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COHOMOLOGY OF LINE BUNDLES ON SCHUBERT VARIETIES-I

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Abstract. The aim of this paper is to begin a study of the cohomology modules $H^i(X(w), \mathcal{L}_\lambda)$ for *non-dominant weights* λ on Schubert varieties $X(w)$ in G/B. The aim is to set-up a combinatorial dictionary for describing the cohomology modules and give criteria for their vanishing. Here $\mathcal{L}_\lambda$ denotes the line bundle on $X(w)$ corresponding to the 1-dimensional representation of B given by the character λ.

1. Introduction

The following notations will be maintained throughout this paper.

Let k be an algebraically closed field of characteristic zero and G a semi-simple, simply connected algebraic group over k. We fix a maximal torus T of G and let $X(T)$ denote the set of characters of T, $W = N(T)/T$ denote the Weyl group of G with respect to T. Let R denote the set of roots of G with respect to T, $B \supseteq T$ the Borel subgroup of G with respect to the set of negative roots $R^- \subseteq R$ and $S = \{\alpha_1, \ldots, \alpha_l\}$ denote the set of simple roots in R^+. Here l is the rank of G. For $\beta \in R^+$ we also use the notation $\beta > 0$. Since G is simply connected it is well known that $X(T)$ is the same as the weight lattice Λ. The element of the Weyl group corresponding to α_i is denoted by s_{α_i}. The positive definite W-invariant form on the weight lattice induced by the Killing form of the Lie algebra of G is denoted by $(\ , \ )$. We use the notation $\langle \ , \ \rangle$ to denote $\langle \nu \ , \alpha \rangle = \frac{2(\nu,\alpha)}{(\alpha,\alpha)}$. Let $x_\alpha, y_\alpha, \alpha \in R^+, h_{\alpha_i}, \alpha_i \in S$, denote a Chevalley basis of the Lie algebra of G. We denote by Λ^+ the set of dominant weights (also sometimes referred to as positive weights) i.e. the set of weights $\lambda \in \Lambda$, such that $\langle \lambda \ , \alpha \rangle \geq 0$ for all $\alpha \in R^+$. If $\langle \lambda \ , \alpha \rangle > 0$ for all α then we say λ is regular dominant. For $w \in W$ let $l(w)$ denote the length of w. Let $w_0 \in W$ denote the longest element of the Weyl group.

For $w \in W$ let $X(w)$ denote the Schubert variety in G/B corresponding to w.

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The aim of this paper is to begin a comprehensive study of the cohomology modules $H^i(X(w),\mathcal{L}_\lambda)$ for non-dominant weights λ. The aim broadly is to set-up a combinatorial dictionary for describing the cohomology modules and give criteria for their vanishing. Here $\mathcal{L}_\lambda$ denotes the line bundle on $X(w)$ corresponding to the 1-dimensional representation of B given by the character λ.

In the case when λ is dominant this problem has been well studied for a number of years. When λ is *non-dominant*, the cohomology modules were studied by Bott (cf [1]) when $X(w)\simeq G/B$ and we have in particular the celebrated Borel-Weil-Bott theorem. Again in the non-dominant case one can still have the Demazure character formula which in particular will give the Euler characteristic of the line bundle $\mathcal{L}_\lambda$ on $X(w)$. (cf [5, II.14.8]). Our work arose out of an attempt to understand the individual cohomology modules of Schubert varieties for non-ample line bundles and their structures as B-modules and to examine whether there was an analogue of the Borel-Weil-Bott theorem for Schubert varieties. In particular, it was of interest to understand the character of the individual cohomology modules as well.

Recall the Borel-Weil-Bott theorem; if λ is a weight such that $(w\cdot\lambda)\in\Lambda^+$ then we have, $H^i(G/B,\mathcal{L}_\lambda)=0$ for $i\neq l(w)$, and $H^{l(w)}(G/B,\mathcal{L}_\lambda)=H^0(G/B,\mathcal{L}_{w\cdot\lambda})$. (See [1] and [3]). Here $(w\cdot\lambda)$ is the usual dot action of w on weights and is given by $w\cdot\lambda=w(\lambda+\rho)-\rho$. In the case when $\lambda+\rho$ is singular, the theorem states that all cohomology modules vanish.

We have the following results which relate the combinatorics of the Weyl group to the cohomology vanishing of Schubert varieties. We have primarily stuck to the characteristic zero case but we believe that the alcove type generalization in positive characteristics due to Andersen for the Borel-Weil-Bott theorem should carry over to provide some very interesting theory. The standard results on the combinatorics of the Weyl group that are used in this paper can be found in (cf [5, II, 1.5]).

We consistently use the following terminology in this paper. A weight λ is said to be *generic* if for all simple roots α one has $|\langle\lambda\ ,\alpha\rangle|\gg 0$. For a generic weight λ it is well known that there is a unique element $\phi\in W$ such that the weights $\phi(\lambda)$ and $\phi\cdot\lambda$ are dominant weights (cf [4, Th. 10.3 (a)]). We then say that λ is a generic weight in the ϕ–chamber. All the results mentioned below hold for *generic* λ. The precise genericity assumptions for each result is given in the statement of the corresponding theorem.

We have the following (cf Theorem 3.3):

Theorem *Let $X(w)$ be a Schubert variety and λ a generic weight in the ϕ–chamber. Let $R^+(w)=\{\alpha\in R^+\mid w(\alpha)\in R^-\}$.*

(1) $H^0(w,\lambda)\neq 0$ if and only if $R^+(w)\cap R^+(\phi)=\emptyset$.
(2) $H^{l(w)}(w,\lambda)=0$ if and only if $R^+(w)\not\subseteq R^+(\phi)$.

The first part of this theorem is due to P.Polo (cf [10] and also the paper of Dabrowski [2] and Remark 3.4).

We also have the following related results for λ a generic weight in the ϕ–chamber :

(1) $H^j(w,\lambda)=0$ for $j>\min(l(w),l(\phi))$.
(2) $H^j(w,\lambda)=0$ for $j\leq l(w)-l(w_0)+l(\phi)-1$.
(3) (Cohomological characterization of the Bruhat order)
$\phi\leq w$ if and only if $H^{l(\phi)}(w,\lambda)\neq 0$.

We call a Schubert variety $X(w)$ a *Schubert variety of distinct type* if the w is a product of the l simple reflections each occurring exactly once, but appearing in any order.

Let $X(w)$ be a *Schubert variety of distinct type* in G/B, for general G. We fix a reduced expression for $w = s_{i_l}s_{i_{l-1}}\dots s_{i_1}$ and denote by w_j the suffix $s_{i_j}\dots s_{i_1}$ of w. We are interested in studying the cohomology modules $H^i(w,\lambda)$, $0 \le i \le l(w)$, for generic λ.

For $1 \le j \le l$ define $A_j = \{\tau^{-1}(\alpha_{i_j}) \mid \tau \le w_{i_{j-1}}\}$.

We say $\lambda(A_j) > 0$ (respectively, $\lambda(A_j) < 0)$) if $\forall\alpha \in A_j, \langle\lambda\ ,\alpha\rangle > 0$ (respectively, $\forall\alpha \in A_j, \langle\lambda\ ,\alpha\rangle < 0$). If there are roots $\alpha,\beta \in A_j$ such that $\langle\lambda\ ,\alpha\rangle < 0$ and $\langle\lambda\ ,\beta\rangle > 0$ then we write it as $\lambda(A_j) \gtrless 0$. Let $a(w) := a(w,\lambda) = \big|\{i|\lambda(A_i) < 0\}\big|$ and $b(w) := b(w,\lambda) = \big|\{i|\lambda(A_i) \gtrless 0\}\big|$. Then for λ generic we have the following:

Theorem *$H^i(w,\lambda) \neq 0$, for all i, $a(w) \le i \le a(w)+b(w)$. $H^i(w,\lambda) = 0$, for all $i < a(w)$ and all $i > a(w)+b(w)$.*

An interesting feature of the work is the isolation of certain Weyl chambers associated to Schubert varieties different from the dominant and the anti-dominant ones which we term the *diagonal* and *anti-diagonal chambers* (cf. Def 7.1). Their cohomology carries some very natural and precise information some of which we have brought out in this paper (cf. §2). Indeed, as an illustration of the failure of the Borel-Weil-Bott theorem for Schubert varieties we have the following cohomological characterization of G/B in terms of the cohomology of line bundles from the *diagonal* and *anti-diagonal chambers.* For precise genericity conditions see Th 7.2.

Theorem *Let $w \in W$ be such that $s_\alpha \le w$ (in the Bruhat order) $\forall\alpha \in S$.*

(1) *Let λ be a generic weight in the w–chamber. i.e. λ is a generic weight in the* diagonal *chamber with respect to w. Suppose that*

$$\#\{j \mid H^j(X(w),\mathcal{L}_\lambda) \neq 0\} = 1.$$

Then $H^{l(w)}(X(w),\mathcal{L}_\lambda) \neq 0$ and $X(w) = G/B$, i.e $w = w_0$ and λ is negative dominant.

(2) *Let μ be a generic weight in the w_0w–chamber. i.e. μ is a generic weight in the* anti-diagonal *chamber with respect to w. Suppose that*

$$\#\{j \mid H^j(X(w),\mathcal{L}_\mu) \neq 0\} = 1.$$

Then $H^0(X(w),\mathcal{L}_\mu) \neq 0$ and $X(w) = G/B$, i.e $w = w_0$ and μ is dominant.

In the course of the work it was realized that for a uniform theory we will have to assume certain *genericity* assumptions for the non-dominant weights and these in most situations can be made very specific to the problem at hand. The case when the weights are *somewhat special*, situated essentially near the walls of some Weyl chambers, the behaviour can be erratic and seems to involve very complicated combinatorics. We have avoided addressing these issues in this paper.

The broad strategy can be termed as a delicate use of the Bott-Samelson inductive machinery, both in its ascending approach as well as the descending approach. One of the key ingredients that we need is a generalization of a lemma due to D. N. Verma (cf. [8] and §9 of this paper).

The layout of the paper is as follows: In §3 we describe the relationship between the vanishing of cohomology modules and the combinatorics of the Weyl group. In §4 we give upper and lower bounds on j for the vanishing of the cohomology modules $H^j(X(w), \mathcal{L}_\lambda)$. In §5 we state our key lemma and discuss our broad strategy for computing cohomology. In §6 we study explicitly the cohomology modules for Schubert varieties of distinct type. In §7 we prove the main results of this paper which gives a cohomological characterization of G/B. In §8 we state a conjecture on the cohomological non-triviality of certain Schubert cohomology modules and we conclude with some remarks. In the appendix we give a self-contained proof of our key lemma.

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2. Preliminaries

Throughout this paper we are concerned with Schubert varieties in G/B.

We denote by U the unipotent radical of B. We denote by P_α the minimal parabolic subgroup of G containing B and s_α. Let L_α denote the Levi subgroup of P_α containing T. We denote by B_α the intersection of L_α and B. Then L_α is the product of T and a homomorphic image of $SL(2)$ in G (cf. [5, II.1.1.4]). We denote this copy of $SL(2)$ in G by $SL(2, \alpha)$.

Given a $w \in W$ the closure in G/B of the B orbit of the coset wB is the Schubert variety corresponding to w, and is denoted by $X(w)$. We recall some basic facts and results about Schubert varieties. A good reference for all this is the bok by Jantzen. (cf [5, II, Chapter 14]).

Let $w = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_n}$ be a reduced expression for $w \in W$. Define

$$Z(w) = \frac{P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_n}}{B \times \ldots \times B},$$

where the action of $B \times \ldots \times B$ on $P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_n}$ is given by $(p_1, \ldots, p_n)(b_1, \ldots, b_n) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \ldots, b_{n-1}^{-1} \cdot p_n \cdot b_n)$, $p_i \in P_{\alpha_i}$, $b_i \in B$. Note that $Z(w)$ depends on the reduced expression chosen for w. It is well known that $Z(w)$ is a smooth B variety and is a resolution for $X(w)$. We denote by ϕ_w the birational surjective morphism

$$\phi_w : Z(w) \longrightarrow X(w).$$

Let $f_n : Z(w) \longrightarrow Z(ws_{\alpha_n})$ denote the map induced by the projection $P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_n} \longrightarrow P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_{n-1}}$. Then we observe that f_n is a $P_{\alpha_n}/B \simeq \mathbf{P}^1$-fibration.

Let V be a B-module. Let $\mathcal{L}_w(V)$ denote the pull back to $X(w)$ of the homogeneous vector bundle on G/B associated to V. *By abuse of notation* we denote the pull back of $\mathcal{L}_w(V)$ to $Z(w)$ also by $\mathcal{L}_w(V)$, when there is no cause for confusion. Then, for $i \geq 0$, we have the following isomorphisms of B-linearized sheaves

$$R^i f_{n*} \mathcal{L}_w(V) = \mathcal{L}_{ws_{\alpha_n}}(H^i(P_{\alpha_n}/B, \mathcal{L}_w(V))).$$

This together with easy applications of Leray spectral sequences is the constantly used tool in what follows. We term this the *descending 1-step construction.*

We also have the *ascending 1-step construction* which too is used extensively in what follows sometimes in conjunction with the descending construction. We recall this for the convenience of the reader.

Let the notations be as above and write $\tau = s_\alpha w$, with $l(\tau) = l(w) + 1$, for some simple root α. Then we have an induced morphism

$$g_1 : Z(\tau) \longrightarrow P_\alpha/B \simeq \mathbf{P}^1,$$

with fibres given by $Z(w)$. Again, by an application of the Leray spectral sequences together with the fact that the base is a $\mathbf{P}^1$, we obtain for every B-module V the following exact sequence of P_α-modules

$$0 \longrightarrow H^1(P_\alpha/B, R^{i-1}g_{1*}\mathcal{L}_w(V)) \longrightarrow H^i(Z(\tau), \mathcal{L}_\tau(V)) \longrightarrow H^0(P_\alpha/B, R^i g_{1*}\mathcal{L}_w(V)) \longrightarrow 0.$$

We also recall the following well-known isomorphisms:

- $\phi_{w*}\mathcal{O}_{Z(w)} = \mathcal{O}_{X(w)}$.
- $R^q\phi_{w*}\mathcal{O}_{Z(w)} = 0$ for $q > 0$.

This together with [5, II. 14.6] implies that we may use the Bott-Samelson schemes $Z(w)$ for the computation and study of all the cohomology modules $H^i(X(w), \mathcal{L}_w(V))$. Henceforth in this paper we shall use the Bott-Samelson schemes and their cohomology modules in all the computations.

If V is a B-module and $\mathcal{L}_w(V)$ is the induced vector bundle on $Z(w)$ we denote the cohomology modules $H^i(Z(w), \mathcal{L}_w(V))$ by $H^i(w, V)$.

In particular if λ is a character of B we denote the cohomology modules $H^i(Z(w), \mathcal{L}_\lambda)$ by $H^i(w, \lambda)$.

Some constructions from Demazure's paper. We recall briefly two exact sequences that Demazure used in his short proof of the Borel-Weil-Bott theorem [3]. We use the same notation as in Demazure. In the rest of the paper these sequences are referred to as Demazure exact sequences

Let α be a simple root and let $\lambda \in X(T)$ be a weight such that $\langle \lambda , \alpha \rangle \geq 0$. For such a λ, we denote by $V_{\lambda,\alpha}$ the module $H^0(P_\alpha/B, \lambda)$. As B-modules we have the following exact sequences: (to obtain the second sequence we need to assume that $\langle \lambda , \alpha \rangle \geq 2$).

Demazure Exact Sequences 2.1.

$$0 \longrightarrow K \longrightarrow V_{\lambda,\alpha} \longrightarrow L_\lambda \longrightarrow 0.$$
$$0 \longrightarrow L_{s_\alpha(\lambda)} \longrightarrow K \longrightarrow V_{\lambda-\alpha,\alpha} \longrightarrow 0.$$

A consequence of the above exact sequences is the following crucial lemma, a proof of which can be found in [3].

Lemma 2.2.

(1) Let $\tau = ws_\alpha$, $l(\tau) = l(w) + 1$. If $\langle \lambda , \alpha \rangle \geq 0$ then for all j, $H^j(\tau, \lambda) = H^j(w, V_{\lambda,\alpha})$.
(2) Let $\tau = ws_\alpha$, $l(\tau) = l(w) + 1$. If $\langle \lambda , \alpha \rangle \geq 0$ then $H^i(\tau, \lambda) = H^{i+1}(\tau, s_\alpha \cdot \lambda)$ and if $\langle \lambda , \alpha \rangle \leq -2$ then $H^i(\tau, \lambda) = H^{i-1}(\tau, s_\alpha \cdot \lambda)$.
(3) If $\langle \lambda , \alpha \rangle = -1$ then $H^i(\tau, \lambda)$ vanishes for all i (cf. Prop 5.2(b) [5]).

Definition 2.3. A finite dimensional B-module V is said to be dual-cyclic if its dual V^* is cyclic as a B-module, i.e, there exists a linear form $f \in V^*$, which generates V^* as a B-module.

3. Cohomology vanishing - combinatorial conditions based on roots

In this section we state combinatorial conditions on when the cohomology modules $H^i(w, \lambda)$ vanish, for a generic λ belonging to the ϕ–chamber. These combinatorial conditions are based on how the elements w and ϕ act on roots. We begin with two lemmas which describe the weights occurring in $H^{l(w)}(w, \lambda)$ for $w \in W$ and λ a weight such that $w_0(\lambda + \rho)$ is regular dominant. Note that for such a w and λ it follows from the Borel-Weil-Bott theorem that $H^{l(w)}(w, \lambda)$ is non-zero.

Lemma 3.1. *Let λ be a weight such that $w_0(\lambda + \rho)$ is regular dominant. Then for all w, every weight in $H^{l(w)}(w, \lambda)$ is in the convex hull of the weights $\tau \cdot \lambda$, $\tau \leq w$.*

Proof. We prove this by ascending induction on $l(w)$. The base case when $w = 1$ is trivial. Let $l(w) = r > 0$. Let $w = s_\alpha \tau$ with $l(w) = l(\tau) + 1$. We may assume by induction that every weight in $H^{l(\tau)}(\tau, \lambda)$ is of the form $\Sigma_{\tau' \leq \tau} c_{\tau'} \tau' \cdot \lambda$ with $c_{\tau'} \geq 0$ and $\Sigma c_{\tau'} = 1$. Since $H^{l(w)}(w, \lambda) = H^1(s_\alpha, H^{l(\tau)}(\tau, \lambda))$ it is clear that every weight ν of $H^{l(w)}(w, \lambda)$ is of the form $\nu = a\mu + (1 - a)s_\alpha \cdot \mu$, for some weight μ of $H^{l(\tau)}(\tau, \lambda)$ with $0 \leq a \leq 1$. Observe that $s_\alpha \cdot \Sigma_{\tau' \leq \tau} c_{\tau'} \tau' \cdot \lambda = \Sigma c_\tau s_\alpha \cdot \tau' \cdot \lambda = \Sigma c_\tau s_\alpha \tau' \cdot \lambda$. Since $\tau' \leq \tau$ implies $s_\alpha \tau' \leq w$ the lemma follows.

Lemma 3.2. *If λ is a weight such that $w_0(\lambda + \rho)$ is regular dominant then for any $w \in W$, the highest weight of $H^{l(w)}(w, \lambda)$ is $w \cdot \lambda$.*

Proof. We prove this by ascending induction on $l(w)$. The base case when $w = 1$ is trivial. Let $l(w) = r > 0$. Let $w = s_\alpha \tau$ with $l(w) = l(\tau) + 1$. We may assume by induction that the highest weight of $H^{l(\tau)}(\tau, \lambda)$ is $\tau \cdot \lambda$. Since $\langle \tau \cdot \lambda \ , \alpha \rangle \leq -2$, it follows that $s_\alpha \cdot \tau \cdot \lambda = w \cdot \lambda$ is a weight of $H^1(s_\alpha, H^{l(\tau)}(\tau, \lambda)) = H^{l(w)}(w, \lambda)$. From Lemma 3.1 any weight μ of $H^{l(w)}(w, \lambda)$ is of the form $\mu = \Sigma_{\tau' \leq w} c_{\tau'} \tau' \cdot \lambda$. Since λ is negative dominant if follows that $\tau' \cdot \lambda \leq w \cdot \lambda$ for $\tau' \leq w$. This completes the proof.

To state our combinatorial conditions we set up some notation. Given w we denote by $R^+(w)$ the set of positive roots sent to negative by w i.e. $R^+(w) = \{\alpha \in R^+ | w(\alpha) \in R^-\}$. Then we have

Theorem 3.3.

(i) Let λ be a weight such that $\phi(\lambda)$ is regular dominant. Then $H^0(w, \lambda) \neq 0$ iff $R^+(w) \cap R^+(\phi) = \emptyset$. Furthermore when $H^0(w, \lambda) \neq 0$ then it is a dual-cylic module with lowest weight $w(\lambda)$.

(ii) Let λ be a weight such that $\phi(\lambda + \rho)$ is regular dominant. Then $H^{l(w)}(w, \lambda) \neq 0$ iff $R^+(w) \subseteq R^+(\phi)$. Furthermore when $H^{l(w)}(w, \lambda) \neq 0$ it is a cyclic module with highest weight $w \cdot \lambda$.

Proof. Proof of (i). The first case of this theorem can also be found in P.Polo (cf [10, Cor. 2.3]) and Dabrowski (cf [2]).

The weight computation is also not hard to check.

Proof of (ii). The same technique of proof can be be used to give a proof of (i) different from that in [10], [2].

Let w, ϕ be a pair of elements in the Weyl group. Let α be a simple root such that $\phi(\alpha)$ and $w(\alpha)$ are both positive or both are negative. For such an α it can be easily checked that $R^+(w) \subseteq R^+(\phi)$ iff $R^+(ws_\alpha) \subseteq R^+(\phi s_\alpha)$.

The proof proceeds by ascending induction on w. The base case when w is 1 is clear.

Let us assume that $R^+(w) \subseteq R^+(\phi)$ and assume the statement has been proved for all Schubert varieties of dimension strictly less that of $X(w)$. Let λ be a weight such that $\phi \cdot \lambda$ is dominant. For a simple root α such that $w(\alpha) < 0$ consider the long exact sequence of cohomology modules

$$\cdots \longrightarrow H^{l(w)-1}(ws_\alpha, V_{s_\alpha \cdot \lambda, \alpha}) \longrightarrow H^{l(w)-1}(ws_\alpha, s_\alpha \cdot \lambda) \longrightarrow 0.$$

Now ws_α has length one less than w and $\phi s_\alpha(s_\alpha \cdot \lambda + \rho)$ is dominant. From the hypothesis and the discussion above it follows that $R^+(ws_\alpha) \subseteq R^+(\phi_\alpha)$. So by induction $H^{l(w)-1}(ws_\alpha, s_\alpha \cdot \lambda)$ is non zero. Hence so is $H^{l(w)-1}(ws_\alpha, V_{s_\alpha \cdot \lambda, \alpha}) = H^{l(w)}(w, \lambda)$.

Interchanging the roles of the pairs (w, ϕ) and $(ws_\alpha, \phi s_\alpha)$ and using a sequence similar to the above, we can prove the converse by descending induction.

To conclude that $H^{l(w)}(w, \lambda)$ is a cyclic module with highest weight $w \cdot \lambda$ we proceed as follows.

We first show that if λ is a weight such $w_0(\lambda+\rho)$ is regular dominant then $H^{l(w)}(w, \lambda)$ is a cyclic module with highest weight $w \cdot \lambda$. That the highest weight is $w \cdot \lambda$ follows from Lemma 3.2. We need to show cyclicity.

Let K_w denote the dualizing class of $X(w)$. Then $K_w = -\rho - \partial X(w)$, where $\partial X(w)$ denotes the boundary of $X(w)$. Then by Serre duality $H^{l(w)}(w, \lambda)^* \simeq H^0(X(w), -\lambda - \rho - \partial X(w))$. Note that this isomorphism is a U–equivariant map. Consider the short exact sequence of U–sheaves

$$0 \longrightarrow (-\lambda - \rho - \partial X(w)) \longrightarrow (-\lambda - \rho) \longrightarrow (-\lambda - \rho)|_{\partial X(w)} \longrightarrow 0.$$

From this we get a U–equivariant inclusion

$$0 \longrightarrow H^0(w, -\lambda - \rho - \partial X(w)) \longrightarrow H^0(w, -\lambda - \rho) \longrightarrow \cdots$$

Since $w_0(\lambda + \rho)$ is regular dominant it follows that $-\lambda - \rho$ is regular dominant. So the dual of the second module $H^0(w, -\lambda - \rho)$ is a Demazure module, which is cyclic. Hence $H^0(w, -\lambda - \rho - \partial X(w))^*$, being a quotient of a cyclic module is also cyclic. This proves that $H^{l(w)}(w, \lambda)$ is cyclic in this case.

To complete the proof we need to show that for w, ϕ, λ satisfying the hypothesis of the theorem, $H^{l(w)}(w, \lambda)$ is a cyclic module with highest weight $w \cdot \lambda$. We show this by induction on $l(w_0) - l(\phi)$. The base case when $\phi = w_0$ follows from the discussion above.

If ϕ is different from w_0 there is a simple root α such that $\phi(\alpha) > 0$. Since $R^+(w) \subseteq R^+(\phi)$ it follows that $w(\alpha) > 0$. Hence $l(ws_\alpha) = l(w) + 1$ and $l(\phi s_\alpha) = l(\phi) + 1$. Furthermore we note that $R^+(ws_\alpha) \subseteq R^+(\tau s_\alpha)$.

From the short exact sequence

$$0 \longrightarrow K \longrightarrow V_{\lambda, \alpha} \longrightarrow \lambda \longrightarrow 0,$$

we get

$$\longrightarrow H^{l(w)}(w, V_{\lambda,\alpha}) \longrightarrow H^{l(w)}(w,\lambda) \longrightarrow 0.$$

Now $H^{l(w)}(w, V_{\lambda,\alpha}) = H^{l(w)}(ws_\alpha, \lambda) = H^{l(w)+1}(ws_\alpha, s_\alpha \cdot \lambda)$. Now $\phi s_\alpha((s_\alpha \cdot \lambda) + \rho)$ is also regular dominant. So by induction, we may assume that $H^{l(w)+1}(ws_\alpha, s_\alpha \cdot \lambda)$ is a cyclic module with highest $ws_\alpha \cdot s_\alpha \cdot \lambda = w \cdot \lambda$. Since $H^{l(w)}(w, \lambda)$ is a quotient of this module it follows that $H^{l(w)}(w, \lambda)$ is also a cyclic module with highest weight $w \cdot \lambda$.

Remark 3.4. We thank the referee for pointing out that the result (ii) in Th 3.3 can be deduced from (i) using Serre duality. From the hypothesis in (ii), using (i) and Serre duality one can conclude that the top cohomology is non zero. However, Serre duality need not be a B-module homomorphism. In fact when $s_\alpha \leq w$ for all simple roots α, then the isomorphism between $H^0(w,\lambda)^*$ and $H^{l(w)}(w, -\lambda + K_w)$ (K_w being the dualizing class of $X(w)$) given by Serre duality is a B-module isomorphism only in the case when $w = w_0$. In all other cases we need to twist by a character of B. For example, when $w = w_0(min, P_I)$, (where $I \subseteq S$ and P_I denotes the parabolic containing B and $s_\alpha, \alpha \in I$) then $H^{l(w)}(w, K_w)$ is a one dimensional B-module given by the character $\Sigma_{\alpha \in i(I)} \omega_\alpha$, where i denotes the Weyl involution. In this case the dualizing class happens to be a line bundle given by a character of B. However, one knows that in general the dualizing class need not be a line bundle.

So, using Serre duality alone, it is not clear how one can conclude that the highest weight of $H^{l(w)}(w, \lambda)$ is $w \cdot \lambda$ in the statement of (ii) in Th 3.3.

Corollary 3.1. *Let $w = \tau\phi$ with $l(w) = l(\tau) + l(\phi)$.*

(1) Let μ be a weight such that $w_o w(\mu)$ is regular dominant. Then $H^0(\phi, \mu) \neq 0$ and it is a dual-cylic B-module with lowest weight $\phi(\mu)$.
(2) Let λ be a weight such that $w(\lambda + \rho)$ is regular dominant. Then $H^{l(\phi)}(\phi, \lambda)$ is non zero and is a cyclic module with highest weight $\phi \cdot \lambda$.

Proof. The first statement follows from Th 3.3(i). The second follows from Th 3.3(ii).

4. Upper and lower bounds on cohomology vanishing

In this section we give general bounds for vanishing of cohomology. For completeness we state a result from Joseph(cf [6, Th. 5.7] (see also [10, Prop. 1.4.2]). An alternative proof of this result can be given along the lines of the proof of Th 4.2.

Theorem 4.1. *[Joseph] Let $X(w)$ be a Schubert variety and λ a generic weight in the ϕ–chamber. Then $H^j(w, \lambda) = 0$ for $j > l(\phi)$. In particular, since cohomology vanishes beyond dimension, we have $H^j(w, \lambda) = 0$ for $j > \min(l(w), l(\phi))$.*

Theorem 4.2. *Let $X(w)$ be a Schubert variety and let $\phi \in W$. If λ is a generic weight such that $\phi(\lambda)$ is* negative-dominant*, then $H^j(w, \lambda) = 0$ for $j \leq l(w) - l(\phi) - 1$. Equivalently, given λ and ϕ such that $\phi(\lambda)$ is* dominant *then $H^j(w, \lambda) = 0$ for $j \leq l(w) - l(w_0) + l(\phi) - 1$.*

Remark 4.3. The genericity assumption on λ here is: $\langle \phi \cdot \lambda, \gamma \rangle > M * l(w_o \phi)$, where M denotes the maximum of $\{\langle \beta, \gamma \rangle | \beta \in R^+, \gamma \in S\}$.

Proof. We prove this by double (descending) induction on $l(w)$ and $l(\phi)$.

The base case is when $w = w_0$. Since $w_0\phi$ takes λ to the dominant chamber it follows from the Borel-Weil-Bott theorem that $H^j(w_0, \lambda)$ is non-zero except when $j = l(w_0) - l(\phi)$, completing the base case of the induction.

Given w of length less than w_0, let us assume that the theorem is true for all τ of length larger than w and all chambers. We now prove it for w by descending induction on the chamber length $l(\phi)$. When $\phi = w_0$, we have $l(w) - l(\phi) - 1 \leq -1$, and the statement is trivially true in this case. Let $l(\phi)$ be less than $l(w_0)$ and λ be in the ϕ–chamber. We need to show that $H^j(w, \lambda) = 0$, for $j \leq l(w) - l(\phi) - 1$.

Suppose there exists a simple root α such that $\phi(\alpha) \in R^+$ and $w(\alpha) \in R^-$. For such a root we have $\langle \lambda \ , \alpha \rangle = \langle \phi(\lambda) \ , \phi(\alpha) \rangle < 0$. From Lemma 2.2 $H^i(w, \lambda) = H^{i-1}(w, s_\alpha \cdot \lambda)$. Moreover the element of the Weyl group which moves $s_\alpha \cdot \lambda$ to the negative dominant chamber is ϕs_α, and this is of length one more than ϕ (since $\phi(\alpha) \in R^+$). By the inductive hypothesis on chamber length for $X(w)$, $H^{i-1}(w, s_\alpha \cdot \lambda)$ is zero whenever $i - 1 \leq l(w) - l(\phi) - 1 - 1$. So the theorem holds in this case.

Since $\phi \neq w_0$, there exists a simple root α such that $\phi(\alpha) > 0$. From the discussion above it remains to prove the theorem in the case that $w(\alpha) > 0$. If $\tau = ws_\alpha$ then τ has length one more than w. Since $\phi(\alpha) \in R^+$ it is the case that $\langle \lambda \ , \alpha \rangle < 0$. Let $\mu = s_\alpha \cdot (\lambda - \alpha) = s_\alpha(\lambda)$. Clearly $\langle \mu \ , \alpha \rangle \gg 0$.

Consider the two Demazure exact sequences 2.1 of B–modules

$$0 \longrightarrow K \longrightarrow V_{\mu,\alpha} \longrightarrow L_\mu \longrightarrow 0,$$

$$0 \longrightarrow L_{s_\alpha(\mu)} \longrightarrow K \longrightarrow V_{\mu-\alpha,\alpha} \longrightarrow 0.$$

From the first exact sequence we get a long exact sequence

$$\cdots \longrightarrow H^{i-1}(w, V_{\mu,\alpha}) \longrightarrow H^{i-1}(w, \mu) \longrightarrow H^i(w, K) \longrightarrow H^i(w, V_{\mu,\alpha}) \longrightarrow \cdots .$$

Because $\langle \mu \ , \alpha \rangle \geq 0$, $H^i(w, V_{\mu,\alpha}) = H^i(\tau, \mu), \forall i$. Since τ has length one more than w we may assume by the inductive hypothesis applied to $X(\tau)$ that $H^i(\tau, \mu)$ vanishes for $i \leq l(w) + 1 - l(\phi) - 1 - 1$. Hence we get $H^{i-1}(w, \mu) \simeq H^i(w, K), \forall i \leq l(w) - l(\phi) - 1$.

Since $s_\alpha(\mu) = \lambda$, the second Demazure exact sequence above gives us a long exact sequence

$$\cdots \longrightarrow H^{i-1}(w, V_{\mu-\alpha}) \longrightarrow H^i(w, \lambda) \longrightarrow H^i(w, K) \longrightarrow H^i(w, V_{\mu-\alpha}) \longrightarrow \cdots .$$

Now $H^j(w, V_{\mu-\alpha}) = H^j(\tau, \mu - \alpha), \forall j$. Furthermore, since τ ends with s_α, from Lemma 2.2, it follows that $H^j(\tau, \mu - \alpha) = H^{j+1}(\tau, \lambda)$, since $s_\alpha \cdot (\mu - \alpha) = \lambda$. By the inductive hypothesis applied to $X(\tau)$, the extreme two terms of the above long exact sequence are zero when $i + 1 \leq l(w) + 1 + l(\phi) - 1$. So when $i \leq l(w) - l(\phi) - 1$, we get $H^i(w, \lambda) \equiv H^i(w, K)$. Combining this with the isomorphism obtained earlier we get $H^{i-1}(w, \mu) \simeq H^i(w, \lambda), \forall i \leq l(w) - l(\phi) - 1$. By the inductive hypothesis on the chamber length for $X(w)$, the left module vanishes for $i - 1 \leq l(w) - l(\phi) - 1 - 1$ and so, for $i \leq l(w) - l(\phi) - 1, H^i(w, \lambda)$ vanishes.

Corollary 4.1. *(Analogue of the Kodaira vanishing theorem for Schubert varieties) If λ is a generic weight which is negative dominant then*

$$H^i(w, \lambda) = \begin{cases} 0 \text{ if } i < l(w). \\ \text{non-zero if } i = l(w). \end{cases}$$

Remark 4.4. Over fields of positive characteristics, this of course, is a well-known important consequence of a theorem of Mehta-Ramanathan on the Frobenius splitting property of the Schubert varieties in G/B. In the characteristic zero case though, we can obtain it as an easy consequence of our result on bounds, but only for generic λ.

We now prove a statement similar in spirit to that in Cor 3.1. So it would seem appropriate that this theorem belongs to the previous section. However its proof uses Th 4.2 which is why it is here.

Theorem 4.5. *(A cohomological characterization of the Bruhat order)*
Let $\phi, w \in W$. If λ is a generic weight in the ϕ–chamber, then

$$\phi \leq w \Longleftrightarrow H^{l(\phi)}(w, \lambda) \neq 0.$$

Proof. Assume $\phi \leq w$. Let $w_0 = \phi\phi_0$ be a reduced expression for w_0. Then since $X(\phi) \subseteq X(w) \subseteq G/B$, we have the following commutative diagram (we denote the pull back of the line bundle L_λ on $X(w)$ to G/B also by L_λ).

$$\begin{array}{ccc} H^{l(w_0)}(G/B, \phi_0^{-1} \cdot \lambda) & \longrightarrow & H^{l(\phi)}(\phi, \lambda) \longrightarrow 0 \\ \| & & \uparrow \\ H^{l(\phi)}(G/B, \lambda) & \longrightarrow & H^{l(\phi)}(w, \lambda) \end{array}$$

Here the map from $H^{l(w_0)}(G/B, \phi_0^{-1} \cdot \lambda)$ to $H^{l(\phi)}(\phi, \lambda)$ is a surjection and is constructed as in the the proof of Th 3.3(ii). That $H^{l(\phi)}(G/B, \lambda)$ is the same as $H^{l(w_0)}(G/B, \phi_0^{-1} \cdot \lambda)$ follows from the Borel-Weil-Bott theorem. The map from $H^{l(\phi)}(G/B, \lambda)$ to $H^{l(\phi)}(w, \lambda)$ is the restriction map. We show that the diagram is commutative by showing that the map from $H^{l(w_0)}(G/B, \phi_0^{-1} \cdot \lambda)$ to $H^{l(\phi)}(\phi, \lambda)$ is the same as the restriction map from $H^{l(\phi)}(G/B, \lambda)$ to $H^{l(\phi)}(w, \lambda)$.

We do this by induction on $l(\phi_0)$ the case when $\phi_0 = 1$ (and so $\phi = w_0$) being trivially true. Now assume $\phi \neq w_0$ and let α be a simple root such that $\phi(\alpha) > 0$. Then ϕs_α has length one more than ϕ. Since $\langle \lambda , \alpha \rangle > 0$ we get a surjective map from $H^{l(\phi)}(\phi, V_{\lambda,\alpha})$ to $H^{l(\phi)}(\phi, \lambda)$. Now $H^{l(\phi)}(\phi, V_{\lambda,\alpha}) = H^{l(\phi)}(\phi s_\alpha, \lambda) = H^{l(\phi)+1}(\phi s_\alpha, s_\alpha \cdot \lambda)$. So we have a surjective map from $H^{l(\phi)+1}(\phi s_\alpha, s_\alpha \cdot \lambda)$ to $H^{l(\phi)}(\phi, \lambda)$. Continuing this process is what gives us the surjective map $H^{l(w_0)}(G/B, \phi_0^{-1} \cdot \lambda)$ to $H^{l(\phi)}(\phi, \lambda)$. Now by induction the map from $H^{l(w_0)}(G/B, (\phi_0^{-1} s_\alpha) \cdot s_\alpha \cdot \lambda) = H^{l(w)}(G/B, \phi_0^{-1} \cdot \lambda)$ to $H^{l(\phi)+1}(\phi s_\alpha, s_\alpha \cdot \lambda)$ is the same as the restriction map from $H^{l(\phi)+1}(G/B, s_\alpha \cdot \lambda)$ to $H^{l(\phi)+1}(\phi s_\alpha, s_\alpha \cdot \lambda)$. Since $H^{l(\phi)+1}(G/B, s_\alpha \cdot \lambda) = H^{l(\phi)}(G/B, s_\alpha \cdot s_\alpha \cdot \lambda) = H^{l(\phi)}(G/B, \lambda)$ we are done.

Since the diagram is commutative it follows that the map from $H^{l(\phi)}(w, \lambda)$ to $H^{l(\phi)}(\phi, \lambda)$ is also surjective. Since $H^{l(\phi)}(\phi, \lambda)$ is non zero from Cor 3.1 we conclude that $H^{l(\phi)}(w, \lambda)$ is also non zero.

To prove the converse we assume that $\phi \not\leq w$ and show that in this case $H^{l(\phi)}(w, \lambda) = 0$.

We prove the statement by double induction on $l(w)$ and $l(\phi)$, in fact by a descending induction on $l(w)$ and $l(\phi)$.

When $w = w_0$, then the statement is vacuously true for every ϕ, since $\phi \leq w_0, \forall \phi \in W$. This proves the base case of the induction statement.

Let $w \neq w_0$ be an element of the Weyl group. We prove the statement for w by descending induction on $l(\phi)$.

The base case is when $\phi = w_0$. Since $w \neq w_0$ the statement is true in this case, since cohomology vanishes beyond the dimension of the variety. Let $\phi \neq w_0$ be an element of the Weyl group such that $\phi \not\leq w$ and let λ be a generic weight in the ϕ–chamber. We show that $H^{l(\phi)}(w, \lambda) = 0$.

Since $\phi \neq w_0$ there exists a simple root $\alpha \in R^+$ such that $\phi(\alpha) \in R^+$. If, for this α, $w(\alpha) \in R^-$ then from Lemma 2.2 we have $H^{l(\phi)}(w, \lambda) = H^{l(\phi)+1}(w, s_\alpha \cdot \lambda)$. Now, by our choice of λ, the weight $s_\alpha \cdot \lambda$ is a generic weight in the ϕs_α– chamber since $\phi s_\alpha(s_\alpha \cdot \lambda) = \phi(\lambda) + \phi(\alpha)$, is dominant. Now ϕs_α has length $l(\phi)+1$. Furthermore, since $\phi \not\leq w$ and $w(\alpha) \in R^-$ but $\phi(\alpha) \in R^+$, we also have $\phi s_\alpha \not\leq w$. So by induction on the chamber length applied to $X(w)$ we have $H^{l(\phi)+1}(w, s_\alpha \cdot \lambda) = 0$. This completes the proof in this case. We next consider the case when $w(\alpha) \in R^+$. Let $\tau = w s_\alpha$. Then $l(\tau) = l(w) + 1$. Consider the two Demazure exact sequences of B–modules

$$0 \longrightarrow K \longrightarrow V_{\lambda,\alpha} \longrightarrow L_\lambda \longrightarrow 0,$$

$$0 \longrightarrow L_{s_\alpha(\lambda)} \longrightarrow K \longrightarrow V_{\lambda-\alpha,\alpha} \longrightarrow 0.$$

From the first exact sequence we get the long exact sequence

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^{l(\phi)}(w, V_{\lambda,\alpha}) & \longrightarrow & H^{l(\phi)}(w,\lambda) & \longrightarrow & H^{l(\phi)+1}(w,K) & \longrightarrow & H^{l(\phi)+1}(w, V_{\lambda,\alpha}) & \longrightarrow \cdots \\
 & \| & & \| & & \| & & \| & \\
\cdots \longrightarrow & H^{l(\phi)}(\tau,\lambda) & \longrightarrow & H^{l(\phi)}(w,\lambda) & \longrightarrow & H^{l(\phi)+1}(w,K) & \longrightarrow & H^{l(\phi)+1}(\tau,\lambda) & \longrightarrow \cdots .
\end{array}$$

Since λ is in the ϕ–chamber the rightmost term in the above sequence vanishes from Th 4.1. Moreover $\phi \not\leq \tau$, for it will otherwise imply that $\phi \leq w$, since $\phi(\alpha) \in R^+$. Since τ has length one more than w, it follows by induction applied to $X(\tau)$ that the leftmost term in the above long exact sequence is also zero. So we get, $H^{l(\phi)}(w, \lambda) \simeq H^{l(\phi)+1}(w, K)$.

From the second Demazure exact sequence we get the long exact sequence

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^{l(\phi)+1}(w, s_\alpha(\lambda)) & \longrightarrow & H^{l(\phi)+1}(w,K) & \longrightarrow & H^{l(\phi)+1}(w, V_{\lambda-\alpha,\alpha}) \cdots \\
 & \| & & \| & & \| \\
\cdots \longrightarrow & H^{l(\phi)+1}(w, s_\alpha(\lambda)) & \longrightarrow & H^{l(\phi)+1}(w,K) & \longrightarrow & H^{l(\phi)+1}(\tau, \lambda-\alpha) \cdots .
\end{array}$$

Since we have assumed λ to be a generic weight in the ϕ–chamber, both λ and $\lambda - \alpha$ are moved to the dominant chamber by ϕ. So the rightmost term in the above long exact sequence is zero from Th 4.1. Now $s_\alpha(\lambda)$ is in the ϕs_α–chamber. Furthermore $\phi s_\alpha \not\leq w$, since $\phi \not\leq w$ and $\phi(\alpha) > 0$. Since ϕs_α has length one more than ϕ, by induction on the chamber length applied to $X(w)$, the leftmost term in the above long exact sequence is also zero. Hence so is $H^{l(\phi)+1}(w, K)$. Since from the preceding paragraph $H^{l(\phi)+1}(w, K)$ is isomorphic to $H^{l(\phi)}(w, \lambda)$ we are done.

5. Some remarks on indecomposable B_α–modules

Recall the definitions of P_α, L_α and B_α from §2. We remind the reader that for a simple root α and a B–module V, the notation $H^i(\alpha, V)$ refers to the cohomology module $H^i(P_\alpha/B, \mathcal{L}_{s_\alpha}(V))$. If V is a B_α–module, then $\mathcal{L}(V)$ denotes the associated vector bundle on L_α/B_α.

We have the following lemma which plays the key role in our inductive procedures.

Lemma 5.1. *Let V be a finite dimensional B–module and let $res_{B_\alpha} V \simeq \oplus V_i$ be its decomposition as indecomposable B_α–modules. Then, the cohomology modules $H^j(P_\alpha/B, \mathcal{L}_{s_\alpha}(V))$ with its natural P_α–structure is isomorphic to $\bigoplus_i W_i \otimes H^j(L_\alpha/B_\alpha, \mathcal{L}(\chi_i))$ as an L_α–module.*

Here the isomorphisms $V_i \simeq (W_i \otimes \chi_i)$ are as in Cor 9.1, and W_i is an irreducible L_α–module and χ_i a character of B_α.

Proof. We first note the following isomorphism of homogeneous spaces, namely

$$P_\alpha/B \simeq L_\alpha/B_\alpha.$$

Since V is a B–module, by restricting it to its B_α–structure and denoting the associated vector bundle on L_α/B_α by $\mathcal{L}(V)$, we have the following identification of the cohomology modules:

$$H^j(P_\alpha/B, \mathcal{L}_\alpha(V)) \simeq H^j(L_\alpha/B_\alpha, \mathcal{L}(V)) \qquad (*)$$

Now by Cor 9.1, using the direct sum decomposition of $V \simeq \oplus W_i \otimes \chi_i$, we can take cohomology on either side by applying the functor $H^j(L_\alpha/B_\alpha, -)$ to get the isomorphism $H^j(L_\alpha/B_\alpha, \mathcal{L}(V)) \simeq \bigoplus_i W_i \otimes H^j(L_\alpha/B_\alpha, \mathcal{L}(\chi_i))$. (We note that since W_i are in fact L_α–modules, $H^j(L_\alpha/B_\alpha, \mathcal{L}(W_i \otimes \chi_i)) \simeq W_i \otimes H^j(L_\alpha/B_\alpha, \mathcal{L}(\chi_i))$.) This together with $(*)$ proves the Lemma.

Let $\lambda \in \Lambda$, $w \in W$. Let $w = s_{i_k} s_{i_{k-1}} \dots s_{i_1}$ be a reduced expression for w. Let w_j denote the suffix $s_{i_j} s_{i_{j-1}} \dots s_{i_1}$. The above Lemma 5.1 gives us an inductive procedure to compute the cohomology modules $H^j(w, \lambda)$. At the first step depending upon whether $\langle \lambda \, , \alpha_{i_1} \rangle$ is greater or less than zero, either $H^0(w_1, \lambda)$ or $H^1(w_1, \lambda)$ (or neither) survives. As noted above these modules get a B–structure, so in particular a $B_{\alpha_{i_2}}$–structure. In order to compute $H^i(w_2, \lambda)$, we need to understand the decomposition of the cohomology modules created in the first step into $B_{\alpha_{i_2}}$ indecomposable components, and understand how these various pieces contribute to cohomology at the second step.

The problem thus reduces to understanding, for each B_α–module, its decomposition into indecomposable B_α-components, and to understand the contribution to cohomology of a given indecomposable module. The first of these steps gets increasingly complicated as the length of w increases. We carry it out in a few special cases, in the next section. The second step is easier and is handled by the lemma given below. (cf §2)

Lemma 5.2. *Let V be a finite dimensional indecomposable B_α–module. By Cor. 9.1, we may assume that $V \simeq V_{\mu,\alpha} \otimes \chi$ with $\langle \mu, \alpha \rangle = dimV - 1$. Let $\lambda = \mu + \chi$. Then we have:*

(i) *If* $\langle\lambda\ ,\alpha\rangle \geq \langle\mu,\alpha\rangle$ *then* $H^0(L_\alpha/B_\alpha, \mathcal{L}(V))$ *is non-zero. It is a highest weight module and a lowest weight module. The highest weight is* λ *and lowest weight is* $s_\alpha(\lambda)$*.*

(ii) *If* $\langle\lambda\ ,\alpha\rangle < \langle\mu,\alpha\rangle - 1$ *then* $H^1(L_\alpha/B_\alpha, \mathcal{L}(V))$ *is non-zero. It is a highest weight module with highest weight* $s_\alpha \cdot (\lambda - \langle\mu,\alpha\rangle\alpha)$*.*

(iii) *If* $\langle\lambda\ ,\alpha\rangle = \langle\mu,\alpha\rangle - 1$ *then* $H^i(L_\alpha/B_\alpha, \mathcal{L}(V))$ *is zero for* $i = 0, 1$*.*

Proof. The proof follows by observing that the cohomology module $H^i(L_\alpha/B_\alpha, V)$ is isomorphic to $V_{\mu,\alpha} \otimes H^i(L_\alpha/B_\alpha, \mathcal{L}(\chi))$ and that $\langle\lambda\ ,\alpha\rangle = \langle\mu\ ,\alpha\rangle + \langle\chi\ ,\alpha\rangle$. The vanishing or non-vanishing of the cohomology depends only upon $\langle\chi\ ,\alpha\rangle$.

6. Cohomology modules for Schubert varieties of distinct type

In this section we prove results about *Schubert varieties of distinct type.* Recall that these are products of the l simple reflections each occurring exactly once, but appearing in any order.

Fix a reduced expression for $w = s_{i_l} s_{i_{l-1}} \dots s_{i_1}$. Denote by w_j the suffix $s_{i_j} \dots s_{i_1}$ of w. Note that in this case, the Bott-Samelson resolution coincides with the Schubert variety $X(w)$. In this case one can give a complete description of all the cohomology modules $H^i(w, \lambda), 0 \leq i \leq l(w)$. The reason for this is the following simple observation.

Observation 6.1. *Assume that* $V = H^i(w_j, \lambda)$ *is non-zero for some* i*. Then* V *is a direct sum of one-dimensional* $B_{\alpha_{i_{j+1}}}$*–modules.*

Proof. Since V is a B module by Prop 9.1 it decomposes into a direct sum of cyclic $B_{\alpha_{i_{j+1}}}$–modules. Now since each simple reflection appears exactly once in w, the reflection $s_{i_{j+1}}$ does not appear in w_j. So if v_1 and v_2 are two weight vectors in $H^i(w_j, \lambda)$ with weights μ_1 and μ_2, the coefficient of $\alpha_{i_{j+1}}$ in the expression $\mu_1 - \mu_2$ is zero. The observation now follows since every cyclic $B_{\alpha_{i_{j+1}}}$–module with dimension greater than or equal to two will have two weight vectors whose weights differ by a multiple of $\alpha_{i_{j+1}}$.

This observation gives us a simple inductive procedure to compute the cohomology modules at each stage. For instance in order to compute $H^i(w_j, \lambda)$ for some i, j we may assume by induction that we know how to compute $H^i(w_{j-1}, \lambda)$ and $H^{i-1}(w_{j-1}, \lambda)$. We break up these two modules into one dimensional, indecomposable $B_{\alpha_{i_j}}$ modules. It is then easy to compute and also to conclude which of these various one dimensional pieces contribute to the i-th cohomology of w_j.

We also define a certain weight $\tau[\lambda]$ for each $\tau \leq w$ and λ . When λ is dominant $\tau[\lambda] = \tau(\lambda)$, is an extremal weight of $H^0(w, \lambda)$. In the case when λ is non-dominant and generic we show that there is a non negative integer r such that $\tau[\lambda]$ will be a weight in $H^r(w, \lambda)$.

To define $\tau[\lambda]$ we define for $0 \leq k \leq l$ weights $\tau_k[\lambda]$ by the following procedure. Let $\tau = s_{j_r} \dots s_{j_1}$ be a reduced expression for τ. Recall that $w = s_{i_l} \dots s_{i_1}$.

Define $\tau_0(\lambda) := \lambda$.

For $k = 1$ to l
    if $s_{i_k} \leq \tau$ then
        if $\langle\tau_{k-1}[\lambda], \alpha_{i_k}\rangle < 0$ then
            $\tau_k[\lambda] := s_{i_k} \cdot \tau_{k-1}[\lambda]$.

else

$\tau_k[\lambda] := s_{i_k}(\tau_{k-1}[\lambda])$.

else

if $\langle \tau_{k-1}[\lambda], \alpha_{i_k}\rangle < 0$ then

$\tau_k[\lambda] := \tau_{k-1}[\lambda] + \alpha_{i_k}$.

else

$\tau_k[\lambda] := \tau_{k-1}[\lambda]$.

endfor

Set $\tau[\lambda] := \tau_l(\lambda)$.

Remark 6.2. Starting with λ we can define another sequence of weights $\tau_k(\lambda)$ using the same algorithm as above but replacing the term $\tau_{k-1}(\lambda) + \alpha_{i_k}$ with $\tau_{k-1}(\lambda)$ and replacing the term $s_{i_k} \cdot \tau_{k-1}[\lambda]$ with $s_{i_k}\tau_{k-1}(\lambda)$. Define τ_k to be the product of those simple reflections $s_{i_j} \leq \tau$ and $s_{i_j} \leq w_k$ (naturally this product is taken in the same order that these reflections appear in w_k). Then $\tau_k \leq w_k$ is an element of the Weyl group and since each reflection appears only once in w, this expression for τ_k is reduced. If λ is dominant then for this definition of τ_k, $\tau_k(\lambda)$ is an extremal weight of $H^0(w_k, \lambda)$ and $\tau(\lambda) = \tau_l(\lambda)$ is an extremal weight of $H^0(w_k, \lambda)$. Further $\tau(\lambda)$ is independent of the reduced expression for τ and w.

When λ is a generic non-dominant weight we have statements similar in spirit with $\tau_k[\lambda]$ playing the role of $\tau_k(\lambda)$.

Remark 6.3. Since λ is assumed to be generic, the inequality $\langle \tau_{k-1}[\lambda], \alpha_{i_k}\rangle < 0$ in the algorithm above can be replaced by $\langle \tau_{k-1}[\lambda], \alpha_{i_k}\rangle \leq -2$.

Observe that if τ is a product of commuting reflections it is clear from the procedure outlined above that the weight $\tau[\lambda]$ is independent of the reduced expression for τ.

In fact we observe:

Observation 6.4. *For all λ, $\tau \leq w$, $\tau[\lambda]$ is independent of the reduced expression for τ.*

Proof. We prove the statement by induction on $l(\tau)$, the base case when $l(\tau) = 1$ being trivially true.

Fix a reduced expression for $w = s_{i_l} \dots s_{i_1}$. Let $s_{j_r} \dots s_{j_1}$ and $s_{k_r} \dots s_{k_1}$ be two reduced expressions for τ, $\tau \leq w$.

First suppose that $s_{j_1} = s_{k_1}$. We may assume w.l.o.g that $s_{i_1} = s_{j_1}$. Further w.l.o.g. we may also assume that $\langle \lambda\ , \alpha_{j_1}\rangle > 0$ so that $\tau_1[\lambda] = s_{i_1}(\lambda)$. Let $w' = s_{i_l} \dots s_{i_2}$ and $\lambda' = s_{i_1}(\lambda)$. Then $s_{j_r} \dots s_{j_2}$ and $s_{k_r} \dots s_{k_2}$ are two reduced expressions for $\tau' = \tau s_{j_1} \leq w'$. By the induction hypothesis we may conclude that $\tau'[\lambda']$ is independent of the reduced expression of τ'. Since $\tau[\lambda] = \tau'[\tau_1[\lambda]] = \tau'[\lambda']$ we are done.

If $s_{j_1} \neq s_{k_1}$ it must be the case that these reflections commute. If they do not commute then since the reflections s_{j_1}, s_{k_1} appear exactly once in both τ and w, $\tau(\alpha_{j_1}) < 0$ iff $\tau(\alpha_{k_1}) > 0$. This contradicts the fact that $\tau(\alpha_{j_1}) < 0$ and $\tau(\alpha_{k_1}) < 0$.

We may conclude then that τ has a reduced expression $\tau = \tau'\tau''$ where τ'' is a product of commuting reflections. Let w'' be the the smallest suffix of w for which $\tau'' \leq w''$ and let $w = w'w''$. From the paragraph preceding this observation and the induction hypothesis applied to the pair τ'', w'', we conclude that $\tau''[\lambda]$ is independent of the reduced expression of τ''. Again by induction applied to the pair τ, w', we conclude that $\tau[\lambda] = \tau'[\tau''[\lambda]]$ is independent of the reduced expression of τ'. This concludes the proof.

Observation 6.5. *Let $\tau \le w$. Let $r = |\{k \le l | \langle \tau_{k-1}[\lambda], \alpha_{i_k}\rangle < 0\}|$. Then $\tau[\lambda]$ is a weight in $H^r(w, \lambda)$.*

Proof. For $t = 1$ to l, we prove by induction on t that $\tau_t[\lambda]$ is a weight of $H^{r_t}(w_t, \lambda)$. Here $r_t = |\{k \le t | \langle \tau_{k-1}[\lambda], \alpha_{i_k}\rangle < 0\}|$.

Suppose $\langle \lambda, \alpha_{i_1}\rangle > 0$ (respectively, $\langle \lambda, \alpha_{i_1}\rangle < 0$). Then $H^0(s_{i_1}, \lambda)$ (respectively, $H^1(s_{i_1}, \lambda)$) is non zero. In the first case $\lambda, \lambda - \alpha_{i_1}, \ldots, s_{i_1}(\lambda)$ are weights of $H^0(s_{i_1}, \lambda)$ and in the other case $\lambda + \alpha_{i_1}, \lambda + 2\alpha_{i_1}, \ldots, s_{i_1} \cdot \lambda$ are weights of $H^1(s_{i_1}, \lambda)$. Since $\tau_1[\lambda]$ is either equal to λ or $s_{i_1}(\lambda)$ in the first case and since $r_1 = 0$ we are done in this case. In the other case $\tau_1[\lambda]$ is either equal to $s_{i_1} \cdot \lambda$ or $\lambda + \alpha_{i_1}$. Further $r_1 = 1$ in this case and again we are done. This completes the base case of the induction.

Assume by induction that $\tau_{t-1}[\lambda]$ is a weight in $H^{r_{t-1}}(w_{t-1}, \lambda)$. If $\langle \tau_{t-1}[\lambda], \alpha_{i_t}\rangle < 0$ then, since λ is generic, both $s_{i_t} \cdot \tau_{t-1}[\lambda]$ and $\tau_{t-1}[\lambda] + \alpha_{i_t}$ are weights of $H^{r_{t-1}+1}(w_t, \lambda)$. Since $r_t = r_{t-1} + 1$ in this case and since $\tau_t[\lambda]$ is either $s_{i_t} \cdot \tau_{t-1}[\lambda]$ or $\tau_{t-1}[\lambda] + \alpha_{i_t}$ we are done. The case when $\langle \tau_{t-1}(\lambda), \alpha_{i_t}\rangle > 0$ is handled similarly.

Remark 6.6. From the proof of the above Obs 6.5 it follows that $\tau_k[\lambda]$ is a weight in $H^{r_k}(w_k, \lambda)$ for r_k as defined in the above proof.

If λ is a generic weight it is easy to see that the sign of $\langle \tau_k[\lambda]\ , \alpha_{i_{k+1}}\rangle$ will be the same as that of $\langle \tau_k(\lambda)\ , \alpha_{i_{k+1}}\rangle$ where $\tau_k(\lambda)$ is as defined in Rem 6.2.

So for all practical purposes we may replace the weight $\tau_k[\lambda]$ in $H^{r_k}(w_k, \lambda)$ by the weight $\tau_k(\lambda)$ i.e. in so far as computation is concerned, for λ generic, instead of computing with $\tau_k[\lambda]$ we may compute with $\tau_k(\lambda)$.

Next we would like to give exact bounds on the the least and topmost non-vanishing cohomology module of w. The index of the least and topmost non-vanishing cohomology module is dictated by some interesting combinatorics. To describe this we set up some notation. We continue to use the same reduced expression for w.

For $1 \le j \le l$ define $A_j = \{\tau^{-1}(\alpha_{i_j}) \mid \tau \le w_{j-1}\}$.

We say $\lambda(A_j) > 0$ (respectively, $\lambda(A_j) < 0$) if $\forall \alpha \in A_j, \langle \lambda\ , \alpha\rangle > 0$ (respectively, $\forall \alpha \in A_j, \langle \lambda\ , \alpha\rangle < 0$). If there are roots $\alpha, \beta \in A_j$ such that $\langle \lambda\ , \alpha\rangle < 0$ and $\langle \lambda\ , \beta\rangle > 0$ then we say $\lambda(A_j) \gtrless 0$.

Then we have (assuming λ is generic),

Theorem 6.7.
Let $a(w) := a(w, \lambda) = \big|\{i | \lambda(A_i) < 0\}\big|$ and $b(w) := b(w, \lambda) = \big|\{i | \lambda(A_i) \gtrless 0\}\big|$. Then

(i) $H^i(w, \lambda) = 0$, $\forall i$, $i < a(w)$ and $H^i(w, \lambda) = 0$, $\forall i$, $i > a(w) + b(w)$.

(ii) $H^i(w, \lambda) \ne 0$, $\forall i$, $a(w) \le i \le a(w) + b(w)$, when G is of type different from *D or E.*

Proof. Proof of (i) We continue to use the notation $\tau_k[\lambda], \tau[\lambda]$ from before.

Let μ be the product of those simple reflections s_{i_j}'s (taken in the order in which they appear in w) for which either $\lambda(A_{i_j}) > 0$ or $\lambda(A_{i_j}) \gtrless 0$. Let ν be the product of those simple reflections s_{i_j}'s (taken in the order in which they appear in w) for which either $\lambda(A_{i_j}) < 0$ or $\lambda(A_{i_j}) \gtrless 0$.

We claim that $\mu_k[\lambda]$ (respectively, $\nu_k[\lambda]$) is the lowest weight (respectively, the highest weight) in the least non-vanishing cohomology module $H^{r_k}(w_k, \lambda)$ (respectively, in the topmost non-vanishing cohomology module). Furthermore we show that $\mu_k[\lambda]$ (respectively, $\nu_k[\lambda]$) is less than (respectively, greater than) every weight appearing in any non-zero cohomology module $H^r(w_k, \lambda), 0 \leq r \leq k$. We do this by induction on k.

When $k = 1$, if $\lambda(A_1) > 0$ then $H^0(s_{i_1}, \lambda)$ is non-zero and $H^1(s_{i_1}, \lambda)$ vanishes. Furthermore $\mu_1[\lambda] = s_{i_1}(\lambda)$ is the lowest weight in $H^0(s_{i_1}, \lambda)$ and $\nu_1[\lambda] = \lambda$ is the highest weight in $H^0(s_{i_1}, \lambda)$ as required. On the other hand if $\lambda(A_1) < 0$ then (assuming λ is generic) $H^1(s_{i_1}, \lambda)$ is non-zero and $\nu_1[\lambda] = s_{i_1} \cdot \lambda$ is the highest weight in $H^1(s_{i_1}, \lambda)$ and $\mu_1[\lambda] = \lambda + \alpha_{i_1}$ is the lowest weight in $H^1(s_{i_1}, \lambda)$. This completes the base case.

By induction we may assume that $\mu_k[\lambda]$ is the lowest weight of the least non-vanishing cohomology $H^{r_k}(w_k, \lambda)$ for some r_k depending upon μ. We may also assume that $\nu_k[\lambda]$ is the highest weight of the topmost non-vanishing cohomology $H^{t_k}(w_k, \lambda)$ for some t_k depending upon μ.

Assume $\lambda(A_{k+1}) > 0$. Now since λ is generic, from Rem 6.6, the computation $\langle \mu_k[\lambda] \, , \alpha_{k+1} \rangle$ has the same sign as $\langle \mu_k(\lambda) \, , \alpha_{k+1} \rangle$, with μ_k as defined in Rem 6.2. Since $\mu_k \leq w_k$, $\mu_k^{-1}(\alpha_{i_{k+1}}) \in A_{k+1}$ and so $\langle \mu_k(\lambda) \, , \alpha_{k+1} \rangle > 0$. Hence $\langle \mu_k[\lambda], \alpha_{i_{k+1}} \rangle > 0$. By the definition of μ, $s_{i_{k+1}} \leq \mu$. And so $\mu_{k+1}[\lambda] = s_{i_{k+1}}(\mu_k[\lambda])$. Further $\mu_{k+1}[\lambda]$ is now the lowest weight in $H^{r_k}(w_{k+1}, \lambda)$. Clearly $H^i(w_{k+1}, \lambda)$ is zero for all $i \leq r_k$. Since for every other weight μ' of $H^r(w_k, \lambda)$, $0 \leq r \leq k$, $\mu' > \mu_k[\lambda]$ it follows that $s_{i_{k+1}}(\mu_k[\lambda]) \leq s_{i_{k+1}}(\mu')$. So $\mu_{k+1}[\lambda]$ is the lowest among all weights appearing in any non-zero cohomology module $H^r(w_{k+1}, \lambda)$ for $0 \leq r \leq k+1$.

Again since λ is generic, from Rem 6.6, the computation $\langle \nu_k[\lambda] \, , \alpha_{k+1} \rangle$ has the same sign as $\langle \nu_k(\lambda) \, , \alpha_{k+1} \rangle$. Here $\nu_k \leq w_k$ is as defined in Rem 6.2. Since $\nu_k \leq w_k$, $\nu_k^{-1}(\alpha_{i_{k+1}}) \in A_{k+1}$ and so $\langle \nu_k(\lambda) \, , \alpha_{k+1} \rangle > 0$. Hence $\langle \nu_k[\lambda], \alpha_{i_{k+1}} \rangle > 0$. By the definition of ν, $s_{i_{k+1}} \not\leq \nu$. And so $\nu_{k+1}[\lambda] = \nu_k[\lambda]$. Further $\nu_{k+1}[\lambda]$ is now the highest weight in $H^{t_k}(w_{k+1}, \lambda)$ and $H^i(w_{k+1}, \lambda) = 0$ for all $i > t_k$. An argument similar to what we did for $\mu_{k+1}[\lambda]$ shows that $\nu_{k+1}[\lambda] > \mu'$ for all μ', where μ' is a weight of some non-zero cohomology module $H^r(w_{k+1}, \lambda)$ $0 \leq r \leq k+1$.

The case when $\lambda(A_{k+1}) < 0$ is similar to the above. In this case the index of the least non-vanishing cohomology increases by one and the index of the topmost non-vanishing cohomology module also increases by 1. The new lowest weight is the old lowest weight $\mu_k[\lambda]$ (follows by induction) plus $\alpha_{i_{k+1}}$. Since in this case $s_{i_{k+1}} \not\leq \mu$, it follows that the $\mu_{k+1}[\lambda] = \mu_k[\lambda] + \alpha_{i_{k+1}}$. A similar argument shows that the new highest weight is $s_{i_{k+1}} \cdot \nu_k[\lambda]$. This is exactly what $\nu_{k+1}[\lambda]$ is, since $s_{i_{k+1}} \leq \nu$.

The last case is when $\lambda(A_{k+1})\rangle \gtrless 0$. Then we claim that $\langle \nu_k[\lambda], \alpha_{i_{k+1}} \rangle < 0$ and $\langle \mu_k[\lambda], \alpha_{i_{k+1}} \rangle > 0$. We prove that $\langle \nu_k[\lambda], \alpha_{i_{k+1}} \rangle < 0$. The argument for $\langle \mu_k[\lambda], \alpha_{i_{k+1}} \rangle > 0$ is similar.

Assume $\langle \nu_k[\lambda], \alpha_{i_{k+1}} \rangle \geq 0$. Since $\nu_k[\lambda]$ is the highest among all weights appearing in any cohomology module $H^r(w_k, \lambda)$ for any r, it follows that every other weight ν' occurring in $H^r(w_k, \lambda)$, $0 \leq r \leq k$, is of the form $\nu' = \nu_k[\lambda] - \Sigma_{j \leq k} c_j \alpha_{i_j}$ with $c_j \geq 0$. So $\langle \nu', \alpha_{i_{k+1}} \rangle \geq \langle \nu, \alpha_{i_{k+1}} \rangle \geq 0$ (we notice that none of the $\alpha'_{i_j} s$ in the above sum is equal to $\alpha_{i_{k+1}}$ since this is the first occurrence of $s_{i_{k+1}}$). Now for every $\tau \leq w_k$, $\tau[\lambda]$ is a weight of some non-zero cohomology module $H^j(w_k, \lambda)$ for some j. And so $\langle \tau[\lambda], \alpha_{i_{k+1}} \rangle \geq 0$. From Rem 6.6 it follows that for λ generic, $\langle \tau(\lambda), \alpha_{i_{k+1}} \rangle > 0$. So assuming $\langle \nu_k[\lambda], \alpha_{i_{k+1}} \rangle \geq 0$ we would have $\lambda(A_{k+1}) \geq 0$ contradicting our assumption that $\lambda(A_{k+1})\rangle \gtrless 0$.

In this case then the index of the topmost non-vanishing cohomology module increases and the index of the least non-vanishing cohomology module remains the same. It follows that the new highest (respectively, lowest) weight is $s_{i_{k+1}} \cdot \nu_k[\lambda]$ (respectively, $s_{i_{k+1}}(\mu_k[\lambda])$. But this is exactly $\nu_{k+1}[\lambda]$ (respectively, $\mu_{k+1}[\lambda]$) since $s_{i_{k+1}} \leq \nu$ (respectively, $s_{i_{k+1}} \leq \mu$).

Since the index of the least non-vanishing cohomology module increases whenever $\lambda(A_{i+1}) < 0$, and this happens $a(w, \lambda)$ times it follows that $H^{a(w)}(w, \lambda) \neq 0$ and $H^i(w, \lambda) = 0, i < a(w)$.

Since the index of the topmost non-vanishing cohomology module increases every time $\lambda(A_{k+1})\rangle \gtreqless 0$ or $\lambda(A_{k+1})\rangle < 0$, and the number of times this happens is $a(w) + b(w)$, we have $H^{a(w)+b(w)}(w, \lambda) \neq 0$ and $H^i(w, \lambda) = 0, i > a(w) + b(w)$.

Proof of (ii).

Suppose $\tau \leq w$ and $s_{i_j}\tau \leq w$. We say $\tau[\lambda]$ and $(s_{i_j}\tau)[\lambda]$ are distance d apart in cohomology if $\tau[\lambda]$ is a weight in $H^k(w, \lambda)$ and $(s_{i_j}\tau)[\lambda]$ is a weight in $H^j(w, \lambda)$ where $|j - k| = d$.

Then we have

Observation 6.8. *Suppose $\tau \leq w$ and $s_{i_j}\tau \leq w$. Then the distance between the weights $\tau[\lambda]$ and $(s_{i_j}\tau)[\lambda]$ is at most 2.*

Proof. We assume that $\tau^{-1}(\alpha_{i_j}) > 0$. W.l.o.g. we fix a reduced expression for $w = s_{i_l} \ldots s_{i_1}$ in such a way that the reflection $s_{\alpha_{i_j}}$ appears to the left of the reflections occurring in τ in the fixed reduced expression of w. For $k \leq j - 1$, $\tau_k(\lambda) = (s_{i_j}\tau)_k[\lambda]$. When the group G is of type different from D and E there are at most two simple reflections which do not commute with s_{i_j}. So the number of $m > j$ for which the signs of $\langle \tau_{m-1}(\lambda), \alpha_{i_m} \rangle$ and $\langle (s_{i_j}\tau)_{m-1}[\lambda], \alpha_{i_m} \rangle$ are different, is at most two.

Remark 6.9. We note that the non-vanishing result i.e (ii) of the above theorem has been shown here only when the group G is not of type D or E.

We now show that for a generic λ, $H^i(w, \lambda) \neq 0$ for $a(w) \leq i \leq a(w) + b(w)$.

Given w and λ, for every $\tau \leq w$ we determine that i such that $\tau[\lambda]$ is a weight in $H^i(w, \lambda)$. If for every i such that $a(w) \leq i \leq b(w)$, there is a $\tau \leq w$ such that $\tau[\lambda]$ is a weight in $H^i(w, \lambda)$ we are done.

Defining μ and ν as in the proof of (i) it follows from the proof of (i) $\nu[\lambda]$ is the highest weight and $\mu[\lambda]$ is the lowest weight among all weights in any non-zero cohomology module.

Let $\nu = s_{\beta_k} \ldots s_{\beta_1}$ let $\mu = s_{\gamma_j} \ldots s_{\gamma_1}$ be reduced expressions for ν and μ as a product of simple reflections. Consider two sequences of weights the first being $[\lambda], s_{\beta_1}[\lambda], (s_{\beta_2}s_{\beta_1})[\lambda], \ldots, \nu[\lambda]$ and the second being $[\lambda], (s_{\gamma_1})[\lambda], (s_{\gamma_2}s_{\gamma_1})[\lambda], \ldots, \mu[\lambda]$. By the Obs 6.5 these are all weights of some cohomology module $H^i(w, \lambda)$. Furthermore $\mu[\lambda]$ is a weight in $H^{a(w)}(w, \lambda)$ and $\nu[\lambda]$ is a weight in $H^{a(w)+b(w)}(w, \lambda)$. Gluing these two sequences of weights at $[\lambda]$ we get a sequence of weights which start at a weight in $H^{a(w)}(w, \lambda)$ and end at a weight in $H^{a(w)+b(w)}(w, \lambda)$. By Obs 6.8 above, adjacent weights in this sequence are at distance at most two apart.

Let $r + 1$ be an index for which $H^{r+1}(w, \lambda)$ has no weight vector of the form $\tau[\lambda]$, for any $\tau \leq w$. Since $\nu[\lambda]$ is a weight in $H^{b(w)}(w, \lambda)$ and $\mu[\lambda]$ is a weight in $H^{a(w)}(w, \lambda)$ it follows that $a(w) < r + 1 < a(w) + b(w)$. From the sequence constructed above, we may conclude w.l.o.g that there are elements of the Weyl group $s_{i_k}\phi$ and ϕ such that $\phi[\lambda]$ is a weight in $H^r(w,\lambda)$ and $(s_{i_k}\phi)[\lambda]$ is a weight in $H^{r+2}(w,\lambda)$ (here either $s_{i_k}\phi\mu$ or $s_{i_k}\phi \leq \nu$). We may also assume w.l.o.g that the simple reflections occurring in ϕ appear to the right of s_{i_k} in the fixed reduced expression of w.

Clearly the weights $\phi_j[\lambda]$ and $(s_{i_k}\phi)_j[\lambda]$ are the same for all $j < k$. Moreover the sign of $\langle \phi_{k-1}[\lambda], \alpha_{i_k}\rangle$ determines both $\phi_k[\lambda]$ and $(s_{i_k}\phi)_k[\lambda)]$. W.l.o.g $\phi_k[\lambda] = \phi_{k-1}[\lambda]$ and $(s_{i_k}\phi)_k[\lambda] = s_{i_k}(\phi_k[\lambda])$. Clearly $\phi_k[\lambda]$ and $(s_{i_k}\phi)_k[\lambda]$ are weights in the same cohomology module at the k-th stage. We may also assume that $(s_{i_k}\phi)_k[\lambda] > \phi_k[\lambda]$.

Since the final distance between $\phi[\lambda]$ and $(s_{i_k}\phi)[\lambda]$ is two, there exist indices $m < n \leq l$ for which the distance between the weights $\phi_m[\lambda]$ and $(s_{i_k}\phi)_m[\lambda]$ (respectively, $\phi_n[\lambda]$ and $(s_{i_k}\phi)_n[\lambda]$) becomes one (respectively, two). W.l.o.g. $\langle \phi_{m-1}[\lambda]\ , \alpha_{i_m}\rangle > 0$ and $\langle (s_{i_k}\phi)_{m-1}[\lambda]\ , \alpha_{i_m}\rangle < 0$. Likewise we may assume that $\langle \phi_{n-1}[\lambda]\ , \alpha_{i_n}\rangle > 0$ and $\langle (s_{i_k}\phi)_{n-1}[\lambda]\ , \alpha_{i_n}\rangle < 0$.

From stage $p = k+1$ to $p = m-1$ if $\langle (s_{i_k}\phi)_{p-1}[\lambda]\ , \alpha_{i_p}\rangle > 0$ then $\langle \phi_{p-1}[\lambda]\ , \alpha_{i_p}\rangle > 0$ since the simple reflection s_{i_p} commutes with s_{i_k}. Also since s_{i_p} commutes with s_{i_k} for all p in this range and since $s_{i_p} \nleq \phi$ it is easy to see that $(s_{i_k}\phi)_{m-1}[\lambda] - (s_{i_k}\phi)_k[\lambda] = \phi_{m-1}(\lambda) - \phi_k[\lambda]$. And this difference is some integral combination of $\alpha_{i_p}, p = k+1,\ldots,m-1$. From this we get $(s_{i_k}\phi)_{m-1}[\lambda] = \phi_{m-1}[\lambda] + \langle \phi_{k-1}[\lambda]\ , \alpha_{i_k}\rangle\alpha_{i_k}$.

At the m–th stage we have assumed that $\langle (s_{i_k}\phi)_{m-1}[\lambda]\ , \alpha_{i_m}\rangle < 0$ and that $\langle \phi_{m-1}[\lambda]\ , \alpha_{i_m}\rangle > 0$. Suppose $\phi_{m-1}[\lambda]$ is a weight of $H^q(w_m,\lambda)$ and $(s_{i_k}\phi)_{m-1}[\lambda]$ is a weight of $H^{q+1}(w_m,\lambda)$. Then since for all weights $\mu \in S_m = \{\phi_{m-1}[\lambda] + \langle \phi_{m-1}[\lambda]\ , \alpha_{i_m}\rangle\alpha_{i_k}, \phi_{m-1}[\lambda] + (\langle \phi_{m-1}[\lambda]\ , \alpha_{i_m}\rangle - 1)\alpha_{i_k}, \ldots, \phi_{m-1}[\lambda]\}$, $\langle \mu\ , \alpha_{i_m}\rangle > 0$ the set of weights in S_m are all weights of $H^q(w_m,\lambda)$ at the m–th stage. By a similar reasoning the set of weights $T_m = \{\phi_{m-1}[\lambda] + (\langle \phi_{m-1}[\lambda]\ , \alpha_{i_m}\rangle + 2)\alpha_{i_k} + \alpha_{i_m},\ \ldots,$ $(s_{i_k}\phi)_{m-1}[\lambda] + \alpha_{i_m}\}$ are all weights of the cohomology module $H^{q+1}(s_{i_m}\ldots s_{i_1},\lambda)$ at the m–th stage.

Since λ is generic, for all subsequent reflections s_{i_p} in between s_{i_m} and s_{i_n}, since s_{i_p} commutes with s_{i_k} by choice of m, n the weights in S_m and T_m remain together (they may get modified by an additive factor of α_{i_p} for $m < p < n$; w.l.o.g we may ignore this additive factor since it is irrelevant for computation with α_{i_n}). And so we may assume that the distance between the set of weights in S_m and T_m remains one. At the $n-1$ th stage let the weights in T_m be weights of $H^{s+1}(w_{n-1},\lambda)$ and let S_m be weights of $H^s(w_{n-1},\lambda)$.

Let us analyze what happens when the reflection s_{i_n} is encountered. At this stage $\langle (s_{i_k}\phi)_{m-1}[\lambda] + \alpha_{i_m}\ , \alpha_{i_n}\rangle < 0$ and so $s_{i_n}\cdot((s_{i_k}\phi)_{m-1}[\lambda] + \alpha_{i_m})$ is now a weight of $H^{s+2}(w_n,\lambda)$. Also by assumption $\langle \phi_{m-1}[\lambda]\ , \alpha_{i_n}\rangle > 0$ and so $\phi_{m-1}[\lambda]$ is now a weight of $H^s(w_n,\lambda)$.

Now if the size of S_m and T_m is at least three either $\langle \phi_{m-1}[\lambda] + (\langle \phi_{m-1}[\lambda]\ , \alpha_{i_m}\rangle + 2)\alpha_{i_k} + \alpha_{i_m}, \alpha_{i_n}\rangle > 0$ or $\langle \phi_{m-1}[\lambda] + \langle \phi_{m-1}[\lambda]\ , \alpha_{i_m}\rangle\alpha_{i_k}, \alpha_{i_n}\rangle < 0$. So one of these will be a weight of $H^{s+1}(w_n,\lambda)$.

And this remains a weight (upto an additive factor of $\alpha'_{i_j}s, j > n$) sandwiched between the cohomology modules containing the weights $\phi[\lambda]$ and $s_{i_k}\phi[\lambda]$ in the end.

One can ensure by taking λ generic that the cardinality of S_m and T_m is at least three. This completes the proof.

6.1. Chambers for which only one cohomology survives

As is evident Schubert varieties of distinct type are particularly easy to work with. For such Schubert varieties it seems possible to identify all chambers for which only one cohomology module is non zero. We prove this in a particular case, in typeA_l, when $w = s_l s_{l-1} \ldots s_1$, a product of simple reflections such that s_i is connected to s_j iff $|i - j| = 1$. For such a w (which we call a *chain*) we show that there are 2^l chambers for which exactly one cohomology module is non zero.

We first show

Lemma 6.10. *Let λ be generic weight such that $s_1 s_2 \ldots s_i(\lambda) \in \Lambda^+$. Then $H^j(w, \lambda)$ is non zero exactly when $j = 1$.*

Proof. Since $\{\alpha \in R^+ : \langle \lambda \,, \alpha \rangle < 0\} = \{\alpha_i, \alpha_i + \alpha_{i-1}, \ldots, \alpha_i + \alpha_{i-1} + \ldots + \alpha_1\} = A_i$, we have $\lambda(A_i) < 0$. For the remaining A_j's we have $\lambda(A_j) > 0$, since $A_i \cap A_j = \emptyset$. The lemma now follows from Th 6.7.

From the proof of the Th 6.7(i) it is clear that upto the i-th stage the only non-zero cohomology is H^0. At the i-th stage $H^1(s_1 \ldots s_1, \lambda)$ is non-zero and every other cohomology module vanishes. And from then on upto the l-th stage the only non-trivial cohomology module is H^1, so that finally $H^1(w, \lambda)$ is the only non zero cohomology.

To obtain the other chambers where only one cohomology survives for the chain case we make use of the above lemma.

For this part of the proof it will convenient to let w_i denote $s_1 s_2 \ldots s_i$. Fix a subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, \ldots, l\}$. W.l.o.g we let $l \geq i_k > i_{k-1} > \ldots i_1 \geq 1$.

Then we have

Theorem 6.11. *Let λ be generic such that $w_{i_k} w_{i_{k-1}} \ldots w_{i_1}(\lambda) \in \Lambda^+$. Then $H^j(w_l, \lambda)$ is non-zero exactly when $j = k$. Conversely if λ is a generic weight such that only $H^k(w_l, \lambda)$ is non zero then there exists a k element subset of $\{i_k, \ldots i_1\}$ of $\{1, \ldots, l\}$ such that $w_{i_k} w_{i_{k-1}} \ldots w_{i_1}(\lambda) \in \Lambda^+$.*

Proof. The proof is exactly like in the lemma above. It can be checked that

$$\{\alpha \in R^+ : \langle \lambda \,, \alpha \rangle < 0\} = A_{i_1} \cup A_{i_2} \cup A_{i_3} \ldots \cup A_{i_k}.$$

Again since $A_j \cap A_k = \emptyset$ whenever $j \neq k$, we have $\lambda(A_j) < 0$ for $j \in \{i_1, \ldots, i_k\}$ and $\lambda(A_j) > 0$, $j \notin \{i_1, \ldots, i_k\}$. From the theorem it follows that only $H^k(w_l, \lambda)$ is non zero.

To prove the converse we observe first that for exactly k of the A_j's $\lambda(A_j) < 0$, and for the remaining A_j's, $\lambda(A_j) > 0$. Denote the k indices for which $\lambda(A_j) < 0$ by $\{i_k, \ldots, i_1\}$ and assume $i_k > \ldots > i_1$. Then if ϕ is an element of W which moves λ to the dominant chamber, it must be the case that ϕ sends the positive roots in $A_{i_1} \cup A_{i_2} \ldots \cup A_{i_k}$ to R^-, and sends the remaining positive roots to R^+. But then ϕ must be $w_{i_k} w_{i_{k-1}} \ldots w_{i_1}$.

7. Cohomological characterization of G/B

We now come to the main theorem of this paper which gives a cohomological characterization of G/B. In fact it characterizes G/B as a Schubert variety in terms of its cohomology.

Let $w \in W$ be such that $s_\alpha \leq w$ (in the Bruhat order) $\forall \alpha \in S$. Let $m(w)$ be the maximum among the coefficients of the simple roots which occur in the expression $w(\lambda) - w \cdot \lambda (= \rho - w(\rho))$. Let $s(w)$ denote the cardinality of the set $\{\tau \mid \tau \leq w\}$.

Definition 7.1.
Let $w \in W$. Then we say that a weight λ (respectively, μ) is in the *diagonal* (respectively, *anti-diagonal*) chamber with respect to w if $(w \cdot \lambda) \in \Lambda^+$ (respectively, $(-w\mu) \in \Lambda^+$).

Note that if λ is in the diagonal (respectively, anti-diagonal) chamber with respect to w such that $w(\lambda+\rho)$ is regular dominant (respectively, $w_0 w(\lambda)$ is regular dominant) it follows from Cor 3.1 that $H^{l(w)}(w,\lambda)$ (respectively, $H^0(w,\lambda)$) is non zero.

Theorem 7.2. *Let $w \in W$ be such that $s_\alpha \leq w$ (in the Bruhat order) $\forall \alpha \in S$. Suppose that*

$$\#\{j \mid H^j(w,\lambda) \neq 0\} = 1,$$

with λ a generic weight in the w–chamber given by the following condition:

$$\langle w\lambda \ , \alpha \rangle > m(w)s(w) \ for \ all \ simple \ roots \ \alpha.$$

Then $w = w_0$ or equivalently $X(w) = G/B$.

We have the following corollary to the above theorem.

Corollary 7.1. *Let $w \in W$ be arbitrary. Suppose that*

$$\#\{j \mid H^j(w,\lambda) \neq 0\} = 1,$$

with λ a generic weight as in Th. 7.2. Then $X(w) = Q/B$ for some parabolic subgroup $Q \supset B$.

Proof. (Of Cor. 7.1) Let $I = \{\alpha \in S | s_\alpha \leq w\}$. Let Q be the parabolic defined by this set of simple roots. Let H be the semi-simple subgroup of Q such that $B \cap H$ is a Borel subgroup of H (there is a canonical such subgroup H). Note that $X(w)$ is naturally a sub-variety of $H/(B\cap H) \simeq Q/B$. Restricting λ to $B \cap H$ and observing that it satisfies the same conditions of the Th. 7.2 for the semi-simple group H, the corollary follows.

Before we come to the proof of the Th 7.2 we establish some combinatorial conditions on the weights which appear in any cohomology.

For τ an element of the Weyl group we recall that $R^+(\tau) = \{\alpha \in R^+ \mid \tau(\alpha) \in R^-\}$. Then we have

Lemma 7.3. *Let $w = \tau\phi$ such that $l(w) = l(\tau) + l(\phi)$. Then $R^+(\phi) \subseteq R^+(w)$.*

Proof. Let $w = s_{\alpha_k} s_{\alpha_{k-1}} \dots s_{\alpha_1}$ be a reduced expression for w. Then $R^+(w) = \{\alpha_1, s_{\alpha_1}(\alpha_2), s_{\alpha_1}s_{\alpha_2}(\alpha_3), \dots, s_{\alpha_1} \dots s_{\alpha_{k-1}}(\alpha_k)\}$. So if $w = \tau\phi$ then $R^+(\phi) \subseteq R^+(w)$ since one can choose a reduced expression $w = s_{\alpha_k} s_{\alpha_{k-1}} \dots s_{\alpha_1}$ such that $\phi = s_{\alpha_j} s_{\alpha_{j-1}} \dots s_{\alpha_1}$ for some j.

Let $\mathcal{X}(w,\lambda)$ denote the following set of weights:

$$\mathcal{X}(w,\lambda) = \cup_{i=0}^{i=l(w)} \{chH^i(w,\lambda)\}.$$

Here $chH^i(w,\lambda)$ denotes the set of weights of the i–th cohomology module $H^i(w,\lambda)$, regarded as a T module.

Lemma 7.4. *Let $\mu \in \mathcal{X}(w,\lambda)$. Then*

(i) μ is in the convex hull of $\{\tau(\lambda)|\tau \leq w\}$.
(ii) If λ is dominant $\mu \geq w(\lambda)$ under the usual ordering.
(iii) There exists a unique maximal $\gamma(w,\phi)$ in $\{\tau\phi|\tau \leq w\}$.
(iv) There exists a unique minimal element in $\{\tau\phi \mid \tau \leq w\}$.

Proof. (i) We do this by induction on the length of w. When w is equal to a simple reflection s_α then the weights in $\mathcal{X}(w,\lambda)$ are either $\lambda, \lambda-\alpha, \ldots, s_\alpha(\lambda)$ or $\lambda+\alpha, \lambda+2\alpha, \ldots, s_\alpha \cdot \lambda$. In either case it is clear that the weights are in the convex hull of $\{\lambda, s_\alpha(\lambda)\}$.

Let us assume that the lemma is true for all ϕ of length less than w. Say $w = s_\alpha\phi$. Let μ be a weight in $\mathcal{X}(w,\lambda)$. If μ comes from a weight $\mu' \in \mathcal{X}(\phi,\lambda)$ such that $\langle \mu' \,, \alpha\rangle \geq 0$ then $\mu = \mu' - t\alpha$, where $0 \leq t \leq \langle \mu' \,, \alpha\rangle$. Clearly μ is in the convex hull of μ' and $s_\alpha(\mu')$. Now we can write μ' as a convex combination of weights $\gamma(\lambda)$ where $\gamma \leq \phi$ i.e. $\mu' = \sum_{\gamma\leq\phi} a_\gamma\gamma(\lambda)$. Since each such $\gamma \leq w$ as well, μ' can be written as a convex combination of weights in $\{\gamma(\lambda) \mid \gamma \leq w\}$. Now $s_\alpha(\mu') = \sum_{\gamma\leq\phi} a_\gamma s_\alpha\gamma(\lambda)$. Since $\gamma \leq \phi$, either $s_\alpha\gamma \leq w$ or $s_\alpha\gamma \leq \phi \leq w$. So $s_\alpha(\mu)$ is in the convex combination of weights $\{\tau(\lambda) \mid \tau \leq w\}$. Hence μ is in the convex combination of weights of the desired type.

The proof in the case when μ comes from a weight $\mu' \in \mathcal{X}(\phi,\lambda)$ such that $\langle \mu' \,, \alpha\rangle < 0$ is similar.

(ii) This is well known from the identification of the dual $H^0(X(w),\lambda)^*$ with a subspace of the Weyl module (cf . [5, II.14.19.3]). We give the proof below since it follows easily from (i).

Write $\mu = \sum_{\gamma\leq w} a_\gamma\gamma(\lambda)$. Subtracting $w(\lambda)$ from this we have

$$\mu - w(\lambda) = \sum_{\gamma\leq w} a_\gamma(\gamma(\lambda) - w(\lambda)).$$

Since λ is dominant and $\gamma \leq w$ each term in the right hand is in the dominant chamber, completing the proof.

(iii) We do this by induction on the length of w the case when $l(w) = 1$ being trivial. If $w = s_\alpha\tau$ with $l(w) = l(\tau)+1$ then by induction $\{\tau_1\phi \mid \tau_1 \leq \tau\}$, has a unique maximal element say $\gamma_0\phi$; i.e $\gamma_0\phi \geq \tau_0\phi, \forall\tau_0 \leq \tau$.

We claim that if $s_\alpha\gamma_0\phi > \gamma_0\phi$ then $\gamma(w,\phi) = s_\alpha\gamma_0\phi$, otherwise $\gamma(w,\phi) = \gamma_0\phi$. Suppose it is the case that $s_\alpha\gamma_0\phi > \gamma_0\phi$. Take $w'\phi$, $w' \leq w$. If $w' \leq \tau$ then from the maximality of $\gamma_0\phi$ it follows that $w'\phi < \gamma_0\phi < s_\alpha\gamma_0\phi$. If $w' \not\leq \tau$, then since $w' \leq w$, it follows that $s_\alpha w' \leq \tau$. Then we have $s_\alpha w'\phi < \gamma_0\phi < s_\alpha\gamma_0\phi$, where the first inequality follows from the maximality of γ_0. From this we get $w'\phi \leq s_\alpha\gamma_0\phi$, completing the induction step in this case.

Suppose on the other hand that $\gamma_0\phi > s_\alpha\gamma_0\phi$. If $w' \leq w$ then if $w' \leq \tau$ we are done by induction. Otherwise $s_\alpha w' \leq \tau$ and so $s_\alpha w'\phi \leq \gamma_0\phi$. It follows that $w'\phi \leq \gamma_0\phi$ as well.

(iv) We do this by induction on $l(w)$. If $w = s_\alpha\tau$, with $l(w) = l(\tau)+1$, then by induction $\{\tau_1\phi \mid \tau_1 \leq \tau\}$ has a minimal element, say $\gamma_0\phi$. Then it is easy to see that either $\gamma_0\phi$ or $s_\alpha\gamma_0\phi$ satisfies the claim. The calculations are exactly as in part (iii) above.

7.1. Proof of Theorem 7.2

The proof of Th 7.2 is by contradiction.

Let w and λ be given such that λ belongs to the w -chamber. Let us write $w_0 = s_{\beta_1}s_{\beta_2}\dots s_{\beta_r}w$, with β_i simple roots and $l(w_0) = l(w) + r$. For $1 \le i \le r$, we denote by τ_i the element of the Weyl group $s_{\beta_i}s_{\beta_{i-1}}\dots s_{\beta_1}w_0$. We show

Claim 7.5. Assume that the conclusion in Th 7.2 is false. Suppose w and λ are such that $H^j(w,\lambda)$ is non-zero for only one j. Then in fact $j = l(w)$. Furthermore we claim that for $1 \le i \le r$, there exists a non-zero weight vector in $H^{l(w)}(\tau_i,\lambda)$ of weight $\tau_i(w\cdot\lambda)$. In particular, $w(w\cdot\lambda)$ is a weight in $H^{l(w)}(w,\lambda)$.

Proof. That $j = l(w)$ follows from Cor 3.1(2) and the hypothesis. So $H^j(w,\lambda)$ is also non-zero only when $j = l(w)$.

By the Borel-Weil-Bott theorem $H^j(w_0,\lambda)$ vanishes for all j except $j = l(w)$. Hence it follows that $H^j(w_0,\lambda)$ is created from $H^j(w,\lambda)$ by a series of H^0's i.e. $H^{l(w)}(w_0,\lambda) = H^0(\beta_1,(H^0(\beta_2,\dots H^0(\beta_r, H^{l(w}(w,\lambda))\dots)))$.

By the Borel-Weil-Bott theorem we know that $H^{l(w)}(w_0,\lambda)$ is an irreducible module with highest weight $w\cdot\lambda$ and lowest weight $w_0(w\cdot\lambda)$. Furthermore these weights occur with multiplicity one. Fix a weight vector of weight $w\cdot\lambda$. Denote by $v(j)$ the unique vector v (upto scalar) of weight $\tau_j(w\cdot\lambda)$ in $H^{l(w)}(w_0,\lambda)$.

For $0 \le j \le r-1$ we consider the sequence of evaluation maps

$$H^0(\beta_{j+1}, H^{l(w)}(\tau_{j+1},\lambda)) \longrightarrow H^{l(w)}(\tau_{j+1},\lambda). \tag{1}$$

These maps are all non-zero. Since $H^1(\beta_{j+1}, H^{l(w)-1}(\tau_{j+1},\lambda)) = 0$, the left hand side is precisely $H^{l(w)}(\tau_j,\lambda)$. Composing such maps we get, for $1 \le j \le r$, a non-zero map f_j,

$$f_j : H^{l(w)}(w_o,\lambda) \longrightarrow H^{l(w)}(\tau_j,\lambda). \tag{2}$$

We prove by induction on j that the image of $v(j)$, $f_j(v(j))$ is non-zero. This gives us the desired vector of weight $\tau_j(w\cdot\lambda)$ in $H^{l(w)}(\tau_j,\lambda)$.

Moreover we also show that, $y_{\beta_{j+1}}(f_j(v(j)))$ is zero, implying that $f_j(v(j))$ is an extremal weight vector in the $B_{\beta_{j+1}}$ indecomposable piece containing it.

For the base case (i.e. $j = 1$) we consider the cohomology module $H^{l(w)}(\tau_1,\lambda)$ as a B_{β_1} module and look at its B_{β_1} indecomposable components. The map f_1 is B_{β_1}–equivariant. The vector of weight $w_0(w\cdot\lambda)$ is in a unique B_{β_1} indecomposable summand W in $H^{l(w)}(w_o,\lambda)$. It is the lowest weight in that W and the highest weight in W is $v(1)$ of weight $s_{\beta_1}w_0(w\cdot\lambda)$. If V is the indecomposable B_{β_1}–summand in $H^{l(w)}(\tau_1,\lambda)$ such that $H^0(\beta_1,V) \supseteq W$ it is clear that the image $f_1(v(1))$, of $v(1)$ under the evaluation map, $H^0(\beta_1,V) \mapsto V$ is non zero.

If $y_{\beta_2}f_1(v(1))$ were non zero, the map f_1 being B–equivariant, $y_{\beta_2}v(1)$ would be non zero and there would be a vector of weight $\tau_1(w\cdot\lambda) - \beta_2$ in $H^{l(w)}(w_0,\lambda)$. But then $\tau_1^{-1}(\tau_1(w\cdot\lambda) - \beta_2)$ would be a weight in $H^{l(w)}(w_0,\lambda)$. Since $\tau^{-1}(\beta_2) < 0$, $\tau_1^{-1}(\tau_1(w\cdot\lambda) - \beta_2) > w\cdot\lambda$ contradicting the fact that $w\cdot\lambda$ is the highest weight of $H^{l(w)}(w_0,\lambda)$. This completes the base case of the induction.

Assume that the statement has been proven for $j = k$. We will prove it for $j = k+1$. We may assume by induction that $f_k(v(k))$ is non-zero and it is a weight vector of weight

$\tau_k(w \cdot \lambda)$ in $H^{l(w)}(\tau_k, \lambda) = H^0(\beta_{k+1}, H^{l(w)}(\tau_{k+1}, \lambda))$. Furthermore it is killed by $y_{\beta_{k+1}}$. So in the $B_{\beta_{k+1}}$–equivariant map in (1) above (with $j = k$) it follows that $f_k(v(k))$ is the lowest weight vector in the indecomposable $B_{\beta_{k+1}}$ summand containing it. Now $\langle \tau_k(w \cdot \lambda) \ , \beta_{k+1} \rangle \leq 0$. Since $H^{l(w)}(\tau_k, \lambda)$ is a $P_{\beta_{k+1}}$ module, $s_{\beta_{k+1}} \tau_k(w \cdot \lambda) = \tau_{k+1}(w \cdot \lambda)$ is a weight in $H^{l(w)}(\tau_k, \lambda)$. It is in fact the highest weight vector in the indecomposable summand containing $f_k(v(k))$. It follows like in the base case that the image of the vector of weight $\tau_{k+1}(w \cdot \lambda)$ and, so also, of $v(k+1)$ is non zero. So $f_{k+1}(v(k+1))$ is a vector of weight $\tau_{k+1}(w \cdot \lambda)$ in $H^{l(w)}(\tau_{k+1}, \lambda)$.

If $y_{\beta_{k+2}} f_{k+1}(v(k+1))$ were non zero like in the base case of the induction, this would give us a vector of weight $\tau_{k+1}(w \cdot \lambda) - \beta_{k+2}$ in $H^{l(w)}(w_0, \lambda)$. Applying τ_{k+1}^{-1} to this would contradict the fact that $w \cdot \lambda$ is the highest weight in $H^{l(w)}(w_0, \lambda)$. This completes the proof.

In particular, the above proof implies that if $H^{l(w)}(w, \lambda)$ is the only non-zero cohomology for the diagonal chamber then $w(w \cdot \lambda)$ is a weight in $H^{l(w)}(w, \lambda)$. This proves Claim 7.5.

Claim 7.6. Under the hypothesis of Th 7.2, if $w(w \cdot \lambda)$ is a weight in $H^{l(w)}(w, \lambda)$ then $\forall \tau \leq w, \ \tau(w \cdot \lambda)$ is a weight in $H^{l(w)}(w, \lambda)$.

Proof. Let v be the vector of weight $w(w \cdot \lambda)$ in $H^{l(w)}(w, \lambda)$. As in the proof of Th. 4.5 we have a surjective B-map

$$H^{l(w)}(w_0, \lambda) \longrightarrow H^{l(w)}(w, \lambda) \longrightarrow 0. \tag{3}$$

Dualizing the above, we get a B-map

$$0 \longrightarrow H^{l(w)}(w, \lambda)^* \longrightarrow H^{l(w)}(w_0, \lambda)^*. \tag{4}$$

We identify $H^{l(w)}(w_0, \lambda)^*$ with $H^0(w_0, w \cdot \lambda)^* = V(-w_0(w \cdot \lambda))$, the Weyl module with highest weight $-w_0(w \cdot \lambda)$.

Consider the vector v^* of weight $-w(w \cdot \lambda)$ dual to v in $H^{l(w)}(w, \lambda)^*$. From the fundamental theorem of Demazure on Schubert modules, the B-span of the image of v^* in $V(-w_0(w \cdot \lambda))$ is precisely $H^0(w, w \cdot \lambda)^*$. Since the above map (4) is an injection, it follows that we have an inclusion

$$0 \longrightarrow H^0(w, w \cdot \lambda)^* \longrightarrow H^{l(w)}(w, \lambda)^*. \tag{5}$$

By the above mentioned theorem one knows that all weights of the form $\{-\tau(w \cdot \lambda), \tau \leq w\}$ are in $H^0(w, w \cdot \lambda)^*$, and hence these weights are all in $H^{l(w)}(w, \lambda)^*$, and this completes the proof.

Recall that $m(w)$ is the maximum among the coefficients of the simple roots which occur in the expression $w(\lambda) - w \cdot \lambda$. Let $s(w)$ denote the cardinality of the set $\{\tau \mid \tau \leq w\}$. Then we have the following Proposition.

Proposition 7.7. *Let $w \in W$ such that $s_\alpha \leq w$, for all simple roots α. Let λ be in the w–chamber such that $\langle w\lambda \ , \alpha \rangle > m(w)s(w)$, for all simple roots α. Then there exists a $\tau_0 \leq w$ such that $\tau_0(w \cdot \lambda)$ is not in the convex hull of the set of weights $\{\tau\lambda \mid \tau \leq w\}$.*

Proof. Consider the two sets of elements of the Weyl group $C = \{\tau w^{-1} \mid \tau \leq w\}$ and $D = \{\tau \mid \tau \leq w\}$. Since w is not equal to w_0, there is a simple root α such that $w(\alpha) > 0$. Hence $s_\alpha w^{-1}$ has length one more than w and so cannot belong to D. However the hypothesis that a reduced expression of w involves all the simple roots implies that $s_\alpha w^{-1} \in D$. Since D and C have the same cardinality it follows therefore that there is an element in $\tau_0 \in D$ which is not in C. We claim that $\tau_0(w \cdot \lambda)$ is not in the convex hull of weights in the set $\{\tau(\lambda) \mid \tau \leq w\}$.
Assume to the contrary that it does. Then there is an expression $\tau_0(w \cdot \lambda) = \sum_{\tau \leq w} c_\tau \tau(\lambda)$, with $0 \leq c_\tau \leq 1, \sum_{\tau \leq w} c_\tau = 1$. Rewriting this as

$$\tau_0(w \cdot \lambda) = \sum_{\tau \leq w} c_\tau \tau w^{-1}(w(\lambda)),$$

we get

$$\begin{aligned} w \cdot \lambda = \sum_{\tau \leq w} c_\tau \tau_0^{-1} \tau w^{-1}(w(\lambda)) = & \sum_{\phi \neq id} c_\phi \phi(w(\lambda)) \\ = \sum_{\phi \neq id} c_\phi (w(\lambda) - \sum_{\alpha} m_{\phi,\alpha} \alpha) \\ = w(\lambda) - \sum_{\phi \neq id} c_\phi (\sum_{\alpha \in S} m_{\phi,\alpha} \alpha). \end{aligned}$$

Given that there are at most $s(w)$ terms in the last summation above and the sum of the coefficients c_ϕ is 1, there is a term such that $c_{\phi_0} \geq \frac{1}{s(w)}$. Isolating this term we get

$$c_{\phi_0} \sum_{\alpha \in S} m_{\alpha,\phi_0} \alpha \leq w(\lambda) - w \cdot \lambda.$$

By the genericity of λ it can be seen that each of the coefficients m_{α,ϕ_0} is at least $m(w)S(w)$. So there is an α on the left hand side whose coefficient is at least $\frac{1}{s(w)} m(w)s(w)$ contradicting the fact that the maximum coefficient of an α occurring in an expression of $w(\lambda) - w \cdot \lambda$ is bounded by $m(w)$.

Theorem 7.8. *Let $w \in W$ be such that $s_\alpha \leq w$ (in the Bruhat order) $\forall \alpha \in S$. Let λ be a generic weight in the anti-diagonal chamber with respect to w. Suppose that*

$$\#\{j \mid H^j(w, \lambda) \neq 0\} = 1.$$

Then $w = w_0$ or equivalently $X(w) \simeq G/B$.

Again we have the following corollary whose proof is similar to that of Cor. 7.1.

Corollary 7.2. *Let $w \in W$ such that $X(w) \neq Q/B$ for any parabolic subgroup $Q \supset B$ in G. Let λ be a generic weight in the anti-diagonal weight with respect to w. Then there are at least two values of j for which $H^j(w, \lambda) \neq 0$.*

Proof. (Of Th. 7.8) The proof of this theorem is broadly along the same lines as the previous theorem although there is a *duality* phenomenon which is involved. We skip the proof.

8. Concluding remarks and a conjecture

In this section we conclude with some conjectures which we believe are interesting and non trivial. We also describe some related ongoing work.

Our main conjecture is what we call cohomological non triviality. It appears to be true in some examples in small dimensions. A proof of this conjecture would for instance imply Theorems 7.2 and 7.8. Many other results in this paper can also be deduced from this conjecture. We are working on this.

Conjecture (Cohomological non-triviality of Schubert cohomology modules) *Let $w \in W$ be an element of the Weyl group and let $\alpha \in S$ be a simple root such that $l(s_\alpha w) = l(w) + 1$. Let λ be any generic weight. If the cohomology module $H^i(w, \lambda)$ is non zero then it is* **cohomologically non-trivial** *when considered as a B_α-module. More precisely, if $H^i(w, \lambda)$ is non zero then both the $H^0(\alpha, H^i(w, \lambda))$ and $H^1(\alpha, H^i(w, \lambda))$ cannot simultaneously vanish.*

Remark 8.1. Assume that we are working in type A_l. Let α denote the middle root in the Dynkin diagram corresponding to A_l. Let P_α denote the minimal parabolic subgroup corresponding to the root α. Let $w_o(P_\alpha)$ denote the longest element of W/W_{P_α}. Suppose $w_0 = w_0(min, P_\alpha)w_o(P_\alpha)$ is a reduced expression for w_0. For small values of l it can be shown that Schubert varieties $X(w_0(min, P_\alpha))$ realize the precise cohomology bounds given by Th 4.1 and Th 4.2. So the bounds in these theorems appear to be tight.

Remark 8.2. The results in this paper hold for Schubert varieties in G/P, for other parabolic subgroups.

Remark 8.3. In the rank 2 case the study of the cohomology modules for all λ and all w is part of the doctoral thesis of K.Paramasamy (cf. [9]).

Remark 8.4. In the Kac-Moody set-up, many of the above results have been extended (cf. [7]). There is also a natural generalization of the notion of LS-paths for such cohomology modules and this is work in progress by the second and third authors.

9. Appendix

In this appendix we give a self-contained proof of a key lemma (this is a generalization of a result attributed to D. N. Verma, (cf. [8]). In this appendix alone we use the following notation: Let L_α be the Levi subgroup of the parabolic subgroup P_α as in §1 and $B_\alpha \subset L_\alpha$ the Borel subgroup containing T ; let $U \subset B_\alpha$ be the unipotent radical.

Proposition 9.1. *If V is a finite dimensional B_α–module then V is direct sum of cyclic B_α–modules each of them generated by weight vectors.*

Proof. Since we are over fields of characteristic 0, we work with the Lie algebra modules. The proof proceeds by induction on $dimV$.

Denote by y_α the nilpotent Lie algebra operator corresponding to the unipotent radical of B_α. Let n be the least positive integer for which $y_\alpha^n = 0$. Therefore, $y_\alpha^{n-1} \neq 0$. Hence, there exists a *weight vector* $v \in V$ such that $y_\alpha^{n-1}(v) \neq 0$. Let $V_1 = \langle v, y_\alpha(v), \ldots, y_\alpha^{n-1}(v)\rangle$. Then, clearly V_1 is a cyclic B_α–submodule generated by a weight vector v.

Let W be a maximal B_α–submodule of V with the property that $V_1 \cap W = (0)$.

We claim that $V = W \oplus V_1$. Then using induction we may assume that W decomposes into a direct sum of cyclic B_α submodules and complete the proof.

It remains to prove that $V = W \oplus V_1$. Clearly it suffices to show that every weight vector in V is in $W \oplus V_1$. We show this by contradiction. Let z be a weight vector, $z \notin W \oplus V_1$.

Denote by y_α, as above, the nilpotent Lie algebra operator corresponding to the unipotent radical of B_α. Since $0 = y_\alpha^n(z) \in W \oplus V_1$ and since $z \notin W \oplus V_1$, there is a smallest integer k for which $y_\alpha^k(z) \in W \oplus V_1$. Clearly $k \geq 1$ and $y_\alpha^{k-1}(z) \notin V_1 \oplus W$. *We proceed to get a contradiction to this assumption.*

Let $y_\alpha^k(z) = w_1 + v_1$ for $w_1 \in W, v_1 \in V_1$. Since $y_\alpha^k(z)$ is a weight vector and the sum $W + V_1$ is a direct sum, it follows that w_1, v_1 are both weight vectors with same weight as that of $y_\alpha^k(z)$. By the definition of V_1, we may rewrite the above expression as $y_\alpha^k(z) = w_1 + c.y_\alpha^s(v)$ for some $s \geq 0$ and some $c \in \mathbf{C}$.

Case 1: $c \neq 0$.

We claim that in fact $s \geq 1$. For otherwise we would have $y_\alpha^k(z) = w_1 + c.v$. Applying the operator y_α^{n-k} to both sides we get $0 = y_\alpha^{n-k}(w_1) + c.y_\alpha^{n-k}(v)$. Since $y_\alpha^{n-1}v \neq 0$ and since $k \geq 1$ it follows that $0 \neq c.y_\alpha^{n-k}(v) = -y_\alpha^{n-k}(w_1)$, contradicting the fact that $W \cap V_1 = (0)$.

So we have $y_\alpha^k(z) = w_1 + c.y_\alpha^s(v)$, with both k and $s \geq 1$. So we may write $w_1 = y_\alpha(y_\alpha^{k-1}(z) - c.y_\alpha^{s-1}(v))$. Let us denote the difference $(y_\alpha^{k-1}(z) - c.y_\alpha^{s-1}(v))$ by d.

Both terms of this vector namely $y_\alpha^{k-1}(z)$ and $y_\alpha^{s-1}(v)$ are weight vectors since z and v are also weight vectors. Further, applying y_α to both of them, we get vectors $y_\alpha^k(z)$, $y_\alpha^s(v)$ both of which have the same weight by an earlier remark. So in fact $d = y_\alpha^{k-1}(z) - c.y_\alpha^{s-1}(v)$ is a weight vector.

Observe that the vector $d \notin W$, in fact, $d \notin V_1 \oplus W$, otherwise we would have $y_\alpha^{k-1}(z) \in V_1 \oplus W$, contradicting the definition of k.

Now consider the subspace $W \oplus \mathbf{C}.d$. Since $y_\alpha d = w_1 \in W$ and since d is a weight vector, it follows that $W \oplus \mathbf{C}.d$ is a B_α–submodule of V. Since $d \notin W$ it follows that $(W \oplus \mathbf{C}.d) \supsetneqq W$. By our choice of W this forces that $(W \oplus \mathbf{C}.d) \cap V_1 \neq (0)$ and hence there are non-zero vectors $w \in W$,and $x \in V_1$ such that $w + c_1.d = x \in V_1$. Note that $c_1 \neq 0$ since $W \cap V_1 = (0)$. But this is a contradiction since we observed above that $d \notin W \oplus V_1$.

Case 2: If $c = 0$, following the same argument as above, we see that $y_\alpha^k(z) = w_1$, i.e $y_\alpha^k(z) \in W_1$. Since $k \geq 1$ we may express $y_\alpha^k(z) = y_\alpha(y_\alpha^{k-1}(z))$. Let $y_\alpha^{k-1}(z) = d$. Then by assumption $d \notin V_1 \oplus W$.

The rest of the argument is exactly as above.

Corollary 9.1. *Let V be an indecomposable B_α–module. Then, there exists a character $\chi : B_\alpha \longrightarrow \mathbf{G}_m$ such that $V \simeq W \otimes \chi$, with W an irreducible L_α–module.*

Proof. By the above Proposition, V is a cyclic B_α–module generated by a weight vector v. Therefore, V is the module $\langle v, y_\alpha(v), \ldots, y_\alpha^{n-1}(v)\rangle$ where $n = dim(V)$. Let μ be the highest weight (i.e weight of v) and ν be the lowest weight of V. Then we see that $\nu = \mu - (n-1)\alpha$. Let $m = \langle \mu, \alpha \rangle$ and let $\chi = (m - n + 1)\omega_\alpha$. Let $W = H^0(L_\alpha/B_\alpha, \lambda)$ where $\lambda = \mu - \chi$. Note that by the argument in the proposition above, the weight λ is a weight for the maximal torus of B_α (or equivalently B).

Then, W is an irreducible L_α–module with highest weight λ. Also, as a B_α–module, W, is isomorphic $V \otimes -\chi$. This proves the corollary.

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