$E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D).$$

The integer l is called the *length of the filtration*. A *parabolic structure* is a quasi-parabolic structure, as above, together with a system of *weights* $\{\alpha_1, \ldots, \alpha_l\}$ such that

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{l-1} < \alpha_l < 1,$$

where the weight α_i corresponds to the subsheaf $F_i(E)$.

We shall denote the parabolic sheaf defined above by (E, F_*, α_*). When there is no scope of confusion it will be denoted by E_*.

For a parabolic sheaf (E, F_*, α_*) define the following filtration $\{E_t\}_{t\in\mathbb{R}}$ of coherent sheaves on X parameterized by $\mathbb{R}$:

$$E_t := F_i(E)(-[t]D), \tag{2.1}$$

where $[t]$ is the integral part of t and $\alpha_{i-1} < t - [t] \leq \alpha_i$, with the convention that $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$.

A *homomorphism* from the parabolic sheaf (E, F_*, α_*) to another parabolic sheaf (E', F'_*, α'_*) is a homomorphism from E to E' which sends any subsheaf E_t into E'_t, where $t \in [0, 1]$ and the filtration are as above.

If the underlying sheaf E is locally free then E_* will be called a parabolic vector bundle. *In this section, all parabolic sheaves will be assumed to be parabolic vector bundles.*

Remark 2.1. The notion of *parabolic degree* of a parabolic bundle E_* of rank r is defined as

$$\mathrm{par}_{\deg}(E_*) := \int_0^1 \deg(E_t)\mathrm{d}t + r \cdot \deg(D). \tag{2.2}$$

Similarly one may define $\mathrm{par}_{\mu}(E_*) := \mathrm{par}_{\deg}(E_*)/r$. There is a natural notion of parabolic subsheaf and given any subsheaf of E there is a canonical parabolic structure that can be given to this subsheaf (cf. [24], [6] for details).

DEFINITION 2.2

A parabolic sheaf E_* is called *parabolic semistable* (*resp. parabolic stable*) if for every parabolic subsheaf V_* of E_* with $0 < \mathrm{rank}(V_*) < \mathrm{rank}(E_*)$, the following holds:

$$\mathrm{par}_{\mu}(V_*) \leq \mathrm{par}_{\mu}(E_*) \;\; (\text{resp. } \mathrm{par}_{\mu}(V_*) < \mathrm{par}_{\mu}(E_*)). \tag{2.3}$$

2.1.1 *Some assumptions* The class of parabolic vector bundles that are dealt with in the present work satisfy certain constraints which will be explained now. In a remark below (see Remark 2.3), we observe that these constraints are not stringent in so far as the problem of moduli spaces is concerned.

(1) The first condition is that all parabolic divisors are assumed to be *divisors with normal crossings*. In other words, any parabolic divisor is assumed to be reduced, its each irreducible component is smooth, and furthermore the irreducible components intersect transversally.
(2) The second condition is that all the parabolic weights are *rational numbers*.
(3) The third and final condition states that on each component of the parabolic divisor the filtration is given by *subbundles*. The precise formulation of the last condition is given in Assumptions 3.2 (1) of [6].

Henceforth, all parabolic vector bundles will be assumed to satisfy the above three conditions.

Remark 2.3. We remark that for the purpose of construction of the moduli space of parabolic bundles the choice of rational weights is not a serious constraint and we refer the reader to Remark 2.10 of [25] for more comments on this.

DEFINITION 2.4

A quasi-parabolic filtration on a sheaf E can also be defined by giving filtration by subsheaves of the restriction $E|_D$ of the sheaf E to each component of the parabolic divisor:

$$E|_D = \mathcal{F}_D^1(E) \supset \mathcal{F}_D^2(E) \supset \cdots \supset \mathcal{F}_D^l(E) \supset \mathcal{F}_D^{l+1}(E) = 0$$

together with a system of weights

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{l-1} < \alpha_l < 1.$$

Let $\mathrm{PVect}(X, D)$ denote the category whose objects are parabolic vector bundles over X with parabolic structure over the divisor D satisfying the above three conditions, and the morphisms of the category are homomorphisms of parabolic vector bundles (which was defined earlier).

The direct sum of two vector bundles with parabolic structures has an obvious parabolic structure. Evidently, $\mathrm{PVect}(X, D)$ is closed under the operation of taking direct sum. We remark that the category $\mathrm{PVect}(X, D)$ is an additive tensor category with the direct sum and the parabolic tensor product operation. It is straight-forward to check that $\mathrm{PVect}(X, D)$ is also closed under the operation of taking the parabolic dual defined in [31].

For an integer $N \geq 2$, let $\mathrm{PVect}(X, D, N) \subseteq \mathrm{PVect}(X, D)$ denote the subcategory consisting of all parabolic vector bundles all of whose parabolic weights are multiples of $1/N$. It is straight-forward to check that $\mathrm{PVect}(X, D, N)$ is closed under all the above operations, namely parabolic tensor product, direct sum and taking the parabolic dual.

2.2 *The Kawamata covering lemma*

Let $D = \sum_{i=1}^{c} D_i$ be the decomposition of the divisor D into its irreducible components.

Take any $E_* \in \mathrm{PVect}(X, D)$ such that all the parabolic weights of E_* are multiples of $1/N$, i.e., $E_* \in \mathrm{PVect}(X, D, N)$.

The ‘covering lemma’ of Y Kawamata (Theorem 1.1.1 of [16], Theorem 17 of [15]) says that there is a connected smooth projective variety Y over $\mathbb{C}$ and a Galois covering morphism

$$p: Y \longrightarrow X \tag{2.4}$$

such that the reduced divisor $\tilde{D} := (p^* D)_{\mathrm{red}}$ is a normal crossing divisor on Y and furthermore, $p^* D_i = k_i N \cdot (p^* D_i)_{\mathrm{red}}$, where k_i, $1 \leq i \leq c$ are positive integers. Let Γ denote the Galois group for the covering map p.

2.3 *The category of* Γ*-bundles*

Let $\Gamma \subseteq \mathrm{Aut}(Y)$ be a finite subgroup of the group of automorphisms of a connected smooth projective variety $Y/\mathbb{C}$. The natural action of Γ on Y is encoded in a morphism

$$\mu: \Gamma \times Y \longrightarrow Y.$$

Denote the projection of $\Gamma \times Y$ to Y by p_2. The projection of $\Gamma \times \Gamma \times Y$ to the i-th factor will be denoted by q_i. A Γ-*linearized vector bundle* on Y is a vector bundle V over Y together with an isomorphism

$$\lambda: p_2^* V \longrightarrow \mu^* V$$

over $\Gamma \times Y$ such that the following diagram of vector bundles over $\Gamma \times \Gamma \times Y$ is commutative:

$$\begin{tikzcd}
q_3^* V \arrow[r, "(q_2, q_3)^* \lambda"] \arrow[dr, "(m \times Id_Y)^* \lambda"'] & (\mu \circ (q_2, q_3))^* V \arrow[d, "(Id_\Gamma \times \mu)^* \lambda"] \\
 & (\mu \circ (m, Id_Y))^* V
\end{tikzcd}$$

where m is the multiplication operation on Γ.

The above definition of Γ-linearization is equivalent to giving isomorphisms of vector bundles

$$\bar{g}\colon V \longrightarrow (g^{-1})^* V$$

for all $g \in \Gamma$, satisfying the condition that $\overline{gh} = \bar{g} \circ \bar{h}$ for any $g, h \in \Gamma$.

A Γ-*homomorphism* between two Γ-linearized vector bundles is a homomorphism between the two underlying vector bundles which commutes with the Γ-linearizations. Clearly the tensor product of two Γ-linearized vector bundles admits a natural Γ-linearization; so does the dual of a Γ-linearized vector bundle. Let $\mathrm{Vect}_\Gamma(Y)$ denote the additive tensor category of Γ-linearized vector bundles on Y with morphisms being Γ-homomorphisms.

As before, $\mathrm{Vect}_\Gamma(Y)$ denotes the category of all Γ-linearized vector bundles on Y. The isotropy group of any point $y \in Y$, for the action of Γ on Y, will be denoted by Γ_y.

2.4 *On local types of* Γ*-bundles*

Recall that since the Γ-action on Y is *properly discontinuous*, for each $y \in Y$, if Γ_y is the isotropy subgroup at y, then there exists an *analytic* neighbourhood $U_y \subset Y$ of y which is Γ_y-invariant and such that for each $g \in \Gamma \backslash \Gamma_y$, $g \cdot U_y \cap U_y = \emptyset$.

DEFINITION 2.5

Let ρ be a representation of Γ in $GL(r, \mathbb{C})$. Then Γ-acts on the trivial bundle $Y \times \mathbb{C}^r$ by $(y, v) \longrightarrow (\gamma y, \rho(\gamma)v)$, $y \in Y$, $v \in \mathbb{C}^r$, $\gamma \in \Gamma$. Following [29] we call this Γ-bundle, the Γ-bundle associated to the representation ρ.

Let $\mathcal{M}_Y(G)$ be the sheaf of germs of meromorphic maps of Y into a complex linear group G.

Now the sheaf $\mathcal{O}_Y(G)$ of germs of holomorphic maps from Y into G is a subsheaf of groups of $\mathcal{M}_Y(G)$.

Note that $\mathcal{O}_Y(G)$ acts on $\mathcal{M}_Y(G)$. The quotient sheaf of sets $\mathcal{O}_Y(G)\backslash\mathcal{M}_Y(G)$ is called the *sheaf of germs of divisors with values in* G and it is denoted by $D_Y(G)$. Since Γ acts on the sheaf $\mathcal{O}_Y(G)$ we see that Γ operates on $D_Y(G)$ as well and hence $D_Y(G)$ becomes a Γ-sheaf. A Γ-invariant section of $D_Y(G)$ is called a (Γ, G)-divisor.

Remark 2.6. Let Θ be a Γ-invariant section of $D_Y(G)$. From the above description of a (Γ, G)-divisor we see that Θ can be defined by the following datum:

An open covering $\{U_i\}$ of Y by Γ-invariant open subsets and $\{f_i\}$, where f_i is a meromorphic map of U_i into G such that $f_i(s^{-1}y) = \lambda_i^s(y) f_i(y)$, $\forall s \in \Gamma$, $\lambda_i^s(y)$ being a holomorphic map of U_i into G and such that $f_j/f_i = g_{ij} \in \mathcal{O}_Y(G)(U_i \cap U_j)$ which satisfies the identity $g_{ij} \cdot g_{jk} \cdot g_{ki} = 1$ on $U_i \cap U_j \cap U_k$.

Remark 2.7. We remark that a to Γ-vector bundle one can naturally associate a $(\Gamma, GL(n))$-divisor in our situation, i.e. where $p\colon Y \longrightarrow X$ is a Kawamata cover and X is therefore projective. This can be seen as follows.

Let E be a Γ-locally free sheaf of rank r on Y. Then the invariant direct image sheaf $F := (p_*(E))^\Gamma$ is of rank r on X. Since X is a projective variety we can choose large n so that $F(m) = F \otimes \mathcal{H}(m)$ is globally generated for $m \geq n$ where $\mathcal{H}$ is an ample line bundle on X. In other words, we can choose r linearly independent sections of $F(m)$ or equivalently, r linearly independent meromorphic sections of F. This gives r independent Γ-invariant meromorphic sections of E on Y.

Observe that giving r independent sections of a vector bundle E is equivalent to giving a section of the underlying frame principal bundle $E_{GL(r)}$. In particular, giving r independent meromorphic Γ-invariant sections of E on Y is equivalent to giving a meromorphic Γ-invariant section f of the underlying $(\Gamma, GL(r))$-principal bundle $E_{GL(r)}$. Let $\{g_{ij}\}$ be the transition functions of $E_{GL(r)}$ subordinate to the Γ-invariant covering U_i of Y. A Γ-invariant meromorphic section f is giving Γ-invariant meromorphic functions $\{f_i\}$, where $f_i\colon U_i \longrightarrow GL(r)$ such that $f_i = g_{ij} f_j$ on $U_i \cap U_j$ and satisfying equivariant properties as in Remark 2.6. In other words, we have a $(\Gamma, GL(r))$-divisor to which the Γ-bundle E is associated. From now on we call it a Γ-divisor associated to Γ-vector bundle E.

We recall the following lemma which is straightforward to check.

Lemma 2.8. *Let P be a (Γ, G)-principal bundle on Y. Then given $y \in Y$ there exists an analytic neighbourhood U_y which is Γ_y-invariant such that $P|_{U_y}$ defined by* 1*-cocycles* $H^1(\Gamma_y, H^0(U_y, \mathcal{O}_Y(G)))$.

We then have the following equivariant local trivialisation lemma (cf. page 141–07 of [12]).

Lemma 2.9. *Let E be a Γ-bundle on Y of rank r. Let $y \in Y$ and let Γ_y be the isotropy subgroup of Γ at y. Then there exists a Γ_y-invariant analytic neighbourhood U_y of y such that the Γ_y-bundle $E|_{U_y}$ is associated to a representation $\Gamma_y \to GL(r)$* (*in the sense of Definition* 2.5).

Proof. For the purpose of notation we write G for $GL(n, \mathbb{C})$. Let $\mathcal{O}_Y(G)_y$ denote the stalk at y of the sheaf of germs of holomorphic maps of Y into G. And let $\mathcal{M}_Y(G)_y$ denote the stalk at y of the sheaf of germs of meromorphic maps from Y to G.

Since we have the exact sequence of sheaf of multiplicative groups

$$1 \longrightarrow \mathcal{O}_Y(G) \longrightarrow \mathcal{M}_Y(G) \longrightarrow \mathcal{D}_Y(G) \longrightarrow 1,$$

we have the following exact sequence of multiplicative groups

$$1 \longrightarrow \mathcal{O}_Y(G)_y \longrightarrow \mathcal{M}_Y(G)_y \longrightarrow \mathcal{D}_Y(G)_y \longrightarrow 1.$$

Hence we have the following long exact sequence of cohomology sets by taking Γ_y-invariants

$$\begin{aligned} 0 \longrightarrow H^0(\Gamma_y, \mathcal{O}_Y(G)_y) &\longrightarrow H^0(\Gamma_y, \mathcal{M}_Y(G)_y) \longrightarrow H^0(\Gamma_y, \mathcal{D}_Y(G)_y) \\ &\xrightarrow{\delta} H^1(\Gamma_y, \mathcal{O}_Y(G)_y) \longrightarrow \dots. \end{aligned}$$

Now we know that E is determined locally by an element of $H^1(\Gamma_y, \mathcal{O}_Y(G)_y)$ i.e. a 1-cocycle by Lemma 2.8.

Note that there is a canonical map from G to $\mathcal{O}_Y(G)_y$, i.e. the constant map $g\colon Y \longrightarrow G$ such that $g(y) = g$ for every $g \in G$. Hence this induces a canonical map

$$\chi\colon H^1(\Gamma_y, G) \longrightarrow H^1(\Gamma_y, \mathcal{O}_Y(G)_y).$$

To prove the lemma we need to show that the map χ is bijective. A 1-cocycle $\{f_\gamma\} \in H^1(\Gamma_y, \mathcal{O}_Y(G)_y)$, which when evaluated at y defines a map j: $H^1(\Gamma_y, \mathcal{O}_Y(G)_y) \longrightarrow H^1(\Gamma_y, G)$. We see that $j \circ \chi =$ identity. Hence it is enough to prove that χ is surjective.

By Remark 2.7, E is defined by a Γ-divisor (note that we need this only locally!). Therefore $\{f_\gamma\}$ comes from $H^0(\Gamma_y, \mathcal{D}_Y(G)_y)$ by the above cohomology exact sequence. In particular, it is represented by a coset $\Theta \in H^0(\Gamma_y, \mathcal{D}_Y(G)_y) \simeq \frac{H^0(\Gamma_y, \mathcal{M}_Y(G)_y}{H^0(\Gamma_y, \mathcal{O}_Y(G)_y}$.

In other words, if $\gamma \longrightarrow \{f_\gamma\}$ is a 1-cocycle representing an element of $H^1(\Gamma_y, \mathcal{O}_Y(G))$, there exists a Γ_y-invariant meromorphic section $\Theta \in \mathcal{M}_Y(G)$, such that $\Theta(\gamma z) = f_\gamma(z)\Theta(z)$ where $z \in U_y$ and $\gamma \in \Gamma_y$.

Define $\varphi(\gamma) := f_\gamma(y)$ for all $\gamma \in \Gamma_y$; then observe that φ defines a representation of Γ_y into $GL(n)$, for

$$\varphi(\gamma_1\gamma_2) = f_{\gamma_1\gamma_2}(y) = f_{\gamma_1}(\gamma_2 y) f_{\gamma_2}(y) = f_{\gamma_1}(y) f_{\gamma_2}(y) = \varphi(\gamma_1)\varphi(\gamma_2)$$

because $\gamma_2 y = y \forall \gamma_2 \in \Gamma_y$.

Define

$$\Psi(z) := \sum_{\gamma \in \Gamma_y} \varphi(\gamma)\Theta(\gamma^{-1}z).$$

Note that as it stands, $\Psi(z)$ is a Γ_y-invariant section of $\mathcal{M}_Y(M(n, \mathbb{C}))$.

Then we have

$$\Psi(\alpha z) = \sum_{\gamma \in \Gamma_y} \varphi(\gamma)\Theta(\gamma^{-1}\alpha z) = \sum_{\gamma \in \Gamma_y} \varphi(\alpha)\varphi(\nu^{-1})\Theta(\nu z)$$

by setting $\gamma^{-1}\alpha = \nu$. Hence

$$\Psi(\alpha z) = \varphi(\alpha)\Psi(z) \tag{2.5}$$

Further we have

$$\Psi(z)\Theta^{-1}(z) = \left\{ \sum_{\gamma \in \Gamma_y} \varphi(\gamma) f_{\gamma^{-1}}(z)\Theta(z) \right\} \Theta^{-1}(z) = \sum_{\gamma \in \Gamma_y} \varphi(\gamma) f_{\gamma^{-1}}(z)$$

since $\Theta(\gamma^{-1}z) = f_{\gamma^{-1}}(z)\Theta(z)$. But $f_{\gamma^{-1}}(y) = \varphi(\gamma^{-1})$, therefore we get

$$\Psi(y)\Theta^{-1}(y) = \sum_{\gamma \in \Gamma_y} \varphi(\gamma) f_{\gamma^{-1}}(y) = n_y \,.\, Id,$$

where n_y is a order of the group Γ_y.

We note that $\det(\Psi\Theta^{-1}) \neq 0$ since its evaluation at $y \in \Gamma_y$ is not zero (in fact it is equal to ${n_y}^n$ by the above equation). Hence $\Psi\Theta^{-1} \in H^0(\Gamma_y, \mathcal{O}_Y(G)_y)$. Also since $\Theta \in H^0(\Gamma_y, \mathcal{M}_Y(G)_y)$, it follows that $\Psi \in H^0(\Gamma_y, \mathcal{M}_Y(G)_y)$.

This also shows that both Ψ and Θ define the same divisor locally at y. But the $(\Gamma_y, GL(n))$-bundle defined by Ψ is given by the representation φ by the equation (2.5). Hence χ is surjective. q.e.d

Remark 2.10. The above lemma for Γ-bundles on curves with structure group $GL(r)$ can be found in Proposition 2, pp. 159 of [29] and [12] (see also Remark 2, page 162 of [29]). Here the key property that is used is that U_y and U_y/Γ_y are Stein spaces. This result, for the more general setting of arbitrary compact groups K instead of Γ and for general structure groups can be found in §11 of [13].

2.4.1 *Γ-bundles of fixed local type* We make some general observations on the local structure of Γ-bundles on the Kawamata cover defined in (2.3).

Let $\mathrm{Vect}_{\Gamma}^{D}(Y, N)$ denote the subcategory of $\mathrm{Vect}_{\Gamma}(Y)$ consisting of all Γ-linearized vector bundles W over Y satisfying the following two conditions:

(1) For a general point y of an irreducible component of $(p^* D_j)_{\mathrm{red}}$, the isotropy subgroup Γ_y is *cyclic* of order $|\Gamma_y| = n_y$. Note that in the earlier notation $n_y = k_j N$, where y is the generic point of the irreducible component D_j.
(2) In fact, the action is given by a representation ρ_y of Γ_y given as follows:
(3) The weights are rational numbers $\alpha_i = \frac{m_i}{N}$, with

$$m_1 < m_2 < \cdots m_l < N.$$

Since $n_y = k_j N$, we have $\alpha_i = d_i / n_y$, where $d_i = k_j m_i$, for each i and j.

$$\rho_y(\zeta) = \begin{bmatrix} z^{n_y \alpha_1} \cdot I_1 & & 0 \\ & \ddots & \\ 0 & & z^{n_y \alpha_l} \cdot I_l \end{bmatrix}, \tag{2.6}$$

where

- ζ is a generator of the group Γ_y.
- $\alpha_i = d_i / n_y$, where d_i's are increasing sequence of positive integers such that $d_1 < d_2 < \cdots d_l < n_y$.
- I_j is the identity matrix of order r_j, where r_j is the multiplicity of the weight α_j.
- z is an n_y-th root of unity.

(4) For a general point y of an irreducible component of a ramification divisor for p not contained in $(p^* D)_{\mathrm{red}}$, the action of Γ_y on W_y is the trivial action.
(5) For a *special point* y contained in $(p^* D)_{\mathrm{red}}$, the isotropy subgroup Γ_y contains the cyclic group Γ_n of order n determined by the irreducible component containing y. By the rigidity of representations of finite groups, the Γ_y-module structure on W_y (given by Lemma 2.9) when restricted to $\Gamma_n \subset \Gamma_y$ is given by the matrix (2.6).
(6) At special points y of the ramification divisor for p not contained in $(p^* D)_{\mathrm{red}}$, the restriction of the representation to the generic isotropy is trivial.

DEFINITION 2.11

To describe a Γ-vector bundle in $\mathrm{Vect}_{\Gamma}^{D}(Y, N)$ we need to specify the following datum: (i) an integer N, (ii) α_i's and (iii) r_i's. Altogether we denote this datum by a single letter τ (to be consistent with page 161 of [29]).

Remark 2.12. The reason for calling it local type τ is that, for a Γ-bundle and a point y the generic point of a divisor as above, the structure of the representation defines the bundle E_U for a Γ_y-invariant analytic neighbourhood in Y. Seshadri denoted the collection of representations of the cyclic groups which define the local isomorphism type over an analytic neighbourhood by the letter τ; note that the Γ-bundle defines what is known as an orbifold bundle.

Remark 2.13. We remark that this definition of Γ-bundles of fixed local type easily extends to Γ-torsion-free sheaves since the local action is specified only at the generic points of the ramification divisor.

We note that $\mathrm{Vect}_\Gamma^D(Y, N)$ is also an additive tensor category.

2.4.2 *Parabolic bundles and* Γ*-bundles* In [6] an identification between the objects of $\mathrm{PVect}(X, D, N)$ and the objects of $\mathrm{Vect}_\Gamma^D(Y, N)$ has been constructed. Given a Γ-homomorphism between two Γ-linearized vector bundles, there is a naturally associated homomorphism between the corresponding vector bundles, and this identifies, in a bijective fashion, the space of all Γ-homomorphisms between two objects of $\mathrm{Vect}_\Gamma^D(Y, N)$ and the space of all homomorphisms between the corresponding objects of $\mathrm{PVect}(X, D, N)$. An equivalence between the two additive tensor categories, namely $\mathrm{PVect}(X, D, N)$ and $\mathrm{Vect}_\Gamma^D(Y, N)$, is obtained this way. Since the description of this identification is already given in [6] and [2], it will not be repeated here.

We observe that an earlier assertion that the parabolic tensor product operation enjoys all the abstract properties of the usual tensor product operation of vector bundles, is a consequence of the fact that the above equivalence of categories indeed preserves the tensor product operation.

The above equivalence of categories has the further property that it takes the parabolic dual of a parabolic vector bundle to the usual dual of the corresponding Γ-linearized vector bundle.

Let $W \in \mathrm{Vect}_\Gamma^D(Y, N)$ be the Γ-linearized vector bundle of rank n on Y that corresponds to the given parabolic vector bundle E_*. The fiber bundle

$$\pi \colon P \longrightarrow Y$$

whose fiber $\pi^{-1}(y)$ is the space of all $\mathbb{C}$-linear isomorphisms from $\mathbb{C}^n$ to the fiber W_y, has a the structure of a $(\Gamma, GL(n, \mathbb{C}))$-bundle over Y.

DEFINITION 2.14

A Γ-linearized vector bundle E over Y is called Γ-*semistable* (resp. Γ-*stable*) if for any proper nonzero coherent subsheaf $F \subset E$, invariant under the action of Γ and with E/F being torsionfree, the following inequality is valid:

$$\mu(F) \leq \mu(E)(\text{resp. } \mu(F) < \mu(E)), \tag{2.7}$$

where the slope is as usual $\mu(E) = \deg(E)/r$ and $\deg(E)$ is computed with respect to the Γ-linearised very ample divisor Θ on Y.

The Γ-linearized vector bundle E is called Γ-*polystable* if it is a direct sum of Γ-stable vector bundles of the same slope.

Remark 2.15. The above correspondence between *parabolic bundles* on X and Γ-bundles on Y preserves the semistable (resp. stable) objects as well, where *parabolic semistability* is as in (2.3) (cf. [6]).

Remark 2.16. We remark that it is not hard to check that for Γ-bundles, Γ-semistability (resp. Γ-polystablity) is the same as the usual semistability (resp. polystability). This can be seen from the fact that the *top term of the Harder–Narasimhan filtration* (resp. *the Socle*) are *canonical* and hence invariant under the action of Γ. But we note that a Γ-stable bundle need not be Γ-stable, as can be seen by taking a direct sum of Γ-translates of a line bundle.

Remark 2.17. We make some key observations in this remark where we also note the essential nature of assumptions of characteristic zero base fields.

(1) The notion of Γ-cohomology for Γ-sheaves on Y has been constructed and dealt with in great detail in [11]. These can be realised as higher derived functors of the Γ-*fixed points* – sub-functor $(H^0)^\Gamma$ of the section functor H^0. (We use this notation to avoid Γ^Γ, because we have denoted the finite group by the letter Γ!).
We note immediately that since we work over fields of characteristic zero, the sub-functor $(H^0)^\Gamma \subset H^0$ is in fact a direct summand (by averaging operation). Hence, we see immediately that the higher derived functors of the functor$(H^0)^\Gamma$ are all sub objects of the derived functors of H^0.
(2) When we work with a Kawamata cover as in our case, then we have the following relation between the Γ-cohomology and the usual cohomology on $Y/\Gamma = X$:

$$H^i_\Gamma(Y, \mathcal{F}) = H^i(X, p_*^\Gamma(\mathcal{F})) \quad \forall i.$$

2.4.3 Γ-*bundles and orbifold bundles* We make a few general remarks on the advantages of working with a Kawamata cover Y and Γ-bundles on Y over working with *orbifold bundles* or V-bundles over V-manifolds. Locally, these two notions can be completely identified but for any global construction such as the one which we intend doing, namely a moduli construction, working with a Kawamata cover albeit *non-canonical*, has obvious advantages since it immediately allows us to work with a certain 'Quot' scheme over Y. To recover the moduli of parabolic bundles with fixed quasi parabolic structure, we then simply use the *functorial equivalence of parabolic bundles and* Γ-*bundles of fixed local type*.

2.4.4 Γ-*line bundles and parabolic line bundles* A Γ line bundle on Y is a line bundle L on Y together with a lift of action Γ. The Γ line bundle gives a Γ invariant line bundle L^Γ on X. Let D be a divisor of normal crossing on X. Let $D = \sum_{i=1}^d D_i$ be a decomposition into irreducible components. A parabolic line bundle on (X, D) is a pair of the form $(M, \beta_1, \ldots, \beta_i, \ldots, \beta_d)$ where M is a holomorphic line bundle on X and $0 \leq \beta_i < 1$ is a real number. When we start from a Γ line bundle on Y we get a pair $(L^\Gamma, \beta_1, \ldots, \beta_i, \ldots, \beta_d)$ where β_i is a rational number and it can be written as $\beta_i = m_i/N$. Let $\tilde{D}_i = (p^* D_i)_{\text{red}}$. Then by §2b of [8] we have $L = p^*(L^\Gamma) \otimes \mathcal{O}_Y(\sum_{i=1}^d k_i m_i \tilde{D}_i)$.

Remark 2.18. In our situation, by choice we work with a single weight when we consider Γ-line bundles of fixed local type τ although this may not be absolutely essential.

2.4.5 *Serre duality for* Γ-*line bundles of fixed local type*

DEFINITION 2.19

By a line bundle L of fixed local type τ we mean a *parabolic line bundle* $(L, \alpha_1, \alpha_2, \ldots, \alpha_d)$, where $\alpha_i = \alpha\ \forall i$. In other words, locally, the generic isotropy on the irreducible components of the inverse image of the parabolic divisor acts by a single character namely α. We will write $L^{(\alpha)}$ to specify the character.

Let $L = L^{(\alpha)}$ be a Γ line bundle on Y of type τ. Then by §2.4.4, one knows that $L = p^*(p_*^\Gamma(L)) \otimes \mathcal{O}_Y(\sum k_i m_i \tilde{D}_i)$ where all the m_i can be assumed to be equal to m since we have a single weight α. If $M = M^{(\alpha)}$ is another Γ-line bundle with the same local character type τ, then we have $L^* \otimes M = p^*(p_*^\Gamma(L)^*) \otimes p^*(p_*^\Gamma(M))$. Hence

$$(p_*^\Gamma(L^* \otimes M)) = (p_*^\Gamma(L)^* \otimes (p_*^\Gamma(M)). \tag{2.8}$$

Consider the canonical bundles K_X of X and define the Γ-bundle $K_Y^{(\alpha)}$ as follows:

$$K_Y^{(\alpha)} = p^*(K_X) \otimes \mathcal{O}_Y \left(\sum k_i \tilde{D}_i)m\right). \tag{2.9}$$

Then, we see as above that $p_*^\Gamma(K_Y^{(\alpha)}) = K_X$. We then have the following duality for Γ-line bundles of type τ.

Lemma 2.20. *For Γ-line bundles L of type τ, with local character α, the Γ-line bundle $K_Y^{(\alpha)}$ is the* dualising sheaf. *In other words, we have a canonical isomorphism*:

$$H_\Gamma^i(Y, L^* \otimes K_Y^{(\alpha)}) \simeq H_\Gamma^{n-i}(Y, L)^*$$

for all i.

Remark 2.21. We remark that we have made this statement for Γ-varieties Y of any dimension.

Proof. The proof is straightforward, but we give it for the sake of completeness. Recall the relationship between the Γ-cohomology on Y and the usual cohomology on X (Remark 2.17). We have the following isomorphism (using (2.8)):

$$\begin{aligned} H_\Gamma^i(Y, L^* \otimes K_Y^{(\alpha)}) &\simeq H^i(X, p_*^\Gamma(L^* \otimes K_Y^{(\alpha)}) \\ &\simeq H^i(X, p_*^\Gamma(L)^* \otimes (p_*^\Gamma(K_Y^{(\alpha)})). \end{aligned}$$

Using $p_*^\Gamma(K_Y^{(\alpha)}) = K_X$ we then conclude from the following isomorphism:

$$\simeq H^i(X, p_*^\Gamma(L)^* \otimes K_X) \simeq H^{n-i}(X, p_*^\Gamma(L))^* \simeq H_\Gamma^{n-i}(Y, L)^*,$$

where we use the usual Serre duality on X. q.e.d

The following lemma is important in proving Lemma 2.23. For a proof of this fact see for example page 37 of [9].

Lemma 2.22. *Let L, M are two line bundles over a smooth projective surface Y and Z be a reduced* 0*-dimensional cycle. Then*

$$\mathcal{H}om(M \otimes \mathcal{I}_Z, L) \simeq M^* \otimes L; \qquad \mathcal{E}xt^1(M \otimes \mathcal{I}_Z, L) \simeq \mathcal{O}_Z.$$

Lemma 2.23. *Let $L^{(\alpha_1)}$, $M^{(\alpha_2)}$ be two Γ-line bundles and Z be a reduced Γ-invariant cycle away from ramification locus. Then we have*

$$\dim(\mathrm{Ext}_\Gamma^1(M^{(\alpha_2)} \otimes \mathcal{I}_Z, L^{(\alpha_1)})) = \dim(H_\Gamma^1(M^{(\alpha_2)*} \otimes L^{(\alpha_1)} \otimes K_Y^{(\alpha_1-\alpha_2)}).$$

Proof. We have the following standard exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O} \to \mathcal{O}_Z \to 0. \tag{2.10}$$

Recall the Grothendieck Ext spectral sequence (§5 of [11]). Let $E_2^{p,q} = H_\Gamma^p(Y, \mathcal{E}xt^q(M \otimes \mathcal{I}_Z, L))$. Then the spectral sequence gives a long exact sequence whose initial part is of the form

$$0 \to E_2^{1,0} \to \mathrm{Ext}_\Gamma^1(M^{(\alpha_2)} \otimes \mathcal{I}_Z, L^{(\alpha_1)}) \to E_2^{0,1} \to E_2^{2,0}$$
$$\to \mathrm{Ext}_\Gamma^2(\mathcal{M} \otimes \mathcal{I}_Z, L^{(\alpha_1)}) \to$$

From Grothendieck's spectral sequence (§5 of [11]) we get

$$0 \to H_\Gamma^1(\mathcal{H}om(M^{(\alpha_2)} \otimes \mathcal{I}_Z, L^{(\alpha_1)}) \to \mathrm{Ext}_\Gamma^1(M^{(\alpha_2)} \otimes \mathcal{I}_Z, L^{(\alpha_1)})$$
$$\to H_\Gamma^0(\mathcal{E}xt(M^{(\alpha_2)} \otimes \mathcal{I}_Z, L^{(\alpha_1)}) \to H_\Gamma^2(\mathcal{H}om(M^{(\alpha_2)} \otimes \mathcal{I}_Z, L^{(\alpha_1)})).$$

Using Lemma 2.23 we get

$$0 \to H_\Gamma^1(M^{(\alpha_2)*} \otimes L^{(\alpha_1)}) \to \mathrm{Ext}_\Gamma^1(M^{(\alpha_2)} \otimes \mathcal{I}_Z, L^{(\alpha_1)}) \to H_\Gamma^0(\mathcal{O}_Z)$$
$$\to H_\Gamma^2(M^{(\alpha_2)*} \otimes L^{(\alpha_1)}).$$

Let $P = M^{(\alpha_2)} \otimes L^{(\alpha_1)*} \otimes K_Y^{(\alpha_1-\alpha_2)}$. Then, tensoring (2.10) with P and taking long exact sequences we get $0 \to H_\Gamma^0(P \otimes \mathcal{I}_Z) \to H_\Gamma^0(P) \to H_\Gamma^0(P \otimes \mathcal{O}_Z) \to H_\Gamma^1(P \otimes \mathcal{I}_Z) \to H_\Gamma^1(P) \to 0$.

Taking duals and comparing with 2.4.5 we get a commutative diagram:

$$\begin{tikzcd}
H_\Gamma^1(P)^* \arrow[r] \arrow[d, equal] & H_\Gamma^1(P \otimes \mathcal{I}_Z)^* \arrow[r] & H_\Gamma^0(P \otimes \mathcal{O}_Z)^* \arrow[r] \arrow[d, equal] & H_\Gamma^0(P)^* \arrow[r] \arrow[d, equal] & H_\Gamma^0(P \otimes \mathcal{I}_Z)^* \arrow[d, equal] \\
H_\Gamma^1(M^{(\alpha_2)*} \otimes L^{(\alpha_1)}) \arrow[r] & \mathrm{Ext}_\Gamma^1(M \otimes \mathcal{I}_Z, L^{(\alpha_1)}) \arrow[r] & H_\Gamma^0(\mathcal{O}_Z) \arrow[r] & H_\Gamma^2(M^{(\alpha_2)*} \otimes L^{(\alpha_1)}) \arrow[r] & H_\Gamma^0(P \otimes \mathcal{I}_Z)
\end{tikzcd}.$$

The downward equality follows from Γ-Serre duality (Lemma 2.20). Hence by using 'five lemma' we conclude

$$\dim(\mathrm{Ext}_\Gamma^1(M^{(\alpha_2)} \otimes \mathcal{I}_Z, L^{(\alpha_1)})) = \dim(H_\Gamma^1(M^{(\alpha_2)} \otimes L^{(\alpha_1)*} \otimes K_Y^{(\alpha_1-\alpha_2)})).$$

q.e.d

Remark 2.24. In fact after little bit of work one can prove that the spaces are canonically isomorphic, but for our purpose we need only the dimensional equality.

Lemma 2.25. *Let G be a Γ-locally free sheaf on Y and F be a coherent sheaf on X. Then $p_*^\Gamma(p^*(F) \otimes G) \simeq F \otimes p_*^\Gamma(G)$.*

Proof. Since the lemma is about isomorphism of sheaves, it is enough to check at local level. So we can restrict our attention to $\mathrm{Spec}(B) \subset Y, \mathrm{Spec}(A) \subset X$ such that Γ acts to $\mathrm{Spec}(B)$ and $A = B^\Gamma$. Let $F\ |_{\mathrm{Spec}(A)} = \tilde{M}$, where M is a finitely generated A-module (need not be locally free) and $G\ |_{\mathrm{Spec}(B)} = \tilde{B^r}$, where r is the rank of G. So locally the left-hand side becomes $(M \otimes_A B \otimes_B B^r)^\Gamma = \bigoplus_r (M \otimes_A B^\Gamma) = M^r$. Locally right-hand side also becomes $M \otimes (B^r)^\Gamma = M^r$. q.e.d

3. Towards the construction

3.1 On determinant line bundles

We briefly recall the basic definitions for the convenience of the reader. Let Y be an irreducible smooth projective variety equipped with a very ample $\mathcal{O}_Y(1)$. Let $K(Y)$ be the Grothendieck algebra of classes of coherent sheaves. Let θ be the class in $K(Y)$ of the structure sheaf $\mathcal{O}_\Theta$ of a hyperplane section $\Theta \subset Y$. This algebra is equipped with a quadratic form $q\colon u \mapsto \chi(u^2)$. This form is calculated in terms of the rank and the Chern classes of u. For example, if Y is a smooth projective surface, and if $u \in K(Y)$ is of rank r, and the Euler characteristic χ, we have

$$q(u) = 2r\chi + c_1^2 - r^2\chi(\mathcal{O}_Y).$$

The kernel $\ker(q)$ comprises of the classes which are *numerically equivalent to zero*. We work with the quotient:

$$K_{\mathrm{num}}(Y) = K(Y)/\ker(q).$$

For a smooth projective surface Y, $K_{\mathrm{num}}(Y) \simeq \mathbb{Z} \times H^2(Y, \mathbb{Z}) \times \mathbb{Z}$ and this isomorphism is by giving (r, c_1, χ).

Recall that if $\mathcal{F}$ is a flat family of coherent sheaves on Y parametrized by a scheme S, then $\mathcal{F}$ defines an element $[\mathcal{F}] \in K^0(S \times Y)$, the Grothendieck group of $S \times Y$ generated by locally free sheaves. We may then define the homomorphism from the Grothendieck group of coherent sheaves on Y given by

$$\lambda_{\mathcal{F}}\colon K(Y) \longrightarrow \mathrm{Pic}(S)$$

as follows: For $u \in K(Y)$, $\lambda_{\mathcal{F}}(u) = \det(\mathrm{pr}_{1!}(\mathcal{F} \cdot \mathrm{pr}_2^*(u))$, where $\mathcal{F} \cdot \mathrm{pr}_2^*(u)$ is the product in $K(S \times Y)$ and $\mathrm{pr}_{1!}\colon K^0(S \times Y) \to K^0(S)$ associates to each class u the class $\sum_i (-1)^i R^i \mathrm{pr}_{1*}(u)$.

We observe that this has a collection of functorial properties for which we refer to page 179 of [14].

Let Y be a smooth projective surface. Fix a class $c \in K_{\mathrm{num}}(Y)$, i.e. the rank r, the first Chern class $c_1 = \mathcal{O}_Y$ and the Euler characteristic χ. This in particular fixes c_2 as well. Fix also the very ample divisor Θ on Y and a base point $x \in Y$. Let $\theta = [\mathcal{O}_\Theta] \in K(Y)$. Define for each i:

$$u_i(c) := -r \cdot \theta^i + \chi(c \cdot \theta^i) \cdot [\mathcal{O}_x] \tag{3.11}$$

(cf. page 183 of [14]).

3.2 *Projective Γ-frame bundle*

We make some general remarks on the general construction of Γ-frame bundle associated to a Γ-vector bundle. This is a generalization of the classical frame bundle construction but will be needed in the construction of the moduli space. Let Y be a scheme of finite type with a *trivial* Γ-action. Let F be a Γ-locally free $\mathcal{O}_Y$ module of rank r and assume that each fibre F_y is a Γ-module and the Γ-module structures are isomorphic at different points. Let W be a finite dimensional vector space of dimension r which is a Γ-module isomorphic to the Γ-module F_y for any $y \in Y$. Denote by $\mathcal{O}_Y(W)$ the trivial rank r sheaf modelled by W. With this added structure, we have a canonical group namely, $H = \mathrm{Aut}_\Gamma(W) \subset GL(W)$, which acts on $\mathcal{O}_Y(W)$ by automorphisms which preserve the Γ-structure.

Let $\mathbb{H}\mathrm{om}_\Gamma(\mathcal{O}_Y(W), F) := \mathrm{Spec}(\mathrm{Sym}(\mathcal{H}om_\Gamma(\mathcal{O}_Y(W), F))^*) \to Y$ be the geometric Γ-vector bundle that parametrizes all Γ-homomorphisms from $\mathcal{O}_Y(W)$ to F. Let $\Phi(F) := \mathbb{I}\mathrm{som}_\Gamma(\mathcal{O}_Y(W), F) \subset \mathbb{H}\mathrm{om}_\Gamma(\mathcal{O}_Y(W), F)$ be the open subscheme which parametrizes all Γ-isomorphisms and let $\pi \colon \Phi(F) \to Y$ denote the canonical projection.

Then we observe that H acts on $\Phi(F)$ by composition and π is a principal bundle with structure group H. Indeed, the Γ-structure on F gives a natural reduction of structure group of the frame bundle associated to F (which by the usual construction is a principal $GL(W)$-bundle).

Similarly, if PH is the image of $H \subset GL(W)$ in $PGL(W)$, then one can construct projective PH-bundle by taking image of $\Phi(F)$ in $\mathrm{Proj}(\mathrm{Sym}(\mathcal{H}om_\Gamma(\mathcal{O}_Y(W), F)^*))$. We term the image of $\Phi(F)$ the *projective Γ-frame bundle* over Y associated to the Γ-bundle F.

3.3 *The determinant line bundle*

The aim of this section is to construct a line bundle on the Quot scheme which parametrizes the objects we need. This will be a natural *determinantal bundle* as in the Donaldson construction.

Recall that our aim is to construct the moduli space of μ-semistable bundles with Γ-structure and the notion of μ-semistability in the higher dimensional setting (in our case the surface Y) is not a GIT notion; in fact, the GIT semistable will be the Gieseker semistable bundles.

Since Γ-semistability is the same as the usual semistability for torsion free sheaves (cf. Remark 2.16) we observe that the family of Γ-semistable sheaves with fixed Hilbert polynomial is bounded (Theorem 3.3.7 of [14]).

Let $\mathcal{E}$ be a torsion free Γ-coherent sheaf over a smooth projective surface Y of rank r and P be any polynomial in $\mathbb{Q}[z]$. Let $\mathrm{Quot}(\mathcal{E}, P)$ be the Quot scheme which parametrizes all quotients of $\mathcal{E}$ with fixed Hilbert polynomial P. Let $\mathcal{F}$ denote the universal quotient sheaf of $\mathcal{O}_{\mathrm{Quot}(\mathcal{E},P)} \otimes \mathcal{E}$ on $Y \times \mathrm{Quot}(\mathcal{E}, P)$. Let Q denote the subscheme of $\mathrm{Quot}(\mathcal{E}, P)$ whose closed points correspond to torsion-free sheaves with fixed topological data (c_1, c_2, r) (note that fixing Hilbert polynomial for a family of sheaves gives only finitely many choices for the triples (c_1, c_2, r) and $\mathcal{F}\mid_{Q\times Y}$ be universal quotient sheaf on $Q \times Y$. Let L be the determinantal line bundle $\lambda_{\mathcal{F}}(u)$. Since Γ is acting on $\mathcal{E}$ and Y, Γ acts on Q in the natural manner:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "{[q]}"] & \mathcal{F}_q \\ \mathcal{E} \arrow[u, "\gamma^*"] \arrow[ur] & \end{tikzcd},$$

where γ^* is the canonical pull back. Let $Q^\Gamma \subset Q$ be the set of all Γ-invariant points of Q which is a nonempty subset (!), and it gets a *closed subscheme structure* for, if $\mathrm{Spec}\, A$ is an affine open subset of Q then $Q^\Gamma \cap \mathrm{Spec}\, A$ is a closed subsheame of $\mathrm{Spec}\, A$. In fact it is given by an ideal $I = \{f - \gamma f \mid f \in A, \gamma \in \Gamma\}$.

Let $P_c(m) = \chi(c(m))$ be the Hilbert polynomial associated to the fixed class $c \in K_{\mathrm{num}}(Y)$, where $c(m) := c \cdot [\mathcal{O}_Y(m)]$. Let $\mathcal{E} = V \otimes \mathcal{O}_Y(-m)$ where V is a vector space of dimension $P_c(m)$.

DEFINITION 3.1

We say that a coherent sheaf F on Y is m-regular if higher cohomology group $H^i(Y, \mathcal{F}(m-i))$ vanishes for all $i \geq 1$.

We choose m large enough so that all quotients are m-regular.

Notation 3.2. Let $P = P_c(m)$ and let $Q = \mathrm{Quot}(\mathcal{E}, P)$. Let Q^Γ denote the closed subscheme of Γ-fixed points. Let $\mathcal{R} \subset Q$ (resp. $\mathcal{R}^\Gamma \subset Q^\Gamma$) be the locally closed subscheme of all μ-semistable quotients (resp. (Γ, μ)-semistable quotients) of $\mathcal{E}$ with fixed topological data (r, c_1, c_2) *and fixed determinant* $\mathcal{Q}$. We observe that giving the topological data is giving a class $c \in K_{\mathrm{num}}(Y)$.

Because of m-regularity (Definition 3.1) we have $V \simeq H^0(\mathcal{F}_q(m)) \simeq \mathrm{k}^{P_c(m)}$. The group $\mathrm{Aut}(V)$ acts naturally on the scheme Q.

Notation 3.3. Let us denote by G the group $SL(V)$ and by H the subgroup $\mathrm{Aut}_\Gamma(V) \cap G$ i.e. the subgroup of G which are Γ-automorphisms as well. We will use this notation through out this paper.

Remark 3.4. The group $\mathrm{Aut}_\Gamma(V)$ is a direct product of full linear groups and in particular connected and reductive. The group H is also therefore connected and reductive To see this, observe that we can decompose V as a Γ-module into its isotypical decomposition. This decomposition gives the choice of a torus in $SL(V)$ and the group H is the centraliser of this torus; indeed, H is the Levi subgroup associated to the parabolic subgroup given by the decomposition. This implies that H is connected and reductive. The group $\mathrm{Aut}_\Gamma(V)$ is similarly the Levi subgroup in the bigger group $GL(V) = \mathrm{Aut}(V)$.

The group H (resp. G) acts on the scheme $\mathcal{R}^\Gamma$ (resp. $\mathcal{R}$) by automorphisms. The universal quotient $\mathcal{F}$ allows us to construct a G-linearised line bundle $\mathcal{N}$ on $\mathcal{R}$ given as follows:

$$\mathcal{N} := \lambda_{\mathcal{F}}(u_1(c)),$$

where $u_i(c)$ is defined as in (3.11). Denote by $\mathcal{M}$ the restriction of this line bundle to $\mathcal{R}^\Gamma$. That is

$$\mathcal{M} = \mathcal{N}|_{\mathcal{R}^\Gamma}. \tag{3.12}$$

Let $\mathcal{R}^\Gamma(D, N)$ be the subset $\mathcal{R}^\Gamma$ consisting of Γ-torsion-free sheaves of fixed local type.

Remark 3.5. By the rigidity of representation of finite groups, it follows that $\mathcal{R}^\Gamma(D, N)$ is both *open and closed* in $\mathcal{R}^\Gamma$. Moreover, it is easily seen that $\mathcal{R}^\Gamma(D, N)$ is also invariant under the action of H.

Remark 3.6. By definition, the line bundle $\mathcal{M}$ comes with a canonical H-linearisation.

Then we have the following:

Lemma 3.7 (Lemma 8.2.4 of [14]).

1. *If $s \in \mathcal{R}^\Gamma$ is a point such that for a general high degree Γ-invariant curve C, $\mathcal{F}_s \mid_C$ is semistable then there exists an integer $N > 0$ and an H-invariant section $\tilde{\sigma} \in H^0(\mathcal{R}^\Gamma, \mathcal{M}^N)^H$ such that $\tilde{\sigma}(s) \neq 0$.*
2. *If s_1 and s_2 are two points in $\mathcal{R}^\Gamma$ such that for a general high degree Γ-invariant curve C, $\mathcal{F}_{s_1} \mid_C$ and $\mathcal{F}_{s_2} \mid_C$ are both semistable but not S-equivalent or one of them is semistable but other is not then there is a H-invariant section $\tilde{\sigma}$, in some tensor power of $\mathcal{M}$ which separates these two points (i.e. $\tilde{\sigma}(s_1) = 0$ but $\tilde{\sigma}(s_2) \neq 0$).*

Proof. The proof (following ideas from Le Potier [20]) is largely following the exposition in Huybrechts–Lehn [14]. But we give all the main steps in the argument even at the risk of repetition. This is because there are certain distinctive points in this setting which needs to be highlighted, especially those relating to the projective Γ-frame bundle and the morphism to the quot scheme of Γ-bundles on a curve. In a sense these are precisely the points which distinguish the possible Γ-structures on a given semistable bundle.

Since Γ-semistability is same as the usual semistability, one gets a general high degree smooth curve $C \in \mid a\Theta \mid^\Gamma$, $a \gg 0$, such that, $\mathcal{F} \mid_{\mathcal{R}^\Gamma \times C}$ produces a family of generically semistable sheaves on C with fixed topological data $(r, \mathcal{Q} \mid_C)$. Recall that $\mathcal{Q}$ is the fixed determinant for objects in $\mathcal{R}^\Gamma$ (see (3.2)). The fact that it is a generic family of semistable sheaves on C is because of openness of semistability property (cf for example [28]). Let U be a nonempty open subset of $\mathcal{R}^\Gamma$ such that $\mathcal{F} \mid_{U \times C}$ is a flat family of semistable sheaves on C.

Recall that we have fixed a class $c \in K^\Gamma_{\mathrm{num}}(Y)$. Let $c \mid_C$ be its pull-back (or restriction) in $K^\Gamma_{\mathrm{num}}(C)$. Note that $c \mid_C$ is completely determined by its rank r and the line bundle $\mathcal{Q}|_C$.

Recall that $P_c(m) = \chi(c(m))$ is the Hilbert polynomial associated to the fixed class $c \in K^\Gamma_{\mathrm{num}}(Y)$, where $c(m) := c \cdot [\mathcal{O}_Y(m)]$. Let $P'(n) := P_{c|_C}(n)$. Then, by computing the Euler characteristic from the exact sequence of sheaves obtained by *restriction to the curve* C, we see that P' is given by the equation $P'(n) = P_c(n) - P_c(n-a)$, since $C \in \mid a\Theta \mid^\Gamma$.

Let $\mathcal{H}' = \mathcal{O}_C(-m')^{P'(m')}$ and $Q^\Gamma_C \subset \mathrm{Quot}^\Gamma_C(\mathcal{H}', P')$ be the closed subset of quotients with determinant $\mathcal{Q} \mid_C$. Observe that $\mathcal{H}'$ can be identified with $W \otimes \mathcal{O}_C(-m')$, where W is a vector space of dimension $P'(m')$.

Denote by G_1 the group $SL(W)$ and by H_1 the subgroup of G_1 given by

$$H_1 = G_1 \cap \mathrm{Aut}_\Gamma(W).$$

As remarked earlier (Remark 3.4), the group H_1 is also *connected and reductive*.

We also have a natural H_1-action on Q^Γ_C by automorphisms.

Let $\mathcal{O}_{Q_C{}^\Gamma} \otimes \mathcal{H}' \twoheadrightarrow \tilde{\mathcal{F}}'$ be the universal quotient and $L_C = \lambda_{\tilde{\mathcal{F}}'}(u_0(c \mid_C)$ (see (3.11) for the definition of $u_0(c)$).

One can check that $L_C \cong \det(p_{Q^\Gamma_C *}(\tilde{\mathcal{F}}'))$. If m' is sufficiently large the following holds:

(1) Given a point $[q\colon \mathcal{H}' \twoheadrightarrow \tilde{\mathcal{F}}'_q] \in Q^\Gamma_C$, the following assertions are equivalent:

- (a) $\tilde{\mathcal{F}}'_q$ is Γ-semistable sheaf and $W \simeq H^0(C, \tilde{\mathcal{F}}'_q(m'))$.
- (b) $[q]$ is a semistable point in Q^Γ_C for the action of H_1 with respect to the linearization of L_C, i.e., there is an integer ν and a H_1-invariant section $\sigma \in H^0(C, L^\nu_C)^{H_1}$ such that $\sigma([q]) \neq 0$.

(2) Two points $[q_i\colon \mathcal{H}' \to \tilde{\mathcal{F}}'_{q_i}]$; $i = 1, 2$ are separated by H_1-invariant sections if and only if either both are semistable points but $\tilde{\mathcal{F}}'_{q_1}$ and $\tilde{\mathcal{F}}'_{q_2}$ are not S-equivalent or else, one of them is semistable and other is not semistable.
(3) $\tilde{\mathcal{F}} := \mathcal{F}\,|_{\mathcal{R}^\Gamma \times C}$ is m'-regular (Definition 3.1) with respect to $\mathcal{R}^\Gamma$.

Note that $p_*(\tilde{\mathcal{F}}(m'))$ is a Γ-locally free $\mathcal{O}_{\mathcal{R}^\Gamma}$ sheaf of rank $P'(m')$. The group H_1 acts on Q_C^Γ. Let $\pi\colon \tilde{\mathcal{R}^\Gamma} \to \mathcal{R}^\Gamma$ be the associated PH_1-bundle, i.e. the *projective* Γ*-frame bundle* (by (1) above, the conditions required in (3.2) hold good here). From the H-action on $\mathcal{R}^\Gamma$, we see that $\tilde{\mathcal{R}^\Gamma}$ gets an H-action as well.

The projective Γ-frame bundle $\tilde{\mathcal{R}^\Gamma}$ parametrizes a quotient $\mathcal{O}_{\tilde{\mathcal{R}^\Gamma}} \otimes \mathcal{H}' \longrightarrow \pi^*\tilde{\mathcal{F}} \otimes \mathcal{O}_\pi(1)$. So it gives rise to H_1-equivariant morphism $\phi_{\tilde{\mathcal{F}}}\colon \tilde{\mathcal{R}^\Gamma} \longrightarrow Q_C^\Gamma$. We note that $\tilde{\mathcal{R}^\Gamma}$ also carries an H-action on it induced from $\mathcal{R}^\Gamma$. So $\tilde{\mathcal{R}^\Gamma}$ carries an $(H_1 \times H)$–action. So one gets the following diagram

$$\begin{array}{ccc} \tilde{R}^\Gamma & \xrightarrow{\phi_{\mathcal{F}}} & Q_C^\Gamma \\ \Big\downarrow{\scriptstyle \pi} & & \\ R^\Gamma & & \end{array}.$$

We now use the computations involving determinant bundles in 8.2 of [14] and the functoriality of the determinant bundle and note the fact that all the families involved which are defined over the schemes $\mathcal{R}^\Gamma$ and Q_C^Γ, are just the pull-backs of the ones on the usual quot scheme. It therefore follows that the relation obtained in 8.2 of [14] hold verbatim over the projective Γ-frame bundle $\tilde{\mathcal{R}^\Gamma}$ as well.

We note that, since the projective Γ-frame bundle $\tilde{\mathcal{R}^\Gamma}$ is the reduction of structure group of the usual projective frame bundle over $\mathcal{R}$ restricted to $\mathcal{R}^\Gamma$, $\tilde{\mathcal{R}^\Gamma}$ is a closed subscheme of $\tilde{\mathcal{R}}$ over $\mathcal{R}^\Gamma$. Thus, if $\mathcal{M}$ is as in (3.12), we have

$$\phi^*_{\tilde{\mathcal{F}}}(L_C)^{\deg(C)} \simeq \pi^*(\mathcal{M})^{a^2\deg(Y)}.$$

If s is a H_1-invariant section of $L_C^{\nu\deg(C)}$ for some $\nu > 0$, $\phi^*_{\tilde{\mathcal{F}}}(s)$ is a $(H_1 \times H)$-invariant section i.e. an element of $H^0(\tilde{\mathcal{R}^\Gamma}, \phi^*_{\tilde{\mathcal{F}}}(L_C)^{\nu\deg(C)})^{H_1\times H} = H^0(\tilde{\mathcal{R}^\Gamma}, \pi^*(\mathcal{M})^{\nu a^2\deg(Y)})^{H_1\times H}$.

Since $\pi\colon \tilde{\mathcal{R}^\Gamma} \to \mathcal{R}^\Gamma$ is a principal PH_1-bundle, the section $\phi^*_{\tilde{\mathcal{F}}}(s)$ will descend to give an element in $H^0(\mathcal{R}^\Gamma, \mathcal{M}^{\nu a^2\deg(Y)})^H$. In other words, for each $\nu > 0$, we get a linear (injective) map:

$$s_{\mathcal{F}}\colon H^0(Q_C^\Gamma, L_C^{\nu\deg(C)})^{H_1} \to H^0(\mathcal{R}^\Gamma, \mathcal{M}^{\nu a^2\deg(Y)})^H.$$

Now let $\mathcal{F}_q$ be a point in $\mathcal{R}^\Gamma$, i.e. a Γ-semistable torsion free sheaf. By the Orbifold Mehta–Ramanathan restriction theorem (Theorem 7.2) it follows that there exists a curve C as above such that the restriction $\mathcal{F}_q|_C$ is in Q_C^Γ. Hence, by the usual GIT and Seshadri's theorem, there exists a section $s \in H^0(Q_C^\Gamma, L_C^{\nu\deg(C)})^{H_1}$ for some $\nu > 0$ which is *non-zero* at the point $\mathcal{F}_q|_C$.

Following the map $s_{\mathcal{F}}$ we get a section in $H^0(\mathcal{R}^\Gamma, \mathcal{M}^{\nu a^2 \deg(Y)})^H$ which is non-zero at $\mathcal{F}_q$ proving the lemma. q.e.d

We have the following immediate corollary from the first part of Lemma 3.7.

COROLLARY 3.8

There exists an integer $\nu > 0$ such that the line bundle $\mathcal{M}^\nu$ on $\mathcal{R}^\Gamma$ is generated by H-invariant global sections.

4. Donaldson–Uhlenbeck compactification

The aim of this section is to construct a *reduced algebraic scheme* i.e. a *variety*, which is projective and whose points give the analogue of the underlying set of points of the Donaldson–Uhlenbeck compactification for Γ-bundles on a smooth projective algebraic surface with a Γ-action. This, in conjunction with the Kawamata covering lemma and the general (*parabolic bundles*)-(Γ-*bundles*) correspondence would enable us to construct a projective variety whose underlying set of points parametrizes the natural analogue of Donaldson–Uhlenbeck compactification of the moduli space of *μ-stable parabolic bundles* on a surface X with parabolic structure on a divisor with normal crossings. We also describe the *boundary points* of the compactification in terms of Γ-bundles and 0-cycles on the surface Y (and as a consequence on X as well).

Since $\mathcal{R}^\Gamma$ is a quasi-projective scheme and since $\mathcal{M}$ is *H-semi-ample* (Corollary 3.8, i.e. $\mathcal{M}$ is generated by H-invariant sections), there exists a finite dimensional vector space $A \subset A_\nu := H^0(\mathcal{R}^\Gamma, \mathcal{M}^\nu)^H$ that generates $\mathcal{M}^\nu$; of course, there is nothing canonical in the choice of A.

Let morphism $\phi_A \colon \mathcal{R}^\Gamma \to \mathbb{P}(A)$ be the induced H-invariant morphism defined by the sections in A.

But because of non-uniqueness of A a different choice of subspace of invariant sections gives rise to a different map $\phi_{A'}$ to a different projective space $\mathbb{P}(A')$.

DEFINITION 4.1

We denote the by M_A the schematic image $\phi_A(\mathcal{R}^\Gamma)$ with the canonical reduced scheme structure.

Remark 4.2. By the following result which may be titled H-properness, the variety M_A is *proper* and hence because of its quasi-projectivity it is a projective variety. We note that we use the term *variety* in a more general sense of a *reduced algebraic scheme of finite type* which need not be irreducible. So in what follows we will be working with the $\mathbb{C}$-valued points of M_A.

PROPOSITION 4.3

If T is a separated scheme of finite type over k, and if $\phi \colon \mathcal{R}^\Gamma \longrightarrow T$ is an H invariant morphism then image of ϕ is proper over k.

Remark 4.4. This is a consequence of the Langton type semistable reduction theorem for Γ-torsion free sheaves which we have shown in the Appendix and some general schematic methods (cf Proposition 8.2.5 of [14] for details).

Let A_ν denote the vector space $H^0(\mathcal{R}^\Gamma, \mathcal{M}^\nu)^H$, $\nu \in \mathbb{Z}^+$; and let $A \subset A_\nu$ be a finite dimensional vector space which generates $\mathcal{M}^\nu$.

For any $d \geq 1$, let A^d be the image of the canonical multiplication map $f_d \colon A \otimes, \ldots, \otimes A$ (*d-times*) $\to A_{d\nu}$; in particular $A^1 = A$.

Let A' be any finite dimensional vector subspace of A_{dN} containing A^d. Then clearly the line bundle $\mathcal{M}^{d\nu}$ is also globally generated by H-invariant sections coming from the subspace A' and this is so for any $d \geq 0$.

So we have $A \to A^d \subset A'$, and hence a commutative diagram

$$\begin{tikzcd} M_{A'} \arrow[r, "\pi_{A'/A}"] & M_A \\ \mathcal{R}^\Gamma \arrow[u, "\phi_{A'}"] \arrow[ur, "\phi_A"'] & \end{tikzcd} .$$

Since M_A and M'_A are both projective, the map $\pi_{A'/A}$ is a finite map (pull-back of ample remains ample). So if we fix a A as above we get an inverse system (indexed by $d \geq 1$) of projective varieties $(M_{A'}, \pi_{A'/A})$ dominated by the finite type scheme $\mathcal{R}^\Gamma$.

$$\begin{tikzcd} & \mathcal{R}^\Gamma \arrow[dl] \arrow[d, "\phi_{A'}"] \arrow[dr, "\phi_A"] & \\ \cdots \arrow[r] & M_{A'} \arrow[r, "\pi_{A'/A}"'] & M_A \end{tikzcd} .$$

Hence the inverse limit of the system $(M_{A'}, \phi_*)$ is in fact one of the $M_{A'}$'s where A' is a finite dimensional subspace of $H^0(\mathcal{R}^\Gamma, \mathcal{M}^n)^H$ which generates $\mathcal{M}^n$.

DEFINITION 4.5

We denote this *inverse limit variety* by M_Γ and let $\phi \colon \mathcal{R}^\Gamma \to M_\Gamma$ be the canonical morphism induced by the invariant sections coming from the subspace A' associated to the inverse limit.

Remark 4.6. We will show that the moduli space of isomorphism classes (Γ, μ)-stable locally free sheaves of fixed type τ and *fixed determinant* $\mathcal{Q}$ will be a subvariety of M_Γ. This will allow us to take the closure of the moduli space of stable bundle in M_Γ and give it the reduced scheme structure.

Remark 4.7. The underlying set of points of this projective variety, namely the closure in M_Γ, is precisely the Donaldson–Uhlenbeck compactification of the moduli space of Γ-stable bundles. Indeed, in the case when Γ is *trivial* this is the result of Li and Morgan.

Remark 4.8. Note that this is not a categorical quotient since $\mathcal{M}$ is not ample and is only *semi-ample* (Corollary 3.8), i.e. some power of $\mathcal{M}$ is generated by sections.

Remark 4.9. The reduced scheme has a *weak categorical quotient* property for families parametrized by *reduced* schemes.

4.0.1 *Double duals, associated graded* Let F be a μ-semistable Γ-torsion free sheaf over Y. Let $\mathrm{gr}^{\mu}_{\Gamma}(F)$ be the graded torsion free polystable sheaf associated to its Jordon–Holder filtration. Let F^{**} denote the double dual of $\mathrm{gr}^{\mu}_{\Gamma}(F)$; it is a polystable bundle (since Y is a surface, a reflexive sheaf is locally free). Let $l_F\colon Y \to \mathbb{N}$ be the function given by $x \mapsto l(F^{**}/\mathrm{gr}^{\mu}(F))_x$, which associates an element in $S^l_{\Gamma}(Y)$(length l Γ-cycle) with $l = c_2(F) - c_2(F^{**})$. We denote by Z_F the 0-cycle:

$$Z_F := \sum_{x \in Y} l(F^{**}/\mathrm{gr}^{\mu}_{\Gamma}(F))_x \cdot x$$

Both F^{**} and Z_F are well defined, i.e. they do not depend on the choice of filtration.

4.1 *Points of the moduli*

The main aim of this subsection is to describe the points of the moduli space M_{Γ}. Towards this we have the following theorem.

Let $\mathrm{Quot}(E, l)$ denote the Quot scheme which parametrizes all 0-dimensional quotients of E of length l, where E denotes an arbitrary torsion-free sheaf on Y. If E is a Γ-vector bundle on Y the scheme $\mathrm{Quot}(E, l)$ gets a natural Γ-structure and we can again consider the closed subscheme of Γ-fixed points in $\mathrm{Quot}(E, l)$. We denote this closed subscheme by $\mathrm{Quot}^{\Gamma}(E, l)$. Clearly this scheme parametrizes 0-dimensional Γ-quotients of E of length l.

The l-fold symmetric product $S^l(Y)$ parametrizes 0-cycles on Y of length l; again, since Y is a Γ-surface, by taking the fixed point subscheme we get the scheme $S^l_{\Gamma}(Y)$ of zero dimensional Γ-invariant cycles of length l on Y. There is universal sheaf exact sequence on $Y \times \mathrm{Quot}(E, l)$:

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathrm{Quot}} \otimes E \longrightarrow \mathcal{T} \longrightarrow 0\ , \tag{4.1}$$

where $\mathcal{E}$ is a flat family of torsion-free sheaves on Y parametrized by $\mathrm{Quot}(E, l)$. Similarly, we have a Γ-invariant exact sequence on $Y \times \mathrm{Quot}^{\Gamma}(E, l)$ with $\mathcal{E}$ a family of Γ-invariant torsion-free sheaves on Y.

$$\begin{tikzcd} \mathrm{Quot}^{\Gamma}(E, l) \arrow[r, "\text{inclusion}"'] \arrow[d, "\psi_{\Gamma}"'] & \mathrm{Quot}(E, l) \arrow[d, "\psi"'] \\ S^l_{\Gamma}(Y) \arrow[r, "\text{inclusion}"'] & S^l(Y) \end{tikzcd} \ . \tag{4.2}$$

Remark 4.10. If F is Γ-semistable torsion free sheaf we can construct a family $\mathcal{F}$ parametrized by $\mathbb{P}^1$ such that $\mathcal{F}_{\infty} = gr^{\mu}_{\Gamma}(F)$ and $\mathcal{F}_t = F$ for all $t \in \mathbb{P}^1 - \infty$. This means that $\phi(F) = \phi(gr^{\mu}_{\Gamma}(F))$, where $\phi\colon \mathcal{R}^{\Gamma} \to M_{\Gamma}$ is the canonical morphism. Hence we can restrict to polystable case alone. It is easy to see that double dual of any Γ-sheaf gets a canonical Γ-structure.

Remark 4.11. We follow the notations as in (2.2). Consider the closed subvariety $S^l_{\Gamma}(Y)$ of Γ-invariant cycles on Y. Let $Z \in S^l_{\Gamma}(Y)$ and write $Z = \sum m_i y_i$. Then the points $y \in \mathrm{Supp}(Z)$ can be of the following types:

(1) A point $y \in (Y \backslash D')$, where D' is the ramification divisor of the covering map $p\colon Y \to X$.

(2) A general point y contained in an irreducible component $(p^*D)_{\text{red}}$, the isotropy subgroup Γ_y being the cyclic group Γ_n of order n determined by the irreducible component containing y.
(3) A general point y of an irreducible component of the ramification divisor for p not contained in $(p^*D)_{\text{red}}$.
(4) A special point y contained in $(p^*D)_{\text{red}}$, the isotropy subgroup Γ_y of which contains the cyclic group Γ_n of order n determined by the irreducible component containing y.
(5) A special point y of the ramification divisor for p not contained in $(p^*D)_{\text{red}}$.

Consider a torsion-sheaf T supported at $y \in \text{Supp}(Z)$ of length m. Then we can consider the vector space V of its section of dimension $\dim(V) = m$. We view the vector space V endowed with a Γ_y-module structure. For T_{my} to be a quotient of a Γ-bundle E on Y of local type τ, the Γ_y-module structure on V will have constraints imposed on it arising from the Γ_y-module structure on $E|_{U_y}$ which has already been described in (2.4.1).

Let $Z \in S^l_\Gamma(Y)$ and write $Z = \sum m_i y_i$. For each torsion sheaf T_Z with support Z, fixing a Γ-structure is equivalent to fixing a tuple of representations $(\rho(y_i))$ with $\rho(y_i)\colon \Gamma_{y_i} \to GL(V)$. Moreover, for any $\gamma \in \Gamma$, since $\gamma y_i \in \text{Supp}(Z)$, we further need that the representation $\rho(\gamma y_i)$ is the γ-conjugate to $\rho(y_i)$.

Notation 4.12. For a given tuple of representations $\rho(y_i)$ associated to the points in the support of the cycle Z, we attach a label to the Γ-cycle Z and denote it by $Z(\rho(y_i))$. So an equality $Z_{F_1}(\rho(y_i)) = Z_{F_2}(\rho(y_i))$ means that the support of the cycles coincide and the torsion sheaves $T_{Z_1} \simeq T_{Z_2}$ are identified as Γ-torsion sheaves.

Theorem 4.13. *Let F_i, $i = 1, 2$, be two μ-semistable Γ-torsion free sheaves of rank r on Y with fixed Chern classes c_1 and c_2. Then F_1 and F_2 define the same point in $M_\Gamma^{\mu ss}$ if and only if $F_1^{**} \cong_\Gamma F_2^{**}$ and $Z_{F_1}(\rho(y_i)) = Z_{F_2}(\rho(y_i))$.*

Remark 4.14. This theorem is proved after the proofs of Proposition 4.15 and Lemma 4.17.

PROPOSITION 4.15

Let E be a Γ-polystable vector bundle as above. Then the connected components *of the fibres of the morphism ψ_Γ are indexed by the representation tuple $(\rho(y_i))$ as discussed above in Remark* 4.11.

Proof. Consider $Z \in S^l_\Gamma(Y)$ and let T_Z be the torsion sheaf with support Z. Let $y \in \text{Supp}(Z)$ and let its multiplicity in Z be m. We first observe that for any Γ-torsion free sheaf $F \in \psi_\Gamma^{-1}(Z)$ canonically induces a tuple of representations $\rho(y_i)$ for each of the points $y_i \in \text{Supp}(Z)$.

For the given decomposition of Z let us denote a given representation type on the torsion sheaf T by $T(\rho)$. In other words, we fix the representation types on T for each point $y \in \text{Supp}(Z)$.

Consider a Γ-quotient $q\colon E \to T_Z(\rho)$. We first reduce the study of such quotients to a local question.

- Since Z is a Γ-cycle, if $y \in \text{Supp}(Z)$ so does γy for each $\gamma \in \Gamma$. Furthermore, the multiplicities m at y and γy also coincide.
- Giving a Γ-structure on T_Z is therefore giving Γ_y-structure to T_{my} such that at γy, the $\Gamma_{\gamma y}$-structure is conjugate to the one at y.

- Again, since E is a Γ-bundle, for any $y \in \mathrm{Supp}(Z)$, there is a Γ_y-invariant analytic neighbourhood U_y as in (4.11) such that $E|_{U_y}$ is associated to a representation $\Gamma_y \to GL(r)$. Furthermore, at γy for each $\gamma \in \Gamma$, the local representation is conjugate to the one at y by the element γ.
- Giving a Γ-quotient q as above implies giving quotients $q_i \colon E_i \to T_i(\rho(y_i))$, where E_i are bundles restricted to neighbourhoods of the points in the support of $Z = \sum_i m_i y_i$ and $T_i(\rho(y_i)) = T_{m_i y_i}$ with a fixed Γ_{y_i}-module structure on the torsion sheaf $T_{m_i y_i}$. Further, the quotient map at γy_i is conjugate to the one at y.
- Thus, the problem of studying Γ-quotients reduces to the study of Γ_y-quotients in a Γ_y-invariant neighbourhood of y. In other words, such a quotient is a point in the product of *equivariant punctual quot schemes* which we describe below.

We therefore need to handle the various points in the possible singular loci of Γ-torsion free sheaves as listed in (4.11).

For any point $y \in \mathrm{Supp}(Z)$ with multiplicity m, suppose that $\rho(y) \colon \Gamma_y \to GL(V)$ is already fixed with $\dim(V) = m$. Let $V = \oplus_l \ b_l V(l)$ be the isotypical decomposition as a Γ_y-module, with $V(l)$ denoting irreducible Γ_y-modules.

Consider $E|_{U_y}$ where U_y is an analytic neighbourhood of y as in (2.4.1). Since the bundle $E|_{U_y}$ is associated to a representation $\Gamma_y \to GL(r)$, we get an isotypical decomposition $E|_{U_y} \simeq \oplus_l (\mathcal{O}_{U_y}^{a_l} \otimes V(l))$.

Then, giving a Γ_y-quotient $q \colon E|_{U_y} \to T_{my}$ imposes some natural constraints on V, namely, that the $V(l)$'s that occur in V as a Γ_y-module must also occur in $E|_{U_y}$ with obvious bounds on a_l and b_l. With this out of the way, giving q is equivalent to giving quotients

$$q_{a_l,b_l} \colon \mathcal{O}_{U_y}^{a_l} \to T_{b_l y}$$

twisted by $Id|_{V(l)}$, for each $V(l)$ occurring in V.

Since q_{a_l,b_l} is a torsion quotient without any Γ_y-action, the irreducibility of the *punctual* quot scheme $\mathrm{Quot}(\mathcal{O}_{U_y}^{n_l}, m_l)$ is immediate by the results of Li [22], Baranovsky [3] and Ellingsrud–Lehn [10]. Note that we have this since Y is smooth.

The case when Γ_y is trivial, i.e. where y avoids the ramification is easy to handle. In fact, in this case it follows immediately by the old result quoted above. Thus by the above discussion, it follows that the *equivariant punctual quot scheme* is also *irreducible*.

This implies that, fixing the representation type for the torsion sheaf T_Z gives a connected component of the fibre of ψ_Γ.

COROLLARY 4.16

Let F_1 and F_2 be two Γ-polystable torsion-free sheaves obtained as kernels of two maps in $\mathrm{Quot}^\Gamma(E, l)$ *and lying in the same fibre of the map ψ_Γ. If we have a Γ-isomorphism $F_1^{**} \cong_\Gamma F_2^{**}$, then F_1 and F_2 give the same point in the moduli space if and only if they lie in the same component of the fibre of ψ_Γ given by a representation tuple $\rho(y_i)$.*

Proof. The fact that F_i $(i = 1, 2)$ both correspond to points in $\mathrm{Quot}^\Gamma(E, l)$, and the assumption that $F_1^{**} \cong_\Gamma F_2^{**}$ implies that we have

$$E \cong_\Gamma F_1^{**} \cong_\Gamma F_2^{**}$$

with the Γ-structure on E fixed before.

Let F_1 and F_2 be (non-uniquely) represented by two closed points $q_i \in \mathrm{Quot}^{\Gamma}(E, l)$, $i = 1, 2$. We think of F_i themselves as points in $\mathrm{Quot}^{\Gamma}(E, l)$ when there is no confusion.

If F_1 and F_2 are in a component $S(\rho) \subset {\psi_\Gamma}^{-1}(Z)$. The line bundle $\mathcal{L}^N$ is trivial on the fibre ψ_Γ and hence on each component $S(\rho)$ of the fibre of ϕ_Γ (since it is the restriction of the determinant bundle on the fibre of ψ). Hence F_1 and F_2 go to the same point in the moduli space. Conversely, if F_1 and F_2 lie in different components, since the line bundle $\mathcal{L}^N$ is trivial on each component, one can clearly separate the points F_i by sections of $\mathcal{L}^N$. In other words, they go to distinct points of the moduli space. q.e.d

We need to prove the following lemma to complete the proof of the converse in Theorem 4.13.

Lemma 4.17. *Let F_1 and F_2 be two Γ-polystable torsion free sheaves over Y. Let $a \gg 0$ and $C \in |a\Theta|^{\Gamma}$ be a general Γ-curve (which exists by the Γ-Bertini theorem in the Appendix). Then $F_1|_C \simeq_\Gamma F_2|_C$ if and only if $F_1^{**} \simeq_\Gamma F_2^{**}$, where $F_i^{**} = (gr_\Gamma^{\mu}(F_i))^{**}, i = 1, 2$.*

Proof. We choose an integer a so large such that restriction of each summand of F_1^{**} to any general smooth curve $C \in |a\Theta|^{\Gamma}$ is Γ-stable (see Theorem 7.2 below). Now we choose one such C in such a way that it avoids finite set of singular points of $\mathrm{gr}_\Gamma^{\mu}(F_1)$. We note that $\mathrm{gr}_\Gamma^{\mu}(F_1)|_C$ is a polystable bundle over C. Hence

$$(\mathrm{gr}_\Gamma^{\mu} F_1)|_C \cong \mathrm{gr}_\Gamma^{\mu}(F_1|_C) = (\mathrm{gr}_\Gamma^{\mu}(F_1)^{**})|_C = F_1^{**}|_C.$$

The last equality is due to the fact that 'restriction to C' and 'double duals' commute with each other. Now by uniqueness (up to isomorphism) of Jordan–Holder filtration of Γ-semistable bundle we get $(\mathrm{gr}_\Gamma^{\mu}(F_1))|_C \cong_\Gamma F_1^{**}|_C$. This shows that for a general high degree curve $C \in |a\Theta|^{\Gamma}$, the bundles $F_1|_C$ and $F_2|_C$ are S-equivalent if and only if $F_1^{**}|_C \cong_\Gamma F_2^{**}|_C$.

$$0 \to \mathcal{O}_Y(-C) \to \mathcal{O}_Y \to \mathcal{O}_C \to 0\ .$$

Tensoring the above equation with locally free sheaf $\mathcal{H}om(F_1^{**}, F_2^{**})$ one gets the following long exact sequence:

$$\begin{aligned} 0 &\to H_\Gamma^0(Y, \mathcal{H}om_(F_1^{**}, F_2^{**})(-C)) \to H_\Gamma^0(Y, \mathcal{H}om(F_1^{**}, F_2^{**})) \\ &\to H_\Gamma^0(Y, \mathcal{H}om(Y, \mathcal{H}om(F_1^{**}, F_2^{**})|_C) \\ &\to H_\Gamma^1(Y, \mathcal{H}om(F_1^{**}, F_2^{**})(-C)) \to \end{aligned}$$

We now observe that since we work over fields of characteristic zero by Remark 2.17, we have the following inclusions:

$$H_\Gamma^i(Y, E) \subset H^i(Y, E).$$

Using this and the usual Serre duality for sheaves on Y, we have

$$H_\Gamma^1(Y, \mathcal{H}om(F_1^{**}, F_2^{**})(-C)) \subset H^1(Y, ((\mathcal{H}om(F_1^{**}, F_2^{**})^* \otimes K_Y)(C)) = 0$$

and similarly,

$$H_\Gamma^0(Y, \mathcal{H}om(F_1^{**}, F_2^{**})(-C)) \subset H^2(Y, ((\mathcal{H}om(F_1^{**}, F_2^{**})^* \otimes K_Y)(C)) = 0.$$

The vanishing follows by Serre vanishing theorem, since $\mathcal{H}om(F_1^{**}, F_2^{**})$ is locally free and C is a high degree curve.

Hence we have

$$H^0_\Gamma(Y, \mathcal{H}om(F_1^{**}, F_2^{**})) \cong H^0_\Gamma(Y, \mathcal{H}om(Y, \mathcal{H}om(F_1^{**}, F_2^{**})|_C).$$

This implies that $F_1^{**}|_C \cong_\Gamma F_2^{**}|_C$ if and only if $F_1^{**} \cong_\Gamma F_2^{**}$. q.e.d

Completion of the proof of Theorem 4.13. If $F_1^{**} \ncong_\Gamma F_2^{**}$ then two points in R^Γ goes to two different points in M^Γ. Now suppose $F_1^{**} \cong_\Gamma F_2^{**}$, $Z_{F_1}(\rho(y_i)) \neq Z_{F_2}(\rho(y_i))$. By (4.12) this means that either $Z_{F_1} \neq Z_{F_2}$ or that $Z_{F_1} = Z_{F_2} = Z$, but F_i lie in different connected components of the fibre of ψ_Z.

The second case follows from Corollary 4.16. If the cycles themselves are different then we will show that they go to two different points. Observe that we have the following diagram:

$$\begin{tikzcd} S^l_\Gamma(Y) \arrow[r, "a"'] \arrow[d, "b"'] & M_\Gamma \arrow[d, "\phi"'] \\ S^l(Y) \arrow[r, "c"'] & M \end{tikzcd}.$$

By [14] the map c is a *closed immersion*. Since $S^l_\Gamma(Y)$ is a closed subset of $S^l(Y)$, it follows that b is also a closed immersion and hence the composite $c \circ b = \phi \circ a$ is a *closed immersion*. So by our assumption F_1 and F_2 will go to two different points. This completes the proof of the converse of Theorem 4.13. q.e.d

To realise the construction as a compactification we need to have the following proposition.

PROPOSITION 4.18

The moduli space $M^{\mu s}_\Gamma(\mathcal{Q})$ of isomorphism classes of (Γ, μ)-stable locally free sheaves with fixed determinant $\mathcal{Q}$ on Y, is embedded in the moduli space M_Γ.

Proof. This follows by Lemma 4.17 since $F \simeq F^{**}$ for a stable bundle F. The fact that the inclusion is an embedding can be ensured by choosing C to be of larger degree.

q.e.d

Remark 4.19. Let $M^{\mu s}_\Gamma(r, \mathcal{Q}, c_2)$ denote the moduli space of (Γ, μ)-stable bundles of rank r, fixed determinant $\mathcal{Q}$ and second Chern class c_2. The closure of this moduli space in M_Γ gives the desired Donaldson–Uhlenbeck compactification. This can *set theoretically* be described as a stratified space in terms of (Γ, μ)-polystable *bundles* with decreasing c_2 as follows:

$$\overline{M^{\mu s}_\Gamma(r, \mathcal{Q}, c_2)(\tau)} \subset \coprod_{l \geq 0, \rho} M^{\mu-\mathrm{poly}}_\Gamma(r, \mathcal{Q}, c_2 - l)(\tau) \times S^l_\Gamma(Y)(\rho), \tag{4.3}$$

where $M^{\mu\text{-poly}}_\Gamma(r, \mathcal{Q}, c_2)(\tau)$ denotes the subset representing Γ-polystable locally free sheaves of type τ and $S^l_\Gamma(Y)(\rho)$ consists of zero cycles $Z(\rho(y_i))$ as in (4.12).

Notation 4.20. We denote by $M^{\boldsymbol{\alpha}}_{k,\mathbf{l},\mathbf{r}}$ the *moduli space of parabolic stable bundles of rank* r with specified parabolic datum. The tuple $(\boldsymbol{\alpha}, k, \mathbf{l}, \mathbf{r})$ is defined as follows:

- $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_l)$,
- $\mathbf{l} = (\deg(F_1), \deg(F_2), \ldots, \deg(F_l))$,
- $\mathbf{r} = (\mathrm{rank}(F_1/F_2), \mathrm{rank}(F_2/F_3), \ldots, \mathrm{rank}(F_l/F_{l+1})$,
- k stands for the second Chern class of a vector bundle. Here we follow the notation in [17].

Recall the correspondence (2.15) between the *polystable parabolic bundles* on X with given parabolic datum and par $c_2 = \kappa$ and (Γ, μ)-polystable bundles of type τ on a Kawamata cover Y (see (2.4.2) and (2.4.1)). By the description of the above moduli space $\overline{M_{\Gamma}^{\mu s}(r, \mathcal{Q}, c_2)(\tau)}$ we get an intrinsic description the compactification of the moduli space $M_{k,\mathbf{j},\mathbf{r}}^{\boldsymbol{\alpha}}(r, \mathcal{P}, \kappa)$ *set-theoretically* as a stratified space in terms of moduli space of parabolic μ-polystable bundles with fixed determinant $\mathcal{P}$ and with decreasing $\kappa = \mathrm{par}\, c_2$ as follows:

$$\overline{M_{k,\mathbf{j},\mathbf{r}}^{\boldsymbol{\alpha}}(r, \mathcal{P}, \kappa)} \subset \coprod_{l \geq 0} M_{k',\mathbf{j}',\mathbf{r}}^{\boldsymbol{\alpha}-poly}(r, \mathcal{P}, \kappa - l) \times S^l(X). \tag{4.4}$$

where by $M_{k,\mathbf{j},\mathbf{r}}^{\boldsymbol{\alpha}\text{-poly}}(r, \mathcal{P}, \kappa)$, we mean the set of isomorphism classes of *polystable* parabolic bundles with parabolic datum given by $(\boldsymbol{\alpha}, \mathbf{l}, \mathbf{r}, \mathbf{j})$, fixed determinant $\mathcal{P}$ and with topological datum given by k and κ as mentioned above.

5. Existence of Γ-stable bundles

The aim of this section is to prove the existence of Γ-stable bundles of *rank two* with the assumption of *large* c_2 or what is termed *asymptotic non-emptiness*. The bound on c_2 is dependent on the polarisation unlike the result of Taubes and Gieseker. The strategy is to generalise the classical Cayley–Bacharach property for Γ-bundles and prove the non-emptiness along the lines of Schwarzenberger–Serre in the usual surface case.

We remark that, although the moduli space of parabolic sheaves was constructed on any smooth projective variety (but with the Gieseker notion of semistability), to the best of our knowledge, the non-emptiness of these moduli spaces has not been hitherto established. In this paper we do this over a surface. *As before, we make the following assumptions throughout this section*: Y is a smooth projective Γ-surface which arises as a ramified Kawamata cover of the smooth projective surface X. Let $p\colon Y \longrightarrow X := Y/\Gamma$ as before denote the covering morphism.

Let D denote the parabolic divisor and $D = \sum_{i=1}^{c} D_i$ be the decomposition of the divisor D into its irreducible components. Since we will be primarily interested in *rank two bundles*, we have the following weights:

$$0 \leq \alpha_1 < \alpha_2 < 1,$$

where $\alpha_i = m_i/N$ (notations are as in (2.4.1)). We fix as above a very ample divisor Θ_1 on X and let $\Theta = p^*(\Theta_1)$.

Theorem 5.1. *The moduli space $M_{\Gamma}^{\mu s}(2, \mathcal{Q})$ of Γ-stable bundles of rank two and of type τ and fixed determinant $\mathcal{Q}$, on a smooth projective Γ-surface Y is nonempty if $c_2(E) \gg 0$ and if $\alpha_2 < \frac{2\cdot{\Theta_1}^2}{\sum D_i \cdot \Theta_1}$. Hence, the moduli space of parabolic bundles on X of rank two with given quasi-parabolic structure and with* par $c_2(V) \gg 0$ *is non-empty.*

Remark 5.2. The parabolic stable bundle that is shown to exist will depend on the choice of the polarisation Θ_1 on X.

5.1 *Orbifold Cayley–Bacharach property*

Remark 5.3. In this section we make the assumption that Γ-line bundles that we work with are of type τ.

DEFINITION 5.4

Let Y be a smooth projective Γ-surface. Let $p: Y \longrightarrow X$ be a morphism where $X := Y/\Gamma$ arising from the Kawamata covering lemma. Let $\mathfrak{D}_{Y/X} = \mathfrak{D}$ be the ramification locus in X and R be a subset of codimension two consisting of reduced points of length l such that $R \cap \mathfrak{D} = \emptyset$ in Y. Let $Z = p^*(R)$. Then we term the cycle Z in Y a good Γ-cycle.

Remark 5.5. Let $0 \leq \beta < \alpha < 1$. Consider Γ line bundles $L = L^{(\alpha)}$, and $M = M^{(\beta)}$ on Y and let $P = M \otimes L^* \otimes K_Y^{(\alpha-\beta)}$ (see notation in (2.9)).

By tensoring the standard exact sequence for the ideal sheaf $\mathcal{I}_Z$ by P we have $0 \longrightarrow \mathcal{I}_Z \otimes P \longrightarrow P \longrightarrow \mathcal{O}_Z \otimes P \longrightarrow 0$. This induces the following exact sequence of Γ cohomology groups (for generalities on Γ-cohomology see § 5 of [12]):

$$\begin{aligned} 0 &\longrightarrow H^0_\Gamma(P \otimes \mathcal{I}_Z) \longrightarrow H^0_\Gamma(P) \longrightarrow H^0_\Gamma(P \otimes \mathcal{O}_Z) \\ &\longrightarrow H^1_\Gamma(P \otimes \mathcal{I}_Z) \longrightarrow H^1_\Gamma(P) \longrightarrow H^1_\Gamma(P \otimes \mathcal{O}_Z) = 0. \end{aligned} \tag{5.1}$$

Let $\dim H^0_\Gamma(P) = l_1$. Then by choosing a generic 0-cycle $Z = p^*(R)$ as above such that $l(Z) > l_1$ it is easily seen that we make sure $H^1_\Gamma(P \otimes \mathcal{I}_Z) \neq 0$. This implies that there exists at least one Γ-torsion free sheaf E on Y which is a non-split extension of $M \otimes \mathcal{I}_Z$ by L.

DEFINITION 5.6

Let $0 \leq \beta < \alpha < 1$, and let $L = L^{(\alpha)}$ and $M = M^{(\beta)}$ be two Γ line bundles of type τ on Y and Z be a *good* Γ-cycle. We say that the Γ-triple (L, M, Z), satisfies the *Orbifold Cayley–Bacharach* property, (or in short *OCB*) if the following holds: for any section $s \in H^0_\Gamma(M \otimes L^* \otimes K_Y^{(\alpha-\beta)})$ if the restriction of s to a good Γ-cycle $Z' \subset Z$ is zero implies that $s|_Z = 0$, where $Z' \subset Z$ is a *good* Γ-cycle such that $l(Z') = l(Z) - d$, where $d = |\Gamma|$.

Let $Z' \subset Z$ be good Γ-cycles. Consider the exact sequence of ideal sheaves:

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{I}_{Z'} \longrightarrow \mathcal{O}_B \longrightarrow 0.$$

Tensor this exact sequence with M. By applying the $\mathrm{Hom}_\Gamma(-, L)$-functor to $0 \longrightarrow M \otimes \mathcal{I}_Z \longrightarrow M \otimes \mathcal{I}_{Z'} \longrightarrow M \otimes \mathcal{O}_B \longrightarrow 0$ we get a map

$$\psi_{Z'}: \mathrm{Ext}^1_\Gamma(M \otimes \mathcal{I}_{Z'}, L) \longrightarrow \mathrm{Ext}^1_\Gamma(M \otimes \mathcal{I}_Z, L)$$

of Γ-extensions.

Lemma 5.7. *Let* (L, M, Z) *be a* Γ*-triple which satisfies OCB. Then we have*

$$\cup\, \mathrm{Image}(\psi_{Z'}) \neq \mathrm{Ext}^1_\Gamma(M \otimes \mathcal{I}_Z, L)$$

for all good Γ*-cycles* $Z' \subset Z$ *with* $l(Z') = l(Z) - d$.

By tensoring the exact sequence $0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{I}_{Z'} \longrightarrow \mathcal{O}_B \longrightarrow 0$ with $P = M \otimes L^* \otimes K_Y^{(\alpha-\beta)}$ we get the following exact sequence:

$$\begin{aligned} 0 &\longrightarrow H^0_\Gamma(P \otimes \mathcal{I}_Z) \longrightarrow H^0_\Gamma(P \otimes \mathcal{I}_{Z'}) \longrightarrow H^0_\Gamma(P \otimes \mathcal{O}_B) \\ &\longrightarrow H^1_\Gamma(P \otimes \mathcal{I}_Z) \longrightarrow H^1_\Gamma(P \otimes \mathcal{I}_{Z'}) \longrightarrow H^1_\Gamma(P \otimes \mathcal{O}_B) = 0. \end{aligned} \tag{5.2}$$

Here we note that the assumption that the triple (L, M, Z) satisfies OCB implies that $H^0_\Gamma(P \otimes \mathcal{I}_Z) \cong H^0_\Gamma(P \otimes \mathcal{I}_{Z'})$. Therefore by dualizing we have

$$0 \longrightarrow H^1_\Gamma(P \otimes \mathcal{I}_{Z'})^* \longrightarrow H^1_\Gamma(P \otimes \mathcal{I}_Z)^* \longrightarrow V \longrightarrow 0,$$

where V is the complex vector space invariant under Γ which is precisely the dual of the space of sections of the torsion sheaf $H^0_\Gamma(P \otimes \mathcal{O}_B)$. Note that V is independent of $Z' \subset Z$ and depends only on $l(Z')$. This in particular implies that $H^1_\Gamma(P \otimes \mathcal{I}_{Z'})^* \subsetneq H^1_\Gamma(P \otimes \mathcal{I}_Z)^*$.

Since the finite union of *proper* subspaces of finite dimensional vector spaces is not equal to the vector space (we are over an infinite field!) we have $\cup H^1_\Gamma(P \otimes \mathcal{I}_{Z'})^* \neq H^1_\Gamma(P \otimes \mathcal{I}_Z)^*$. The lemma now follows from Lemma 2.23. q.e.d

Lemma 5.8. *Let* (L, M, Z) *be a* Γ*-triple which satisfies OCB. Then for* $l(Z) \gg 0$, *there exists a* Γ*-extension*

$$0 \longrightarrow L \longrightarrow E \longrightarrow M \otimes \mathcal{I}_Z \longrightarrow 0$$

with E locally free.

Proof. Suppose now that E is not locally free. This implies that the set $\mathrm{Sing}(E)$, namely the singular locus of E where E fails to be locally free, is a 0-cycle $A \subset Z$, where A is a Γ-cycle. Let $a \in A$. Then $p^{-1}(p(a)) = \sum \gamma \cdot a = B \subset A$.

Let T_A denote the torsion sheaf supported at $\mathrm{Sing}(E)$. Note that we have an inclusion of torsion sheaves $T_B \subset T_A$. Therefore we get the following commutative diagram of Γ torsion free sheaves on Y.

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] & E^{**} \arrow[r] & T_A \arrow[r] & 0 \\ 0 \arrow[r] & E \arrow[r] \arrow[u, equal] & E' \arrow[r] \arrow[u] & T_B \arrow[r] \arrow[u] & 0 \end{tikzcd}, \tag{5.3}$$

where E' be the corresponding subsheaf of E^{**} to $\mathcal{O}_B$. Note that since L is locally free the saturation of L in E' is L itself.

We therefore obtain an extension E' of $M \otimes \mathcal{I}_{Z'}$ by L using the above commutative diagram where Z' is the Γ cycle corresponding to the *good* cycle $R' \subset R$ induced by the set $A - B$ and $l(Z') = l(Z) - d$ where d is the order of the group Γ. Also we have the following commutative diagram of Γ-sheaves on Y given by two Γ-sheaves E and E'.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & M \otimes \mathcal{I}_Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & E' \arrow[r] \arrow[d] & M \otimes \mathcal{I}_{Z'} \arrow[r] \arrow[d] & 0 \\
 & & T_B \arrow[r, equal] \arrow[d] & T_B \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} . \tag{5.4}
$$

It is clear from the above two diagrams (5.3) and (5.4) that $\psi_{Z'}(E') = E$. By Lemma 5.7 it follows immediately that there exists locally free sheaves which can be realised as extensions as desired. q.e.d

Now we give the construction of rank two Γ-*stable* vector bundles as a extension of $M \otimes \mathcal{I}_Z$ by $\mathcal{O}_Y$ where M is a Γ line bundle on Y.

Remark 5.9. Let L be a Γ-line bundle on Y and let Z be a *good* Γ-cycle. Therefore, $Z = p^*(R)$ for a cycle $R \subset X$ of distinct reduced points away from $\mathfrak{D}$. Under these conditions we observe the following easy fact (see Lemma 2.25):

$$p_*^{\Gamma}(L \otimes \mathcal{I}_Z) \simeq p_*^{\Gamma}(L) \otimes \mathcal{I}_R.$$

As before, we fix a very ample divisor Θ_1 on X and let $\Theta = p^*(\Theta_1)$ (which is therefore an ample divisor on Y). All our degree computations are with respect to these choices.

5.1.1 *Classical Cayley–Bacharach* Let C be a divisor on X with $-2\Theta_1^2 < C \cdot \Theta_1 \leq 0$. Let $Q = 2\Theta_1 - C$. Then we have the following well known result.

Lemma 5.10. *Let $l \geq h^0(X, Q \otimes K_X)$. Then for a generic* 0*-cycle R in $Hilb^{l+1}(X)$ we have the usual Cayley–Bacharach property for the triple $(\mathcal{O}_X, Q, R)$.*

Proof. For the sake of completeness we briefly indicate a proof. We first observe that for generic choice of $T \in \mathrm{Hilb}^l(X)$, $l \geq h^0(X, Q \otimes K_X)$ implies $h^0(X, Q \otimes K_X \otimes \mathcal{I}_T) = 0$. Let $V_l \subset \mathrm{Hilb}^l(X)$ consist of *reduced* 0-cycles and

$$U_l = \{T \in V_l | h^0(X, Q \otimes K_X \otimes \mathcal{I}_T) = 0\}$$

an *open dense subset* of V_l. Let $\mathcal{T}$ be the universal family in $V_{l+1} \times X$, i.e. $\mathcal{T} = \{(T, x) \in V_{l+1} \times X | x \in \mathrm{Supp}(T)\}$ and consider the surjection $f \colon \mathcal{T} \to V_l$, $f(T, x) = T - x$ and the second projection $p \colon \mathcal{T} \to V_{l+1}$. Observe that $p(\mathcal{T} - f^{-1}(U_l)) \subset V_{l+1}$ is a proper closed subset. Choose $R \in V_{l+1} - p(\mathcal{T} - f^{-1}(U_l))$ implying $p^{-1}(R) \subset f^{-1}(U_l)$ i.e. $\forall x \in \mathrm{Supp}(R)$, $(R - x) \in U_l$, hence $h^0(X, Q \otimes K_X \otimes \mathcal{I}_{R-x}) = 0$, $\forall x \in \mathrm{Supp}(R)$. q.e.d

Remark 5.11. In fact, we observe that this choice of l forces something stronger, namely $H^0(Q \otimes K_X \otimes \mathcal{I}_R) = 0$. Moreover, for any $x \in \mathrm{Supp}(R)$ we even have $H^0(Q \otimes K_X \otimes \mathcal{I}_{R-x}) = 0$ *which implies the Cayley–Bacharach property*. So if both these vanishings hold, we term the triple $(\mathcal{O}_X, Q, R)$ to have the stronger Cayley–Bacharach property.

Lemma 5.12. *There exists a* good Γ*-cycle* $Z_1 = p^*(R_1)$ *in* Y *with* $l(R_1) \geq 4\Theta_1^2$ *having the following property*: *if* $\mathcal{L}$ *is any* Γ*-line bundle on* Y *such that* $h^0_\Gamma(\mathcal{L} \otimes \mathcal{I}_{Z_1})) > 0$ *then* $c_1(\mathcal{L}) \cdot \Theta \geq 2\Theta^2$.

Proof. Let C_1 and C_1 be two smooth curves in $|\Theta_1|$ in X. Choose a set S_1 of $2\Theta_1^2$ distinct points in $S_1 \subset (C_1 - C_2)$ away from $\mathfrak{D}$ the ramification divisor in X. Similarly choose a set $S_2 \subset (C_2 - C_1)$.

Let $R_1 = S_1 \cup S_2$ and let $Z_1 = p^*(R_1)$. Suppose that we have $h^0_\Gamma(\mathcal{L} \otimes \mathcal{I}_{Z_1}) > 0$. Then from the above Remark 5.9 we get $h^0(p_*^\Gamma(\mathcal{L}) \otimes \mathcal{I}_{R_1}) > 0$.

Let $p_*^\Gamma(\mathcal{L}) = \mathcal{L}'$. Observe that $\mathcal{L}$ and $\mathcal{L}'$ are both effective. By an abuse of notation, we will continue to denote by $\mathcal{L}$ and $\mathcal{L}'$ divisors in the linear equivalence of the line bundles.

Suppose that the effective divisor $\mathcal{L}'$ contains C_1 and C_2 as its components. Then

$$c_1(\mathcal{L}') \cdot \Theta_1 \geq 2\Theta_1^2.$$

If $\mathcal{L}'$ does not have C_i for some $i = 1, 2$ then we have

$$c_1(\mathcal{L}') \cdot \Theta_1 = \mathcal{L}' \cap C_i \geq l(S_i) = 2\Theta_1^2.$$

Therefore $c_1(\mathcal{L}') \cdot \Theta_1 \geq 2\Theta_1^2$. Now

$$\begin{aligned} c_1(\mathcal{L}) \cdot \Theta &= \deg_Y(\mathcal{L}) \\ &= (\mathrm{pardeg}(p_*^\Gamma(\mathcal{L}))\ |\Gamma| \geq \deg_X(\mathcal{L}')\ |\Gamma| \geq 2{\Theta_1}^2\ |\Gamma| = 2\Theta^2. \end{aligned}$$

q.e.d

Remark 5.13. Let $\mathcal{Q} \in Pic(Y)$ be a Γ-line bundle obtained as follows: Let Q be a line bundle on X and consider $\mathcal{Q} \simeq p^*(Q) \otimes \mathcal{O}_Y^{(\alpha_2)}$, where by $\mathcal{O}_Y^{(\alpha_2)}$ we mean the trivial bundle $\mathcal{O}_Y$ with a Γ-structure of type τ given by multiplication by the character corresponding to α_2 (see (2.4.1) for notation).

Let $0 \leq \alpha_1 < \alpha_2 < 1$. Then we claim that for a suitable choice of Q on X, we can ensure that the triple $(\mathcal{O}_Y^{(\alpha_1)}, \mathcal{Q}, Z)$ satisfies the orbifold Cayley–Bacharach property with respect to the cycle Z. By definition $Z = p^*(R)$. So we simply need to choose Q on X such that the triple $(\mathcal{O}_X, Q, R)$ has the usual Cayley–Bacharach property which we get by (5.10). This will involve the choice of *generic* R with $l(R) \gg 0$ since we need to avoid the ramification locus. We choose R and Q with the bounds given by Remark 5.1.1 which clearly does the job.

5.1.2 *Choice of* $\mathcal{Q}$ *and degree bounds* Let $\gamma = \alpha_2 \cdot \sum \deg_X(D_i)$, where D_i are the irreducible components of the *parabolic divisor*. We let $Q = 2\Theta_1 - C$, with

$$-2\Theta_1^2 + \gamma < C \cdot \Theta_1 \leq 0.$$

This imposes a condition on the weight α_2 *which we therefore have as hypothesis in* Theorem 5.1 (compare with (5.1.1)).

Let $\mathcal{Q} = p^*(Q) \otimes \mathcal{O}_Y^{(\alpha_2)}$ as in (5.13). Hence

$$\frac{c_1(Q) \cdot \Theta_1}{2} < 2\Theta_1^2. \tag{5.5}$$

Let $d = |\Gamma|$. Then we see that by comparing degrees, we have

$$c_1(\mathcal{Q}) \cdot \Theta = \deg_Y(\mathcal{Q}) = (\mathrm{pardeg}(p_*^\Gamma(\mathcal{Q}))\, d = \{\deg_X(Q) + \gamma\}\, d.$$

The non-trivial contribution of γ occurs since $p_*^\Gamma(\mathcal{Q})$ is a parabolic line bundle with underlying line bundle Q but with *non-trivial parabolic structure*.

Again, since $\deg_X(Q) = 2\Theta_1^2 - C \cdot \Theta_1$, by the bounds for $C \cdot \Theta$ fixed above and an easy computation gives

$$\frac{c_1(\mathcal{Q}) \cdot \Theta}{2} < 2\Theta^2. \tag{5.6}$$

Lemma 5.14. *Let* $\mathcal{Q} \in \mathrm{Pic}(Y)$ *a* Γ*-line bundle of type* τ *as in* (5.13) *and* (5.1.2) *with* α_2 *as in Theorem* 5.1. *Then there is a* Γ*-stable rank two vector bundle* E *of type* τ, *with* $\det(E) \cong \mathcal{Q}$ *and* $c_2(E) = c$.

Proof. First we start with a triple $(\mathcal{O}_Y^{(\alpha_1)}, \mathcal{Q}, Z_2)$ which satisfies the orbifold Cayley–Bacharach property. This exist by what we have already seen (by (5.13) and (5.1.1)). We in fact choose a 0-cycle R_2 in X to satisfy the stronger property as in (5.1.1) and (5.11) and let $Z_2 = p^*(R_2)$.

This gives us a Γ locally free extension E' of $Q \otimes \mathcal{I}_{Z_2}$ by $\mathcal{O}_Y$.

Now we choose a *good* Γ-cycle Z_1 as in Lemma 5.12 and let

$$Z = Z_1 \cup Z_2.$$

Then we observe that the triple $(\mathcal{O}_Y, \mathcal{Q}, Z)$ also satisfies a orbifold Cayley–Bacharach property. This can be seen as follows: if $Z = p^*(R)$, then by (5.13), its enough to see that $(\mathcal{O}_X, Q, R)$ has the usual Cayley–Bacharach property. This is immediate, for if $x \in \mathrm{Supp}(R) = \mathrm{Supp}(R_1) \cup \mathrm{Supp}(R_2)$, then its easy to see that $H^0(Q \otimes K_X \otimes \mathcal{I}_{R-x}) = 0$ since we have assumed the stronger Cayley–Bacharach property for R_2 and moreover, $\mathcal{I}_{R-x} \subset \mathcal{I}_{R_2-x}$ or $\mathcal{I}_{R-x} \subset \mathcal{I}_{R_2}$ depending on whether $x \in \mathrm{Supp}(R_2)$ or not.

Therefore we get a new Γ-locally free extension E:

$$0 \longrightarrow \mathcal{O}_Y^{(\alpha_1)} \longrightarrow E \longrightarrow \mathcal{Q}^{(\alpha_2)} \otimes \mathcal{I}_Z \longrightarrow 0.$$

We now *claim* that any such E is Γ-stable.

To see this, consider any Γ-line subbundle L of E. If L is non-trivial, then composing the inclusion $L \hookrightarrow E$ with the map $E \longrightarrow \mathcal{Q} \otimes \mathcal{I}_Z$ we get a nontrivial Γ-map $f : L \longrightarrow \mathcal{Q} \otimes \mathcal{I}_Z$. This gives a non-zero Γ-section

$$s \in H^0_\Gamma(\mathcal{Q} \otimes L^* \otimes \mathcal{I}_Z).$$

In particular, $h^0_\Gamma(\mathcal{Q} \otimes L^* \otimes \mathcal{I}_Z) > 0$ and as a result $h^0_\Gamma(\mathcal{Q} \otimes L^* \otimes \mathcal{I}_{Z_1}) > 0$. Therefore by Lemma 5.12 we conclude that

$$(c_1(\mathcal{Q}) - c_1(L)) \cdot \Theta \geq 2\Theta^2.$$

Hence, $\mu(L) = c_1(L) \cdot \Theta \leq (c_1(\mathcal{Q}) \cdot \Theta - 2\Theta^2)$. But we know that $\mu(E) = \frac{(c_1(\mathcal{Q}) \cdot \Theta}{2}$. By (5.6) we thus have

$$\mu(L) \leq c_1(\mathcal{Q}) \cdot \Theta - 2\Theta^2 < \frac{(c_1(\mathcal{Q}) \cdot \Theta}{2} = \mu(E).$$

Hence E is Γ-stable and clearly of determinant $\mathcal{Q}$.

Regarding the type of the Γ-stable bundle E of rank two constructed above, we observe that we work with a zero cycle Z coming from the complement of ramification divisor. So the action of Γ on Z is a free action. So it does not affect the type of the extension bundle we constructed.

Therefore, since we start with Γ-line bundles of type τ (see (2.4.1)), by giving a type τ structure to $\mathcal{O}_Y^{(\alpha_1)}$ i.e. the trivial bundle $\mathcal{O}_Y$ with the action of generic isotropies along the irreducible components of the divisor by the character α_1 and similarly $\mathcal{Q} = p^*(Q) \otimes \mathcal{O}_Y^{(\alpha_2)}$. Then we get a rank two stable Γ-vector bundle of type τ via the extension:

$$0 \longrightarrow \mathcal{O}_Y^{(\alpha_1)} \longrightarrow E \longrightarrow \mathcal{Q} \otimes \mathcal{I}_Z \longrightarrow 0.$$

q.e.d

6. Some computations of Kronheimer–Mrowka revisited

In the section, for the sake of simplicity, we work with $D \subset X$ an irreducible divisor as the parabolic divisor. The other notations are as in §2.

6.0.3 *Calculation of the second parabolic Chern class*

Lemma 6.1. *Consider a general parabolic bundle* (E_*, F_*, α_*). *Then we can compute the parabolic Chern classes* E_* *using the following formula on* X. *Let us assume that* $\deg(F_i) = l_i$ *with corresponding weights* α_i *and* $r_i = \operatorname{rank}(F_i/F_{i+1})$. *Then*

$$\operatorname{par} c_1(E) = c_1(E) + \left(\sum_{i=1}^{i=l} r_i\alpha_i\right) D$$

and

$$\operatorname{par} c_2(E) = c_2(E) + \sum_{i=1}^{i=l} r_i\alpha_i (c_1(E) \cdot D) - \sum_{i=1}^{i=l} \alpha_i (l_i - l_{i+1})$$

$$+ \frac{1}{2}\left\{\left(\sum_{i=1}^{i=l} r_i\alpha_i\right) \cdot \left(\sum_{j=1}^{j=l} r_j\alpha_j\right) - \left(\sum_{i=1}^{i=l} r_i\alpha_i^2\right)\right\} D^2.$$

Proof. We can assume without loss of generality that E is a parabolic direct sum of line bundles (L_i, α_i). It is easy to see that $F_i/F_{i+1} = \oplus_{j\in J} L_j|D$ with $\alpha_j = \alpha_i$ and $J \subset I$ where $E = \oplus_{i\in I} L_i$. Then $\operatorname{par} c_2(E) = \sum_{i<j} (c_1(L_i) + \alpha_i D)(c_1(L_j) + \alpha_j D)$.

Hence

$$\operatorname{par} c_2(E) = \sum_{i<j} c_1(L_i)c_1(L_j) + \sum_{i\neq j} c_1(L_i)\alpha_j D + \sum_{i<j} \alpha_i\alpha_j D^2.$$

The first term in the above equation is $c_2(E)$. In the above equation α_i is repeated r_i times and $\sum r_i = r$. We write

$$\sum_{i\neq j}^{r} c_1(L_i)\alpha_j D = \sum_{i=1}^{r} \alpha_i \sum_{j=1}^{r} c_1(L_j) D - \sum_{i=1}^{r} \alpha_i c_1(L_i) D$$

$$= \sum_{i=1}^{l} r_i\alpha_i c_1(E) D - \sum_{i=1}^{l} \alpha_i c_1(F_i/F_{i+1}),$$

where l is the length of the filtration. So, we get the required second term of the formula. For the third term we just note that $\sum_{i \neq j} \alpha_i \alpha_j = 2 \sum_{i<j} \alpha_i \alpha_j$ and by usual manipulation we get the above formula. q.e.d

As in [17], we work with a parabolic vector bundle E of rank two on (X, D) where D is an irreducible smooth divisor with $c_1(E) = 0$ and a filtration $0 \subset \mathcal{L} \subset E|_D$ with a single weight α associated with a line subbundle $\mathcal{L}$. When $E = L \oplus L^*$ with $c_1(E) = 0$ and a filtration $0 \subset \mathcal{L} \subset E|_D$ we get par $c_1(E) = 0$ and

$$\text{par}\, c_2(E) = c_2(E) + 2\alpha \cdot l - \alpha^2 D^2,$$

where $(-l)$ is the degree of the line bundle $\mathcal{L}$ and α is a corresponding weight.

6.0.4 *The boundary points and action* Theorem 8.21 of [17] says that there is a one-to-one correspondence between the set of irreducible connections in the moduli space $M^{\alpha}_{k,l}(X, D)$ of α twisted connections, anti-self dual with respect to the cone-like metric determined by ω, with holonomy parameter $\alpha = a/v$; and the set of stable parabolic $SL(2, \mathbb{C})$ bundles $(\mathcal{E}, \mathcal{L}, \alpha)$ on X, with the same weight α, satisfying $c_2(\mathcal{E}) = k$ and $c_1(\mathcal{L}) = -l$.

We consider Proposition 7.1 of [18], which is the parabolic analogue of the Uhlenbeck compactness lemma. This says that if A_n be a sequence of twisted connections in the extended moduli space $\bar{M}_{k,l}$ over (X, D), and suppose that the holonomy parameters α_n for these connections converge to $\alpha \in (0, 1/2)$. Then there exists a sub-sequence, which we continue to call A_n, and gauge transformations $g_n \in \mathcal{G}$ such that the connections $g_n(A_n)$ converge away from a finite set of points $x_i \subset X$, to a connection A. The solution A extends across the finite set and defines a point in a moduli space $M^{\alpha}_{k',l'}$.

In [17] the difference between (k, l) and (k', l') is accounted for by what *bubbles off* at the points where convergence fails. Thus, for each point of concentration x_j in $X \backslash D$ there is an associated positive integer k_j, and for points of concentration x_i in D there is an associated pair (k_i, l_i) so that $k' = k - \Sigma k_i - \Sigma k_j$ and $l' = l - \Sigma l_i$. In [17] it is remarked that there is no complete interpretation or description of the possible values of the pairs (k_i, l_i) in the *bubbling off*. The key observation made in [17] is that the *action* κ is precisely the quantity which is seen to decrease in the *bubbling off*.

We wish to interpret this phenomenon in the light of the semistable reduction theorem (see Theorem 7.3 in Appendix) as well as the description of the points in the boundary of the Donaldson–Uhlenbeck compactification constructed in this paper.

The analogue of the Uhlenbeck compactness lemma in our setting is the interpretation of the Langton extension in terms of the points of the boundary, i.e. the limit point of the family $E_{(A-p)}$ of parabolic Γ stable sheaves on $(\text{Spec}(A) - p)$ of parabolic Chern class par c_2 coming from Langton criterion is identified with a pair (E_p, Z_p) where the parabolic Chern class of E_p is par $c_2 - s$ where s is the length of the zero cycle Z_p. In other words, the phenomenon of *bubbling off* is seen in the decreasing of the second parabolic Chern class which is precisely the expected description seen in the light of Donaldson's theorem in the *non-parabolic* setting. In the case of rank 2 as in [17], what is termed *action* and denoted by κ is precisely the second parabolic Chern class. We may therefore interpret the second parabolic Chern class as the *action* in all ranks as seen from our construction of the Donaldson–Uhlenbeck compactification.

The invariant par c_2 captures all the information about the invariants (k, l) and (k', l') in the notation of [17], and also the relation between them. Indeed, par c_2 can be written in terms of these k's and l's as we have seen above. And since we use Γ bundles on Y, we observe that par c_2 is able to recover the information about these numbers as we have described earlier. The term *action*, denoted by κ in [17] is nothing but our par c_2. Kronheimer and Mrowka define $\kappa_i = k_i + 2\alpha l_i$, as the action lost at the point of concentration $x_i \in D$. They also give the relation between κ and κ' i.e. $\kappa' = \kappa - \Sigma\kappa_i - \Sigma k_j$, where κ' is the par c_2 of the limiting point in our compactification. Here k_j are the instanton numbers associated with the points of concentration away from D.

6.0.5 *Concluding remarks* In the sequel to this work [1] we prove the *asymptotic irreducibility, asymptotic normality and generic smoothness* of the moduli space of stable parabolic bundles. These generalise the work of O'Grady and Gieseker–Li for the usual moduli spaces of stable bundles on algebraic surfaces.

7. Appendix

7.1 *The Mehta–Ramanathan restriction theorem for orbifold bundles*

The aim of this section is to prove the Mehta–Ramanathan restriction theorem for Γ-sheaves. This in particular gives a different proof of the restriction theorem for parabolic bundles (proven in [4]) but for the type of parabolic bundles which arise as invariant direct images of orbifold bundles. We remark that for the purposes of the geometric study of the moduli spaces of parabolic bundles, our results suffice by the *yoga* of variation of parabolic weights.

7.1.1 *Remark on* Γ*-Bertini* One has the following version of Γ-Bertini theorem needed in the restriction theorem. We omit the proof which is a straightforward generalisation of the usual case.

Theorem 7.1 (Γ-Bertini). *Let $X = Y/\Gamma$. Let us assume that X is smooth and Θ is a pull-back of a very ample divisor Θ_1 on X.*

Let the closed embedding $Y \subset \mathbb{P}^n$ be induced by Θ i.e. $\mathbb{P}^n$, the projective space determined by $|\Theta|$. Then there exists a Γ hyperplane $Z \subseteq \mathbb{P}^n$, not containing Y, and such that the scheme $Z \cap Y$ is regular at every point. Furthermore, the set of hyperplanes with this property forms an open dense subset of $|\Theta|^{\Gamma}$.

7.1.2 *The restriction theorem for orbifold bundles* We have the following Γ-Mehta–Ramanathan restriction theorem from which the parabolic version follows easily.

Theorem 7.2 (Γ-Mehta–Ramanathan theorem). *Let E be a (Γ, μ)-semistable (resp stable) Γ-torsion free sheaf on a smooth projective Γ-variety such that $X = Y/\Gamma$ is also smooth and projective. Then the restriction $E|_{C_k}$ to a general complete intersection Γ-curve C_k of large degree (with respect to the pull-back line bundle Θ as in Bertini above) is (Γ, μ)-semistable (resp. stable).*

Proof. Since (Γ, μ)-semistability for Γ-sheaves is equivalent to the semistability of the underlying sheaf, the non-trivial case is that of stability. The proof can be seen in the following steps:

(1) Let E be (Γ, μ)-polystable. Then the underlying bundle E is μ-polystable. In particular, if E is (Γ, μ)-stable the underlying bundle is μ-polystable (not necessarily

stable). For, if we start with a Γ stable bundle E we can construct a Socle F of E with $\mu(F) = \mu(E)$ which is invariant under all the automorphisms of E, in particular invariant under the group Γ. This contradicts the Γ stability of E.

(2) By the effective restriction theorem of Bogomolov (cf. [14]), for *every* complete intersection curve C in the linear system $|m\Theta|$ (the number m effectively determined), the restriction $E|_C$ is polystable.

(3) By the Γ-Bertini theorem, there always exists a Γ-curve in $|m\Theta|$. Thus, the restriction $E|_C$ to any Γ-curve is a Γ-bundle and also μ-polystable. This implies that $E|_C$ is a (Γ, μ)-polystable bundle on C. For, we take Γ-Socle F of $E|_C$ which is again the Socle of $E|_C$. Now this is μ-polystable proving that $E|_C = F$.

(4) Observe that if E is (Γ, μ)-stable then it is Γ-simple. Here we note that we are not saying that it is *simple*. If not, choose a nontrivial Γ endomorphism which induces a nontrivial Γ subbundle of E with $\mu(F) \geq \mu(E)$ contradicting the (Γ, μ)-stability of E.

(5) By the orbifold version of Enriques–Severi it follows that for sufficiently high degree C which is also a Γ-curve, $E|_C$ is also Γ-simple.

(6) Hence, if E is (Γ, μ)-stable, then for high degree Γ-curve C, the restriction is Γ-simple and (Γ, μ)-polystable (by (1), (2), and (3) above), and hence Γ-stable. q.e.d

7.2 *Valuative criterion for semistable orbifold sheaves*

Let S be an algebraic variety over k. We say a Γ-coherent sheaf E on $X \times S$ (S with trivial Γ action) is a *family of torsion-free sheaves* on X over S if, E is flat over S such that for each $s \in S$ the induced sheaf E_* on $p^{-1}(s)$ is a Γ-torsion free sheaf on X. We say two such families E and E' are equivalent if there is Γ invertible sheaf L on S such that $E \cong E' \otimes \mathrm{p}_2^*(L)$.

Our field k is algebraically closed. Let $k \subseteq R$ be a discrete valuation ring with maximal ideal m generated by a uniformizing parameter π . Let K be the field of fractions of R. Consider the scheme $X_R = X \times \operatorname{Spec} R$. Denote by X_K the generic fiber and by X_k the closed fiber of X_R. Let i be the open immersion $X_K \hookrightarrow X_R$ and j be the closed immersion $X_k \hookrightarrow X_R$.

We can now state the main theorem in this section, namely the semistable reduction theorem for (Γ, μ)-semistable torsion-free sheaves.

Theorem 7.3. *Let E_K be a Γ-torsion free sheaf on X_K. Then there exists Γ-torsion free sheaf E_R on X_R such that over X_K we have $i^*E_R \simeq E_K$ and over the closed fibre X_k the restriction $j^*(E_R)$ is (Γ, μ)-semistable.*

Proof. We remark that we need essentially two additional ingredients in the old proof of Langton to complete our argument. The first one is that without the demand of semistability by Proposition 6 in [19] one firstly obtains a *canonical* extension of E_K to a torsion-free sheaf $\tilde{E}$ on X_R. Since the family E_K on X_K is given to be a Γ-sheaf and since the extension is canonical it follows without much difficulty that the extension also carries an extended Γ-action. In other words, the restriction $j^*(\tilde{E})$ to the closed fibre X_k is also a Γ-torsion free sheaf but which could be μ-unstable.

The second step in Langton's proof is to modify the family successively by carrying out elementary modifications using the first term of the Harder–Narasimhan filtration (the so-called β-subbundle) of the restriction $j^*\tilde{E}$. We again observe that the β-subbundle being canonical is also a Γ-sheaf. In other words, the family remains a Γ-family even after the elementary modifications. That the process ends after a *finite* number of steps is one of the key points in Langton's proof and we see that we achieve a (Γ, μ)-semistable reduction in the process. q.e.d

COROLLARY 7.4

If the generic member of the family E_K is given to be of type τ as a family of Γ-sheaves then so is the closed fibre.

Proof. This is easy to see since the type of the family remains constant in continuous families. q.e.d

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