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An analogue of the Narasimhan–Seshadri theorem in higher dimensions and some applications

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Abstract

We prove an analogue in higher dimensions of the classical Narasimhan–Seshadri theorem for strongly stable vector bundles of degree 0 on a smooth projective variety X with a fixed ample line bundle Θ. As applications, over fields of characteristic 0, we give a new proof of the main theorem from a recent paper by Balaji and Kollár ('Holonomy groups of stable vector bundles', *Publ. RIMS Kyoto Univ.* 44 (2008) 183–211, archiv:math.AG/06001120) and derive an effective version of this theorem; over uncountable fields of positive characteristics, if G is a simple and simply connected algebraic group and the characteristic of the field is bigger than the Coxeter index of G, we prove the existence of strongly stable principal G-bundles on smooth projective surfaces having the holonomy group of the whole of G.

Contents

1. *Introduction*

Let X be a smooth projective variety over an algebraically closed field k of arbitrary characteristic. Let Θ be an ample divisor on X. When $\dim(X) = 1$ and the ground field is $\mathbb{C}$, the Narasimhan–Seshadri theorem establishes an equivalence between the category of irreducible unitary representations of the fundamental group and the category of stable bundles with degree 0. When $\dim(X)$ is arbitrary, for bundles with all $c_i = 0$, this equivalence is proved by Mehta and Ramanathan in [**29**] using restrictions to curves, and stable bundles are realized as irreducible unitary representations of the topological fundamental group.

By an analogue of the Narasimhan–Seshadri theorem, we mean establishing an equivalence between the category of irreducible representations of a certain group scheme associated to

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X and the category of stable bundles on X when the c_i do not necessarily vanish. Such an analogue is not known even over fields of characteristic 0.

The aim of this paper is to define this category, to establish such an equivalence, and to obtain some properties and give a few applications.

To make these considerations more precise and state our first theorem, we need a few definitions. A vector bundle is said to be *strongly semistable* if all its Frobenius pull-backs are semistable (see Definition 2.2). *Over fields of characteristic* 0, the notions of strong semistability and strong stability are the usual notions of Mumford semistability and stability.

We say that a bundle E of degree 0 is *locally free graded* (*lf-graded*) if it is semistable and has a Jordan-Hölder filtration such that the successive quotients are composed of stable locally free sheaves (that is, vector bundles). In particular, each stable bundle of degree 0 is an object in this category. We say that a bundle E of degree 0 is *strongly lf-graded* if all its Frobenius pull-backs $F^l(E)$ are lf-graded (see Definition 2.5).

Our first result is the following (see Theorems 3.8 and 4.11): semistability and stability are with respect to the fixed ample divisor Θ on X.

THEOREM 1. *Let $\mathcal{C}^{\ell f}$ denote the category of strongly lf-graded bundles on X and let $x \in X$. Let $\omega_x : \mathcal{C}^{\ell f} \to \mathrm{Vect}_k$ be the fibre functor that sends any bundle $V \in \mathrm{Obj}(\mathcal{C}^{\ell f})$ to its fibre V_x. Then, the pair $(\mathcal{C}^{\ell f}, \omega_x)$ forms a neutral Tannaka category. Let $\varpi(X, x)$ be the Grothendieck–Tannaka group scheme associated to $(\mathcal{C}^{\ell f}, \omega_x)$. Then there exists a universal $\varpi(X, x)$-torsor $\mathcal{E}$ on X with the following properties.*

(1) *A representation $\rho : \varpi(X, x) \to \mathrm{GL}(n)$ is irreducible if and only if the associated vector bundle $\mathcal{E}(\rho) \simeq \mathcal{E} \times^{\varpi(X,x)} k^n$ is stable.*

(2) *A bundle V is strongly stable of degree 0 if and only if $V \simeq \mathcal{E}(\rho)$ with $\mathrm{Im}(\rho)_{\mathrm{red}}$ irreducible in $\mathrm{GL}(V)$.*

We call $\varpi(X, x)$ the *holonomy group scheme in degree* 0 of the variety X at $x \in X$ (see Definition 3.9).

A few words about the method of proof. A well-known property of strongly semistable bundles of degree 0 is that they are closed under tensor products (cf. Section 3). On the other hand, unlike the case of curves, the category of strongly semistable bundles of degree 0 but with arbitrary higher Chern classes is not an abelian category.

We show that the property of lf-gradedness is equivalent to a certain weak restriction property (WRP) of semistable bundles to smooth divisors (Definition 2.10). Using this WRP, we prove that the subcategory $\mathcal{C}^{\ell f}$ of strongly lf-graded bundles is an abelian category.

We then go on to prove that the category $\mathcal{C}^{\ell f}$ of strongly lf-graded bundle is closed under tensor products (Proposition 3.7). To prove this, we need some results from [**37**, **12**]. Our results are characteristic free and proving the tensor product theorem is a little subtle since *it is not known* if restriction of strongly semistable bundles to even general high-degree CI curves is strongly semistable, and such tensor product theorems usually force representation-theoretic bounds on the characteristic of k (cf. [**38**, **21**, **39**, **5**]).

We define the *holonomy group schemes associated to an lf-graded bundle V* as images of the *holonomy* representations ρ obtained above and denote this by $\mathcal{H}_{x,\Theta}(V)$.

We have termed our theorem an analogue of the Narasimhan–Seshadri theorem for higher-dimensional varieties but unlike the classical theorem (cf. [**33**]), where one 'understands' the category of representations of the fundamental groups, here we have established an equivalence of two categories (namely that of lf-graded bundles and the category of representations of holonomy group schemes), both of which are at present highly intriguing and pose many questions. Nevertheless, we redeem ourselves, by giving a few concrete applications of

Theorem 1, which relate the abstract holonomy groups with representations of the fundamental groups of CI curves in X.

Applications. The first application of our Theorem 1 is in Section 5. In this section we work over the ground field $\mathbb{C}$ and we give a new proof of a result by Balaji and Kollár [4]; in fact, we prove the following (effective) generalization of this result. For details, see Theorems 5.7 and 5.8.

THEOREM 2. *Let $S(r,c)$ be the set of isomorphism classes of polystable bundles E with $\deg(E)=0$ and such that $\operatorname{rank}(E)\leqslant r$ and $c_2(E)\leqslant c$. There is a number $\ell=\ell(r,c)$ such that for any $m>\ell$ and any smooth CI curve $C\in|m\Theta|$,*

$$\mathcal{H}_x(E)=\mathcal{H}_x(E|_C)\quad \forall E\in S(r,c).$$

Further, the group $\mathcal{H}_x(E|_C)$ is the Zariski closure of the Narasimhan–Seshadri representation $\pi_1(C,x)\to \mathrm{GL}(E_x)$ associated to the bundle E.

The technique of proof is different from that of [4]. The geometry goes into proving Theorem 1 and what remains is an interesting application of some representation theory of finite linear groups. We strengthen the results of [4] by making the choices of the CI curves effective and, furthermore, the result holds for *all smooth CI curves* in the linear system.

We then make some remarks on the Kobayashi–Hitchin correspondence. The Donaldson–Uhlenbeck–Yau theorem shows that a stable bundle is equipped with a canonical Einstein–Hermitian connection [42]. Using a result from Biswas [6, 7] the groups $\mathcal{H}_{x,\Theta}(E)$ can be identified with the complexification of the holonomy groups arising from the Einstein–Hermitian connection on E (see Theorem 5.11). Following Simpson we can view the above picture as a triangle.

$$\begin{array}{ccc}\{\text{Stable bundles}\} & \xrightarrow{\ D-U-Y\ } & \{E-H \text{ bundles}\} \\ & \searrow^{\text{Theorem 1}} \qquad \swarrow_{\text{Holonomy groups}} & \\ & \{\mathrm{Rep}_{\mathbb{C}}(\varpi(X))\} & \end{array} \tag{1.1}$$

When all the Chern classes of the bundles vanish, then in the third vertex we can replace $\varpi(X)$ with the fundamental group $\pi_1(X)$ as in the classical Narasimhan–Seshadri theorem. A natural question that arises from this study is, how close can one get to realizing the Einstein–Hermitian metric by the process of restriction to curves? In Donaldson's approach, the restriction to CI curves plays a key role in bounding the Donaldson–Yang–Mills functional.

When E is lf-graded but not stable, the holonomy group can be related to the classical Weil representation of the fundamental groups of curves. We make a few remarks in the context of our theorem (see Theorem 5.12) relating it to some results of Simpson [40, Corollary 3.10].

We recall that similar Tannakian constructions have been made earlier by Nori [34], where he defines the (true) fundamental group scheme of X by realizing it as the Grothendieck–Tannaka group scheme associated to the Tannaka category of *essentially finite bundles* (cf. Proposition 3.10). Nori's group scheme is a natural generalization of the algebraic fundamental group defined by Grothendieck, and coincides with it in characteristic 0. Later, when the ground field was $\mathbb{C}$, Simpson [40] studied the Tannaka category of semiharmonic Higgs bundles (hence with all Chern classes vanishing) and realized profound connections between these Tannaka group schemes and non-abelian Hodge theory. The Tannaka category we define is in a sense the natural generalization of these considerations and works over fields of all characteristics.

The second half of the paper (from Section 6 onwards) is our second application of this construction of the holonomy group scheme.

Let G be an *almost simple, simply connected algebraic group*. We give a construction of μ-stable principal G-bundles on smooth projective surfaces X; in fact we do more; we construct strongly stable bundles *with holonomy group coinciding with* G under the assumption that the second Chern class $c_2(E)$ is *large* and the characteristic of the field k is larger than the Coxeter number h_G of G ($h_G = (\dim(G)/\mathrm{rank}(G)) - 1$) and k is an uncountable algebraically closed field. This solves the existence problem for stable G-bundles in positive characteristics and shows the existence of an open subset of stable G-bundles with holonomy group G. Even over fields of characteristic 0 this result is new and throws light on the geometry of moduli space of principal bundles on surfaces (cf. [**2**, **3**]). In particular, this shows that the moduli spaces of principal sheaves constructed in [**18**] are *non-empty* when X is an algebraic surface. More precisely, we prove the following result (see Theorems 10.3 and 10.6).

Theorem 3. *Let X be a smooth projective surface over an uncountable algebraically closed field k of characteristic* $\mathrm{char}(k) > h_G$. *Let $M_X(G)^s$ denote the moduli space of isomorphism classes of stable principal bundles on X with $c_1 = 0$. Then the set of k-valued points $M_X(G)^s(k)$ is non-empty. Furthermore, there exist strongly stable principal G-bundles on X having the whole of G as holonomy group.*

The basic strategy is as follows. The first step is to get $\mathrm{SL}(2,k)$-bundles on a *general plane curve* with *full holonomy* and this involves drawing on the paper [**9**]. Using this and some deformation theory, we get such $\mathrm{SL}(2,k)$-bundles on the projective plane. Ideas from Donaldson's fundamental paper [**16**] play a key role in lifting bundles from curves to surfaces. Finally, by projecting the arbitrary surface X to the projective plane and pulling back stable bundles, we construct $\mathrm{SL}(2,k)$-bundles E on X, which are strongly stable with full holonomy. In other words, the holonomy representation obtained in the previous theorem surjects onto the structure group of the bundle. The behaviour of the holonomy group scheme under such coverings is a new feature and it plays a key role in what follows as well (see Section 6).

Then we take a principal $\mathrm{SL}(2)$ in G, the existence of which is guaranteed by an extension of Kostant's results on principal three-dimensional subgroups in semisimple groups to positive characteristics (cf. [**2**, **3**, **39**]). Using this principal $\mathrm{SL}(2)$, we extend the structure group of the $\mathrm{SL}(2,k)$-bundle E constructed above to the group G. Thus, we get the non-emptiness of the moduli of stable G-bundles over surfaces under the assumption that $p > h_G$, the Coxeter number of G. We then use deformation techniques as well as the methods developed in Section 6 to prove that there exist strongly stable G-bundles with holonomy group coinciding with G. Note that by the Kobayashi–Hitchin correspondence, over fields of characteristic 0, these correspond to *irreducible* anti-self-dual principal G-connections with full holonomy (cf. [**1**]).

When $\dim(X) = 1$, the existence of strongly stable principal G-bundles with full holonomy was shown on general curves in [**9**]. The existence of such bundles with full holonomy on an arbitrary smooth projective curve is still not known, whereas in striking contrast we have such bundles on all smooth projective surfaces. The lifting of bundles from $\mathbb{P}^2$ to arbitrary surfaces coming from techniques developed in this paper help us achieve this.

The problem of non-emptiness becomes quite involved in positive characteristics since representation theory is amenable for the study of stable bundles only if the bundles behave well under Frobenius pull-backs (see [**37**]). Or else, one may have to impose conditions on the characteristic of the field and even then the stability of the associated bundle is not to be expected. To the best of our knowledge, barring the work of Taubes on 4-manifolds [**41**], there is no other strategy of constructing stable G-bundles on higher-dimensional varieties. However, in the case when $G = \mathrm{GL}(n)$, such a non-emptiness result is shown in arbitrary characteristics in [**25**].

The problem of construction of stable G-bundles remains open for varieties of dimension at least 3. In the context of Hartshorne's conjecture (viz non-existence of stable rank 2 bundles on $\mathbb{P}^n$, for $n \geqslant 7$) this therefore could become more delicate.

Outline. The paper is organized as follows. In Section 2, we define and study the category of lf-graded bundles and prove the equivalence of a certain WRP with lf-graded property. In Section 3 we prove that the category of strongly lf-graded bundles is a neutral Tannaka category. In Section 4 we study holonomy groups in degree 0 associated to a strongly lf-graded bundle of degree 0. We then go on to characterize irreducible representations of the holonomy group schemes of projective varieties. In Section 5 we give some applications of the main theorem in characteristic 0.

In Section 6 we study genuinely ramified morphisms and stability and characterize this property in terms of holonomy group schemes. We close the section by studying the behaviour of holonomy group schemes under étale morphisms. In Sections 7 and 8 we study lifting of stable bundles from curves to surfaces. In Section 9 we construct strongly stable $\mathrm{SL}(2,k)$ bundles with full holonomy on general plane curves with the assumption that k is uncountable. In Section 10 we show the existence of principal bundles on the projective plane, which are strongly stable. We then complete the proof of the non-emptiness of the moduli space of G-bundles on an arbitrary surface. In the last section we make a number of remarks especially in the setting of characteristic 0. In particular, we study the group of connected components of the reductive holonomy group scheme. We conclude the paper with some remarks and questions. *This paper is dedicated to Madhav Nori on his sixtieth birthday.*

2. *Restriction of semistable bundles to divisors*

Let X be a smooth projective variety of $\dim(X) = d$ over an algebraically closed field k with arbitrary characteristic. When $\mathrm{char}(k) = p > 0$, we define the *Frobenius morphism* of X to be $F : X \to X$ such that $F = \mathrm{id}_{|X|}$ as a map of topological spaces and on each open set $U \subset X$, $F^* : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ takes $f \to f^p$ for all $f \in \mathcal{O}_X(U)$. In characteristic 0, we take $F = Id_X$ so that all statements are uniform across characteristics. We recall the following well-known definition.

Definition 2.1. Let Θ be a polarization on X and define the degree of a torsion-free sheaf $\mathcal{F}$ to be $\deg(\mathcal{F}) = c_1(\mathcal{F}) \cdot \Theta^{d-1}$. A torsion-free sheaf V is said to be *semistable* or *stable* if, for every subsheaf $W \subset V$, respectively,

$$\frac{\deg(W)}{\mathrm{rk}(W)} \leqslant \frac{\deg(V)}{\mathrm{rk}(V)}$$

or

$$\frac{\deg(W)}{\mathrm{rk}(W)} < \frac{\deg(V)}{\mathrm{rk}(V)}.$$

Definition 2.2. Let E be a vector bundle on X. We denote by $F^n(E)$ the bundle $(F^n)^*(E)$ obtained by the n-fold iterated pull-back of the Frobenius morphism. Define E to be *strongly semistable* or *strongly stable* if $F^n(E)$ is *semistable* or *stable* for all $n \geqslant 1$.

Remark 2.3. The above definition can be made for principal G-bundles. An easy but important fact that we use repeatedly is the following. Let E be a principal G-bundle. Let $F^r_* : G \to G$ be Frobenius homomorphism at the level of groups. Then $F^r(E) \simeq E(F^r_*(G))$, where by $E(F^r_*(G))$ we mean the associated bundle coming from the homomorphism F^r_*.

2.1. *lf-graded bundles*

We define the basic objects and give their salient properties; these will play a key role in the subsequent sections.

DEFINITION 2.4. We say that a bundle E of degree 0 is *locally free graded* (*lf-graded*) if it is semistable and has the following property: E has a filtration such that the successive quotients are composed of stable locally free sheaves of degree 0 (that is, vector bundles). In particular, each stable bundle of degree 0 is an object in this category.

EXAMPLE 2.1. It is well known that there exist μ-semistable locally free sheaves of rank 2 with Chern classes $c_1 = 0$ and $c_2 = 1$. Let $X = \mathbb{P}^2$. Then we take generic extensions

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow \mathcal{I}_x \longrightarrow 0$$

where $\mathcal{I}_x$ is the ideal sheaf of a point $x \in \mathbb{P}^2$. These extensions are classified by $\mathrm{Ext}^1(\mathcal{I}_x, \mathcal{O}) = \mathbb{C}$, and a non-trivial extension corresponds to a locally free μ-semistable sheaf that is *not* lf-graded.

DEFINITION 2.5 (The category of strongly lf-graded bundles). We say that a bundle E of degree 0 is *strongly lf-graded* if all its Frobenius pull-backs $F^l(E)$ are lf-graded. We denote by $\mathcal{C}^{\ell f}$ the category of strongly lf-graded semistable bundles of degree 0 on X.

LEMMA 2.6. *Extensions of (strongly) lf-graded bundles are (strongly) lf-graded.*

Proof. The proof is immediate from the definitions. □

We recall the effective restriction theorem from [**24**, Theorem 5.2]: let $\Delta(E) = 2rc_2 - (r - 1)c_1^2$ be the discriminant of E, $\beta_r = \beta(X, \Theta, r)$ and $m = \Theta^d$ be as in [**24**, Section 3].

THEOREM 2.7. *Let E be a torsion-free sheaf of rank $r \geqslant 2$, which is stable with respect to Θ. Let k be defined as*

$$k = \left\lfloor \frac{r-1}{r}\Delta(E) \cdot \Theta^{d-1} + \frac{1}{mr(r-1)} + \frac{(r-1)\beta_r}{mr} \right\rfloor .$$

Then $\forall a > k$, and smooth $D \in |a\Theta|$, such that $E|_D$ is torsion-free; then the restriction $E|_D$ is stable on D with respect to $\Theta|_D$.

COROLLARY 2.8. *Let E be lf-graded of rank r and degree 0. Then there exists a constant k such that, for any $a > k$, and any smooth divisor $D \in |a\Theta|$ the restriction $E|_D$ is lf-graded.*

Proof. If E is stable, then by Theorem 2.7, such a k exists. Now we induct on rank and assume that we have an exact sequence

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$$

such that E_2 is stable and E_1 is lf-graded. Then, by induction, there is a k_i for E_i; then $k = \max(k_1, k_2)$ works for E. □

REMARK 2.9. Observe that the bounds are not invariant under the Frobenius morphism, which is why a strong restriction theorem is difficult in general.

2.2. *Weak restriction property*

We examine the behaviour of semistable bundles under restrictions to complete intersection curves. This will give local criteria for lf-gradedness and enable us to prove that lf-graded bundles form an abelian category.

DEFINITION 2.10. Let E be a vector bundle on X and let $p \in X$. The triple (E, X, p) is said to have *weak restriction property* with respect to an ample divisor Θ if the following recursive properties hold.

(1) The bundle E is semistable of degree 0 with respect to Θ.

(2) Given any positive integer m, there exist an $a \geqslant m$ and a smooth divisor D, $D \in |a\Theta|$, with $p \in D$, such that the restricted triple $(E|_D, D, p)$ has WRP on D with respect to the ample divisor $\Theta|_D$.

We say that the bundle E has WRP if the triple (E, X, p) has WRP for every $p \in X$.

REMARK 2.11. Note that on curves (E, C, p) has WRP means that the bundle E is semistable on the curve C of degree 0 and $p \in C$.

The following proposition is the motivation for making this definition, and its proof will need a few lemmas.

PROPOSITION 2.12. *Let X be a smooth projective variety of dimension $d \geqslant 2$ and let E be a bundle such that (E, X, p) has WRP with respect to Θ. Suppose that we have a quotient of degree* 0

$$E \longrightarrow F \longrightarrow 0$$

with F stable and torsion-free. Then F is locally free at $p \in X$.

LEMMA 2.13. *Suppose that we have an exact sequence*

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$$

of bundles of degree 0 *such that (E, X, p) has WRP. Then so do (E_1, X, p) and (E_2, X, p).*

Proof. Immediate from the definition. □

LEMMA 2.14. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d and suppose that we have an exact sequence of modules*

$$0 \longrightarrow N \longrightarrow M \longrightarrow T \longrightarrow 0,$$

where N is a free R-module, M is torsion-free and T a torsion R-module, and such that codimension of $\operatorname{Supp}(T) \geqslant 2$. *Then, $T = 0$ and $N = M$.*

Proof. It is enough to prove this lemma when M is reflexive. To see this, we simply note that, for any torsion-free module, $M \hookrightarrow M^{**}$ is an isomorphism outside codimension at least 2.

The proof is now by induction on d. For $d = 2$, M being reflexive is therefore locally free. Now, since both N and M have same ranks, the locus where the inclusion $N \hookrightarrow M$ is not an isomorphism is a divisor, namely, the vanishing of the determinant. Since the codimension of $\operatorname{Supp}(T) \geqslant 2$, it follows that $T = 0$ and $N = M$.

Assume $d \geqslant 3$. Take a general element $x \in \mathfrak{m} \subset R$ such that the codimension of $\mathrm{supp}(T/xT) \geqslant 2$. Tensoring the exact sequence by R/x, we get that

$$0 \longrightarrow N/xN \longrightarrow M/xM \longrightarrow T/xT \longrightarrow 0$$

is exact since $\mathrm{Tor}_1(T, R/xR)$ is a torsion module and N/xN is free. By induction M/xM is free. Since M is a torsion-free R-module over a domain R, it follows that $x \in \mathfrak{m}$ is R-regular and M-regular and hence, by [**11**, Lemma 1.3.5], we have $\mathrm{proj}\ \dim_R(M) = \mathrm{proj}\ \dim_{R/xR}(M/xM)$. Since M/xM is free and $\mathrm{proj}\ \dim_R(M) = \mathrm{proj}\ \dim_{R/xR}(M/xM) = 0$, this implies that M is free. Hence, again by codimension arguments as above, $T = 0$ and $N = M$. □

PROPOSITION 2.15 (The case when X is surface). *Let X be a smooth projective surface. Let E be a bundle such that (E, X, p) has WRP with respect to Θ. Suppose that we have a quotient of degree* 0

$$E \longrightarrow F \longrightarrow 0$$

with F stable and torsion-free. Then F is locally free at $p \in X$.

Proof. Consider the composite map

$$E \longrightarrow F^{**} \longrightarrow T \longrightarrow 0,$$

where T is a torsion sheaf supported on the singular locus of F. Assume further that $p \in \mathrm{Supp}(T)$. Observe that the canonical inclusion $F \hookrightarrow F^{**}$ is an isomorphism in codimension 1, that is, the codimension of $\mathrm{Supp}(T) \geqslant 2$.

Now since F^{**} is reflexive (and therefore locally free since X is a surface) and stable, it follows that any restriction to a smooth divisor D is torsion-free. Further, by Theorem 2.7, there exists a k such that, for any $a > k$ and every smooth curve $C \in |a\Theta|$ containing p, $F^{**}|_C$ is stable. Since (E, X, p) has the WRP, we can choose C as above so that $E|_C$ is semistable. Now consider the map

$$E|_C \longrightarrow (F^{**}|_C) \longrightarrow T|_C \longrightarrow 0 \tag{2.1}$$

and let $G = \mathrm{Image}(E|_C \to F^{**}|_C)$.

Moreover, since $E|_C$ is semistable of degree 0 and $F^{**}|_C$ is stable of degree 0, it follows that G is semistable of degree 0. Since $G \hookrightarrow F^{**}|_C$ is a map between semistable bundles of same rank and degree 0, and $F^{**}|_C$ is stable, the map is an isomorphism since C is a curve. Hence, $E|_C$ surjects onto $F^{**}|_C$. Therefore $T|_C = 0$. Hence, by Nakayama lemma, $T \otimes k(p) = 0$ implying $p \notin \mathrm{Sing}(F)$, that is, F is locally free at p. □

Proof of Proposition 2.12. The proof is similar to the one given for the case when X is a surface, but there are some important differences. Therefore, at the risk of some repetition, for the sake of clarity we give it in full.

Consider the composite map

$$E \longrightarrow F^{**} \longrightarrow T \longrightarrow 0,$$

where T is a torsion sheaf supported on codimension at least 2. Assume further that $p \in \mathrm{Supp}(T)$.

Now since F^{**} is reflexive, it follows that any restriction to a smooth divisor D is torsion-free. Further, by the stability of F^{**}, by Theorem 2.7, there exists a k such that, for any $a > k$ and every smooth $D \in |a\Theta|$ containing p, $F^{**}|_D$ is stable. Since (E, X, p) has the WRP, we can choose D as above so that $(E|_D, D, p)$ has the WRP. Now consider the map

$$E|_D \longrightarrow (F^{**}|_D) \longrightarrow T|_D \longrightarrow 0 \tag{2.2}$$

and let $G = \text{Image}(E|_D \to F^{**}|_D)$. Then observe that G is torsion-free and hence, by induction on dimension and by Proposition 2.15, G is locally free at $p \in D$.

Further, since $E|_D$ is semistable of degree 0 and $F^{**}|_D$ is stable of degree 0, it follows that G is semistable of degree 0 and the inclusion $G \hookrightarrow F^{**}|_D$ is an isomorphism in codimension 1, that is, codimension of $\text{Supp}(T|_D) \geqslant 2$.

Observe that in the exact sequence

$$0 \longrightarrow G \longrightarrow F^{**}|_D \longrightarrow T|_D \longrightarrow 0$$

since G is free at $p \in D$, by Lemma 2.14, $F^{**}|_D$ is free at $p \in D$ and therefore, $(T|_D) \otimes k(p) = 0$. Hence, by Nakayama lemma, $T \otimes k(p) = 0$ on X.

Since F^{**} is torsion-free, the hypotheses of [**11**, Lemma 1.3.5] apply to F^{**} in the local ring at p. Therefore, since $F^{**}|_D$ is free at $p \in D$, this implies that F^{**} is free at $p \in X$ (see proof of Lemma 2.14 above). This together with the fact that $T \otimes k(p) = 0$, implies that F is free at $p \in X$. □

THEOREM 2.16. *The bundle E is lf-graded of degree* 0 *if and only if E has the WRP.*

Proof. Let E be semistable of degree 0, which is lf-graded. Then, by an immediate application of Lemma 2.8 and an induction on dimension, it follows that E has the WRP. In fact, it has a much stronger restriction property.

Conversely, let E be semistable of degree 0 with the WRP. Suppose that we have an exact sequence

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$$

with stable torsion-free quotient E_2. By Proposition 2.12 we see that E_2 is locally free since (E, X, p) has WRP for every $p \in X$. By Lemma 2.13 it follows that E_i have WRP and are therefore lf-graded by an induction on ranks. Hence, by Lemma 2.6, it follows that E is lf-graded. □

We then have the following key proposition.

PROPOSITION 2.17. *Let $f : V \to W$ be a map of lf-graded bundles on X. Then the map f is of constant rank. In particular,* $\ker(f)$ *and* $\text{coker}(f)$ *are lf-graded.*

Proof. The proposition is obvious when $\dim(X) = 1$, since V and W are semistable of degree 0 on a curve.

We now induct on dimension. Let $p \in X$ and let $r(p)$ be the rank of the map $f_p : V_p \to W_p$. Then, by Lemma 2.8, there exists a smooth divisor D containing p such that $V|_D$ and $W|_D$ are lf-graded. Hence, by an induction on dimension, f_p has constant rank and since X is irreducible $r(p)$ is constant everywhere. Hence both $\ker(f)$ and $\text{coker}(f)$ are locally free and by Lemma 2.13 have WRP at each point of X. The lf-gradedness of kernel and cokernel now follows from Theorem 2.16. □

3. *The Tannaka category of strongly lf-graded bundles*

In the last section we showed that $\mathcal{C}^{\ell f}$ is an abelian category. It is well-known that the tensor product of strongly semistable bundles of degree 0 is strongly semistable [**37**]. In this section we show that $\mathcal{C}^{\ell f}$ is closed under tensor products.

3.1. *Associated bundles and bounding instability degrees*

Let G be a reductive algebraic group over k. Recall that a principal G-bundle E over X is said to be *μ-semistable* (respectively *μ-stable*) if for all parabolic subgroups Q of G, for all reductions $\sigma_Q : U \longrightarrow E(G/Q)$, on *big open* subsets $U \subset X$ (that is, codimension of $X - U \geqslant 2$) and for all ample line bundle L on G/Q, the degree of the bundle $\sigma_Q^* E(L)) \geqslant 0$ (respectively > 0), where the degree is computed using the polarization Θ.

We remark that it suffices to check the conditions for maximal parabolic Q in which case, since $\mathrm{Pic}(G/Q) = \mathbb{Z}$, the choice of L is canonical.

For a section $\sigma : U \to E(G/Q)$ we shall henceforth denote by $\deg(\sigma)$ the number $\deg\ \sigma^*(E(L))$. For $\mathrm{GL}(n)$ this definition coincides with the one for bundles given in Definition 2.1.

We recall the following notations from [**37**].

Let K be a field. Let G be a connected reductive group over K that acts on a projective K-variety M. Let m be a K-rational point of M that is not semistable. Let $P(m)$ be its Kempf instability parabolic defined over the algebraic closure $\overline{K}$. Recall that by the canonical nature of the Kempf parabolic, if it is defined over the separable closure K_s, then it is already defined over K. Therefore, $P(m)$ is always defined over a purely inseparable extension of K.

We now recall the definition of the K_s-scheme $M(P)$, the $\overline{K}$-valued points of which are precisely points of the orbit $O_G(m)$ with associated Kempf parabolic $P(m)$ (see [**37**, Lemma 2.4]). Observe that if $M(P)$ has an L-valued point for a purely inseparable extension L/K, then $P(m)$ is defined over L.

LEMMA 3.1. *Let E be a semistable G-bundle. Let $\rho : G \to H$ be a representation defined over k with connected component of the centre mapping to the centre. Let $P \subset H$ be a maximal parabolic subgroup and $x_0 \in C$ denote the generic point.*

(1) *If σ is a section of $E(H/P)$ such that $\sigma(x_0)$ is semistable with respect to the ample L, then* $\deg(\sigma) \geqslant 0$.

(2) *If σ is a section of $E(H/P)$ such that $\sigma(x_0)$ is unstable, then there exists a positive integer $N(\sigma)$ such that if $F^N(E)$ is assumed to be semistable for any $N \geqslant N(\sigma)$, then* $\deg(F^N(\sigma)) \geqslant 0$. *Hence* $\deg(\sigma) = (1/p^N)\deg(F^N(\sigma)) \geqslant 0$.

Proof. Part (1) is simply [**37**, Proposition 3.10(i)]. Part (2) follows by combining [**37**, Proposition 3.13] and [**37**, Theorem 3.23], where the number $N(\sigma)$ is precisely the pure inseparability degree of the extension L/K_s where the scheme $M(P)$ is defined over L. □

Now we recall the following boundedness result from [**12**, Proposition 4.5].

PROPOSITION 3.2. *There exists an integer N such that, for any K_s-rational point m of M, which is not semistable, the instability flag $P(m)$ is defined over $K_s^{p^{-N}}$.*

3.2. *lf-graded property of associated bundles*

We begin with a few notations. Let $K = k(t)$, the rational function field in one variable. Let K_s be its separable closure and $\overline{K}$ be the algebraic closure. Observe that, for any smooth projective curve C defined over k, the separable closure $k(C)_s$ can be identified with K_s by choosing a finite separable map from C onto $\mathbb{P}^1$.

Definition 3.3. Let G be an affine algebraic group and let E be a principal G-bundle. We say that E has *degree* 0 if, for every character $\chi : G \to \mathbb{G}_m$, the associated line bundle $E(\chi)$ has degree 0 (with respect to the polarization Θ).

Remark 3.4. Let G be an affine algebraic group and let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional G-module. Let E be a principal G-bundle of degree 0. Then the associated vector bundle $E(V)$ has degree 0. Furthermore, we can decompose $V = \oplus V_i$ such that $\rho = \oplus \rho_i$ and $\rho_i : G \to \mathrm{GL}(V_i)$ maps the centre of G to the centre of $\mathrm{GL}(V_i)$.

Theorem 3.5. *Let $\rho : G \to H$ be a representation defined over k with the connected component of the centre mapping to the centre. Then, associated to this representation ρ, there exists a positive integer $l = l(\rho, K_s)$ with the following property: let C be any smooth projective curve, and let E be a G-bundle of degree* 0 *on C. Then the associated H-bundle $E(\rho)$ is semistable of degree* 0 *whenever the lth Frobenius pull-back $F^l(E)$ is* semistable.

Proof. Let E be a semistable G-bundle on C. For $E(\rho)$ to be semistable, we need to check that, for any parabolic $P \subset H$ and ample L on H/P and any section $\sigma : C \to E(H/P)$, we must have $\deg(\sigma) \geqslant 0$, where $\deg(\sigma) = \deg(\sigma^*(E(L)))$.

Now consider $\sigma(x_0) = m$ as a $k(C)$-rational point of $E(H/P)_{k(C)} = M$. Then, by Lemma 3.1, there are two possibilities. If m is a semistable point for the action of $G_{k(C)}$ on M, then $\deg(\sigma) \geqslant 0$. On the other hand if m is an unstable point in M, then the Kempf parabolic $P(m)$ is defined over $K_s^{p^{-N}}$ by Proposition 3.2, where N is dependent only on ρ and K_s and independent of the $k(C)$-rational point m and C. By the canonical nature of $P(m)$ this implies by Galois-descent that it is defined over $k(C)^{p^{-N}}$.

Now take $l = N$ and assume that the Frobenius pull-back $F^l(E)$ is semistable. Then the degree of the section $\deg F^l(\sigma) \geqslant 0$. This proves that $\deg(\sigma) = (1/p^l)\deg(F^l(\sigma)) \geqslant 0$.

This implies that if we assume that $F^l(E)$ is semistable, then $E(\rho)$ is semistable. □

3.3. *The tensor category*

Lemma 3.6. *Let $\rho : \mathrm{GL}(V) \to \mathrm{GL}(W)$ be a representation defined over k. Then there exists a positive integer l with the following property: for any* $\mathrm{GL}(V)$*-bundle E of degree* 0 *on X, if the Frobenius pull-back $F^l(E)$ is lf-graded, then the associated* $\mathrm{GL}(W)$*-bundle $E(\rho)$ is also lf-graded.*

Proof. Choose $l = l(\rho)$ as in Theorem 3.5. Observe that, by Remark 3.4, we may assume that ρ maps the connected component of the centre to the centre.

We prove this by an induction on dimensions. Let $\dim(X) = 2$. We note that since $F^l(E)$ is lf-graded, by Theorem 2.7, there exists an m such that, for all a, with $a > m$, and for any smooth curve $C \in |a\Theta|$, the bundle $F^l(E)|_C$ is semistable of degree 0.

Hence, by Theorem 3.5, $E(\rho)|_C$ is semistable of degree 0 for all such curves and hence $E(\rho)$ has WRP. Thus, by Theorem 2.16, $E(\rho)$ is lf-graded on the surface X.

Now let $\dim(X) = d$ be arbitrary. Then, by Lemma 2.8, since $F^l(E)$ is assumed to be lf-graded, there exists an m such that, for all $a > m$ and smooth divisors $D \in |a\Theta|$, the restriction $F^l(E)|_D$ is lf-graded. Hence by induction on dimension, $E(\rho)|_D$ is lf-graded for all such divisors; that is $E(\rho)$ has WRP. This implies by Theorem 2.16 that $E(\rho)$ is lf-graded. □

Proposition 3.7. *Let E be a strongly lf-graded bundle on X. Then for any representation $\rho : \mathrm{GL}(V) \to \mathrm{GL}(W)$ the bundle $E(\rho)$ is strongly lf-graded.*

Proof. Observe that, for any n, $F^n(E(\rho))$ is also obtained by a representation $F^n(\rho) : H \to G$ (by composing ρ with the Frobenius power map; see Remark 2.3). Since E is strongly lf-graded, we now choose $l = l(F^n(\rho))$ as in Theorem 3.5. Then since $F^l(E)$ is also lf-graded, by Lemma 3.6 it follows that $F^n(E(\rho))$ is lf-graded. This implies that $E(\rho)$ is strongly lf-graded. □

THEOREM 3.8. *Let $\mathcal{C}^{\ell f}$ denote the category of strongly lf-graded bundles of degree* 0 *on X as in Definition* 2.5. *Fix a point $x \in X$. Then the category $(\mathcal{C}^{\ell f}, \omega_x)$, where $\omega_x : \mathcal{C}^{\ell f} \to \mathrm{Vect}_k$ is the evaluation map at $x \in X$, is a neutral Tannaka category.*

Proof. The category $\mathcal{C}^{\ell f}$ is closed under tensor products. This follows by Proposition 3.7. $\mathcal{C}^{\ell f}$ is an abelian category. This is Proposition 2.17. □

DEFINITION 3.9 (Holonomy group scheme in degree 0). We define the holonomy group scheme in degree 0 of X to be the Grothendieck–Tannaka group scheme $\mathrm{Aut}^{\otimes}(\mathcal{C}^{\ell f})$ associated to the Tannaka category $(\mathcal{C}^{\ell f}, \omega_x)$, and we denote it by $\varpi(X, x, \Theta)$.

The *true fundamental group scheme* in the sense of Nori is the Tannaka group scheme associated to the category $\mathcal{N}$ of *essentially finite* bundles on X (see [**34**] and see [**14**, 2.34, p. 146] for the nomenclature).

PROPOSITION 3.10. *The true fundamental group scheme $\pi_1^{\mathrm{true}}(X, x)$ is a quotient of $\varpi(X, x.\Theta)$ for any Θ. More precisely, we have a faithfully flat morphism $q : \varpi(X, x, \Theta) \to \pi_1^{\mathrm{true}}(X, x)$.*

Proof. We have a canonical functor $F : \mathcal{N} \to \mathcal{C}^{\ell f}$ given in the obvious way since any *essentially finite* bundle in the sense of Nori is strongly lf-graded. In fact, any essentially finite bundle has WRP by the strong definition of semistability given in [**34**]. The functor is fully faithful. So we need to check only that any subobject of an essentially finite bundle within the lf-graded category is essentially finite. Now any subbundle or quotient bundle of degree 0 of an essentially finite bundle is essentially finite by [**34**, Proposition 3.7] and we are done by the criterion for faithful flatness of morphisms of group schemes by [**14**, Proposition 2.21], or [**34**, Proposition 5, Appendix] (see also the proof of Theorem 6.10). □

REMARK 3.11. It is known that the maximal pro-étale quotient of $\pi_1^{\mathrm{true}}(X, x)$ is the usual étale fundamental group $\pi_1^{\mathrm{\acute{e}t}}(X, x)$ (see [**34**] or [**14**]).

REMARK 3.12. We observe that the Frobenius morphism induces a morphism of holonomy groups: at the level of categories we have $F^* : \mathcal{C}^{\ell f} \to \mathcal{C}^{\ell f}$ given by $V \mapsto F_X^*(V)$ and this gives the map $F_X^* : \varpi(X) \to \varpi(X)$.

4. *Holonomy group scheme of a strongly lf-graded bundle*

In this section we define and consider representations of the holonomy group scheme associated to strongly lf-graded bundles. We then go on to prove the analogue of the Narasimhan–Seshadri theorem.

LEMMA 4.1. *Let V be a strongly lf-graded vector bundle. Let $(\mathcal{C}(V), \omega_x)$ be the pair where $\mathcal{C}(V)$ is the subcategory of $\mathcal{C}^{\ell f}$ generated by vector bundles of degree 0 arising as subquotients of $\oplus_{a,b} T^{a,b}(V)$, where $T^{a,b}(V) := V^{\otimes a} \otimes (V^*)^{\otimes b}$. Then $(\mathcal{C}(V), \omega_x)$ is a neutral Tannaka category.*

Proof. By Proposition 2.17, $\mathcal{C}(V)$ is a *full subcategory* of $\mathrm{Vect}(X)$, which is closed under tensor products and is also an abelian subcategory of $\mathcal{C}^{\ell f}$. Hence it is a neutral Tannaka category. □

DEFINITION 4.2. Following Nori, we define a strongly lf-graded principal G-bundle as arising from a functor $F : \mathrm{Rep}_k(G) \to \mathcal{C}^{\ell f}$ satisfying the axioms [**34**, p. 77], that is, F is a strict, exact, faithful tensor functor (cf. [**40**]).

DEFINITION 4.3 (Holonomy group scheme of a bundle). Let $\mathcal{H}_{x,\Theta}(V)$ denote the associated Grothendieck–Tannaka group scheme to the category $(\mathcal{C}(V), \omega_x)$. We call $\mathcal{H}_{x,\Theta}(V)$ the holonomy group scheme of the bundle V. Let E be a strongly lf-graded principal G-bundle on X. Then we can define the holonomy group scheme $\mathcal{H}_{x,\Theta}(E)$ associated to E as follows: fix a faithful representation $\rho : G \hookrightarrow \mathrm{GL}(V)$. Define $\mathcal{H}_{x,\Theta}(E) := \mathcal{H}_{x,\Theta}(E(V))$.

REMARK 4.4. Let V be a strongly lf-graded vector bundle and let E be the underlying principal $\mathrm{GL}(V_x)$-bundle. Observe that E gives rise to the natural functor $F_E : \mathrm{Rep}_k(\mathrm{GL}(V_x)) \to \mathcal{C}^{\ell f}$. It is clear that this functor factors through the category $(\mathcal{C}(V), \omega_x)$. The functor F_E immediately gives rise to a morphism of group schemes $\varpi(X, x.\Theta) \to \mathrm{GL}(V_x)$ and one can see that the image is precisely the group scheme $\mathcal{H}_{x,\Theta}(V)$. Thus we have an inclusion $\mathcal{H}_{x,\Theta}(V) \subset \mathrm{GL}(V_x)$. More generally, let E be a strongly lf-graded principal G-bundle. Then one can see in a similar manner that we have an inclusion $\mathcal{H}_{x,\Theta}(E) \subset G$.

REMARK 4.5. Note that by the Tannakian definition of the holonomy group scheme, the structure group of the underlying principal bundle of an lf-graded vector bundle V can always be reduced to $\mathcal{H}_{x,\Theta}(V)$.

REMARK 4.6. As in Remark 3.12, we again observe that the Frobenius morphism induces a morphism of holonomy groups of bundles: at the level of categories we have $F^* : \mathcal{C}(V) \to \mathcal{C}(F^*(V))$ given by $W \mapsto F_X^*(W)$ and this gives the map $F_X^* : \mathcal{H}_{x,\Theta}(F^*(V)) \to \mathcal{H}_{x,\Theta}(V)$. Let $F^* : \mathrm{GL}(V) \to \mathrm{GL}(V)$ be the Frobenius homomorphism of groups. Then as we have seen in Remark 2.3, $\mathcal{H}_{x,\Theta}(F^*(V)) \simeq F^*(\mathcal{H}_{x,\Theta}(V))$. Further, being subgroup schemes of $\mathrm{GL}(V)$, these are finite-type group schemes and hence after sufficiently many Frobenius pull-backs, we get $\mathcal{H}_{x,\Theta}(F^l(V)) \simeq \mathcal{H}_{x,\Theta}(V)_{\mathrm{red}}$.

It is now fairly standard (see [**14**]) to show that the way we have described $\mathcal{H}_{x,\Theta}(E)$ in Definition 4.3 is independent of the choice of the G-module V. By the choice of the base point one can non-canonically identify $\mathcal{H}_{x,\Theta}(E)$ with a subgroup of G. In fact, with some amount of work, one could possibly characterize $\mathcal{H}_{x,\Theta}(E)$ as the 'smallest subgroup scheme' to which the structure group of E can be reduced *admissibly*, that is, preserving the degree 0 property.

DEFINITION 4.7. We say that a strongly lf-graded G-bundle E has full holonomy if the holonomy group scheme $\mathcal{H}_{x,\Theta}(E) \subset G$ is the whole group G itself.

LEMMA 4.8. *A principal G-bundle E is strongly stable with full holonomy if and only if $E(V)$ is strongly stable for every irreducible G-module V. In fact, if $G \to G'$ is an irreducible homomorphism (that is, the image does not lie in any parabolic subgroup), then every strongly stable G-bundle with full holonomy induces a strongly stable G' bundle.*

LEMMA 4.9. *Let E be a strongly lf-graded bundle on X. Then E is strongly stable if and only if the reduced holonomy group $\mathcal{H}_{x,\Theta}(E)_{\rm red} \subset \mathrm{GL}(V)$ is an irreducible subgroup (that is, it does not lie in any parabolic subgroup).*

Proof. The proofs of both these lemmas are identical. Assume that $E(V)$ is strongly stable. By repeated Frobenius pull-backs, it is not hard to see that we may assume that $\mathcal{H}_{x,\Theta}(E)$ is *reduced* (by Remark 4.6). Indeed, the holonomy group scheme for a high Frobenius pull-back is the reduced holonomy group scheme of $E(V)$. Suppose that $\mathcal{H}_{x,\Theta}(E) \subset \mathrm{GL}(V)$ is not irreducible. Then there exists an $\mathcal{H}_{x,\Theta}(E)$-submodule of $E(V)_x$, which by the definition of $\mathcal{H}_{x,\Theta}(E)$ gives a subbundle of $E(V)$ of degree 0, contradicting the stability of $E(V)$.

Conversely, suppose that $\mathcal{H}_{x,\Theta}(E)_{\rm red} \subset \mathrm{GL}(V)$ is *irreducible*. Suppose that $E(V)$ is strongly lf-graded but not strongly stable.

So assume that there exists a stable torsion-free quotient of $F^*(E(V))$ of degree 0. This gives rise to a subbundle $F^*(E(V))$ of degree 0 since the quotient is locally free by Proposition 2.12. This gives an $\mathcal{H}_{x,\Theta}(E)$-submodule of V, contradicting the irreducibility of $\mathcal{H}_{x,\Theta}(E) \hookrightarrow \mathrm{GL}(V)$. □

4.1. *Stable bundles and irreducible representations of the holonomy group scheme*

We consider vector bundles on smooth projective varieties with $\deg_\Theta(V) = 0$ and arbitrary higher Chern classes. We first have the following lemma.

LEMMA 4.10. *There exists a universal $\varpi(X, x, \Theta)$-torsor $\mathcal{E}$ on X.*

Proof. Consider the following functor:

$$\mathrm{Rep}_k(\varpi(X, x, \Theta)) \simeq (\mathcal{C}^{\ell f}, \omega_x) \hookrightarrow \mathrm{Vect}_X,$$

coming from the natural *forget* functor $(\mathcal{C}^{\ell f}, \omega_x) \to \mathrm{Vect}(X)$. This composite functor satisfies the axioms of a fibre functor given in [**34**], implying that we have a universal $\varpi(X, x, \Theta)$-torsor $\mathcal{E}$ on X. □

THEOREM 4.11. *A vector bundle V of rank n, with $\deg_\Theta(V) = 0$, is strongly lf-graded if and only if it arises as $\mathcal{E}(\eta)$, for a rational representation*

$$\eta : \varpi(X, x, \Theta) \longrightarrow \mathrm{GL}(n).$$

Further, $V \simeq \mathcal{E}(\eta)$ is stable if and only if η is an irreducible representation. Moreover, V is strongly stable if and only if all Frobenius pull-backs $(F^n)^*(\eta)(= \eta \circ (F_X^n)^*$ as in Remark 3.12) are irreducible representations.

Proof. It is immediate from the definition of the category $(\mathcal{C}^{\ell f}, \omega_x)$ that given any lf-graded bundle V, we have an inclusion of categories $\mathcal{C}^{\ell f}(V) \subset \mathcal{C}^{\ell f}$ inducing the representation η. Hence any such V arises as $\mathcal{E}(\eta)$, where $\mathcal{E}$ is as in Lemma 4.10, and conversely.

We need to show that last statement about strong stability. We first observe that, for every $\eta : \varpi(X, x, \Theta) \to \mathrm{GL}(n)$, the image $\mathrm{Im}(\eta)$ can be identified with the holonomy group $\mathcal{H}_{x,\Theta}(\mathcal{E}(\eta))$. Note that this is a finite-type group scheme (being a subgroup of $\mathrm{GL}(n)$). The claim in the theorem now follows from Lemma 4.9. □

5. *Relationship with the Narasimhan–Seshadri theorem*

In this section we give the first application of Theorem 4.11 when the ground field is $\mathbb{C}$. The first application gives the main theorem of [**4**] as a corollary of Theorem 4.11. In fact, we give an effective version of the theorem proved in [**4**].

We then make some remarks on the Kobayashi–Hitchin correspondence. The Donaldson–Uhlenbeck–Yau theorem shows that a stable bundle is equipped with a canonical Einstein–Hermitian connection. The algebraic holonomy groups $\mathcal{H}_x(E)$ can be identified with the complexification of the holonomy groups arising from the Einstein–Hermitian connection on E. We conclude the section with a few remarks on the holonomy groups of semistable (non-polystable) bundles on curves and the relation with a result of Simpson, which makes our group scheme more amenable for applications.

5.1. *Stable bundles in characteristic* 0

Let V be polystable of degree 0. In [**4**], by the process of *restriction to high degree curves*, the notion of an *algebraic holonomy group* of V was defined with some characterizing properties. The aim here is to give a completely new proof of the main theorem of [**4**]; in fact we get an effective version of this result (see Theorem 5.8) as a consequence of Theorem 4.11. Philosophically, the issues in addressing this result involving geometry are already taken into the proof of Theorem 4.11. So it is natural to expect that the remaining aspects of the proof should essentially arise out of representation-theoretic considerations. We show that this is indeed the case.

Recall the following classical theorem by Jordan [**22**, p. 114] (see also [**27**] as well as [**43**] where there is an explicit $J(r) = (r + 1)!r^{a\ (\log r)+\ b}$ for suitable constants a, b.)

THEOREM 5.1 [**22**]. *There is a universal constant $J(r)$ such that if $F \subset \mathrm{GL}_r$ is any finite subgroup, then F has an abelian normal subgroup $N \triangleright F$ with $|F/N| \leqslant J(r)$.*

Proof (see [**13**, Chapter 5, Section 36] for details). Since F is finite, we can embed F in the subgroup U_r of unitary matrices. It is ingenious then to show that the subgroup N generated by matrices $\{A \mid ||A - I|| < 1/2\}$ is abelian, where $||A|| = \mathrm{trace}(A\overline{A}^t)$ is the standard norm on $M_r(\mathbb{C})$. It is easy to see that $N \triangleright F$.

If $A_1, A_2, \ldots A_n$ are the co-set representatives of F/N, then $||A_i - A_j|| \geqslant 1/2$, if $i \neq j$. Represent A as a point in $\mathbb{R}^{2r^2}$. Then $||A||^{1/2} = d(0, A)$ in $\mathbb{R}^{2r^2}$. Since each A_i is in U_r, it follows that $d(0, A_i) = \sqrt{r}$. Hence, the A_i can be seen as points on the surface of spheres with radius $\sqrt{r}$. Since $||A_i - A_j|| \geqslant 1/2$, it follows that $d(0, A_i - A_j) \geqslant 1/\sqrt{2}$ so that small spheres of radius $1/\sqrt{8}$ about the A_i are non-overlapping. Placing these spheres in a shell defined by two spheres of radius $\sqrt{r} \pm 1/\sqrt{8}$ and comparing volumes, one can check that the number of such points is bounded in terms of r. The bound using this argument for $J(r)$ is $(\sqrt{8r} + 1)^{2r^2} - (\sqrt{8r} - 1)^{2r^2}$ and is due to Schur. □

PROPOSITION 5.2. *Let $H \subset \mathrm{GL}(r)$ be a reductive subgroup. For any H-module V with $\dim(V) > J(r)$, V is not an irreducible module for any finite subgroup $F \subset H$.*

Proof. Let V be an irreducible representation of H such that $\dim(V) > J(r)$. Then by [**39**, Corollary, p. 62], since $\dim(V) > |F/N|$, it follows that V is not an irreducible representation of F for any finite subgroup $F \subset H$. □

LEMMA 5.3. *Let H be a reductive irreducible subgroup of $\mathrm{GL}(V)$ with $\dim(V) = r$ and let H^o denote the connected component of identity in H. Assume that $D = \mathcal{D}(H^o)$, the derived group of H^o, is non-trivial. Then the symmetric power $\mathrm{Sym}^{J(r)}(V)$, considered as an H-module, contains an irreducible summand of dimension at least $J(r)$.*

Proof. Observe that $D \subset H$ is a non-trivial connected semisimple group. Then V is a D-module and contains an irreducible submodule W with $\dim(W) > 1$. Then, there is a dominant λ on a Borel subgroup of D such that $W \simeq H^0(D/B, L(\lambda))$. Hence it follows that $H^0(D/B, L(J(r)\lambda)) \subset \mathrm{Sym}^{J(r)}(W)$ is a D-submodule and $\dim(H^0(D/B, L(J(r)\lambda))) \geqslant J(r)$. It follows immediately that even as an H-module, $\mathrm{Sym}^{J(r)}(V)$ contains the required irreducible submodules of dimension at least $J(r)$. □

Let E be a stable bundle and let $H = \mathcal{H}_x(E)$ denote the holonomy group of E. Similarly, let $H_C = \mathcal{H}_x(E|_C)$ whenever $E|_C$ is semistable. Observe that by definition $H_C \subset H$.

LEMMA 5.4. *Let E be a stable bundle such that H is a finite group, that is, $H^o = (1)$. Then, for any smooth ample curve $C \subset X$, we have $H_C = H$.*

Proof. Consider $Y = E_H$ the H-reduction. Then Y is a connected étale cover of X with Galois group H. Since C is an ample curve in X, it follows that $E_H|_C$ is also a connected cover of C. On the other hand, the holonomy group of this finite bundle is H_C. Hence the connectedness of $E_H|_C$ implies that $H = H_C$. □

LEMMA 5.5. *Let E be a stable bundle and suppose that $Z(H)^o \neq 1$. Let $C \subset X$ be an ample curve such that $E|_C$ is stable. Then $Z(H_C) = Z(H)$.*

Proof. Observe that $H \subset \mathrm{GL}(E_x)$ is an irreducible subgroup. Thus, if $Z(H)^o \neq 1$ by Schur's lemma, $Z(H) = Z(\mathrm{GL}(E_x))$, that is, *the scalars*, are precisely the centre of H. In this case, $Z(H) \subset H^o$.

On the other hand, characters of H are precisely line bundles of degree 0 in the Tannaka category $\mathcal{C}(E)$. Now a degree 0 non-torsion line bundle on X restricts to a non-torsion line bundle on C by Lefschetz Theorem, since $\mathrm{Pic}(X) \to \mathrm{Pic}(C)$ is injective. This gives non-torsion degree 0 line bundles in the Tannaka category $\mathcal{C}(E|_C)$, that is, a non-torsion character of H_C. But if $Z(H_C)^o = 1$, then since H_C^o has no characters, all characters of H_C will be torsion. Hence, $Z(H_C)^o \neq 1$.

Again since $E|_C$ is also stable, by the reasoning above, it follows that $Z(H_C) =$ scalars and hence $Z(H_C) = Z(H)$. □

Recall from Theorem 2.7 the number $k(E)$ that we term the Bogomolov index of E:

$$k(E) = \left\lfloor \frac{r-1}{r}\Delta(E)\cdot\Theta^{d-1} + \frac{1}{mr(r-1)} + \frac{(r-1)\beta_r}{mr} \right\rfloor.$$

Proposition 5.6. *Let E be a stable* $\mathrm{GL}(V)$*-bundle with* $H = \mathcal{H}_x(E)$*. Assume that* $D = \mathcal{D}(H^o) \neq 1$*. Let* $W = \mathrm{Sym}^{J(r)}(V)$ *and let* $E(W) = \bigoplus A_j$ *be a decomposition into stable bundles. Let* $k_0 = \max_j k(A_j)$ *and let* $\ell > k_0$*. Let* $C \in |\ell\Theta|$ *be a smooth curve. Then* $H_C^o \neq 1$*.*

Proof. Observe that, by Lemma 5.3, there is a summand $W_0 \subset \mathrm{Sym}^{J(r)}(V)$, with $\dim(W_0) > J(r)$ and such that W_0 is an irreducible H-module.

By the assumption on the Bogomolov indices k_0, $E(W_0)|_C$ is stable. This implies that W_0 as an H_C-module is irreducible. By Proposition 5.2 this implies that $H_C^o \neq 1$. □

Theorem 5.7. *Let X be a smooth projective variety and E be a stable* $\mathrm{GL}(V)$*-bundle of rank r. Let* $V_1 = \mathrm{End}(V)$ *and* $V_2 = S^{J(r)}(V)$*. Let* $E(V_1) = \bigoplus_j W_j$ *and* $E(V_2) = \bigoplus_j A_j$ *be the decomposition of* $E(V_i)$*, where* $i = 1, 2$ *with* W_j *and* A_j *as stable bundles. Let ℓ be bigger than the maximum of the Bogomolov indices for all the* W_j *and* A_j*. Let C be a smooth curve in* $|\ell\Theta|$*. Then* $\mathcal{H}_x(E) = \mathcal{H}_x(E|_C)$*.*

Proof. By Lemma 5.4 we have the theorem when $H^o = 1$, that is, H is finite.

If $H^o \neq 1$ but $\mathcal{D}(H^o) = 1$, that is, $H^o = Z(H)$, then $H/Z(H) = F$ for a finite group F. By Lemma 5.4 and Lemma 5.5 it follows that $H = H_C$ in this case.

Let E be stable such that $\mathcal{D}(H^o) \neq 1$. Let C be so chosen in $|\ell\Theta|$, with ℓ as in the theorem. Then, by Proposition 5.6, if $\mathcal{H}_x(E|_C) = H_C$, then $H_C^o \neq 1$.

Consider the adjoint group $H' = H/Z(H)$. Then $\mathrm{Lie}(H)$ is an irreducible H'-module since it is an irreducible $H^o/Z(H)$-module.

Consider the polystable bundle $E(V_1)$. Note that by the stability of E, we have the inclusion $E(\mathrm{Lie}(H)) \subset E(V_1)$ as a stable summand and hence $E(\mathrm{Lie}(H)) = W_j$ for some j. Restricting to C, we get $E|_C(\mathrm{Lie}(H)) = W_j|_C$.

By the same reasoning, observe that $\mathrm{Lie}(H_C) \subset \mathrm{Lie}(H)$ is an irreducible submodule for the action of H_C. Note that $\mathrm{Lie}(H_C) \neq (0)$ since $H_C^o \neq 1$.

Thus, it follows that $E|_C(\mathrm{Lie}(H_C)) \subset E|_C(\mathrm{Lie}(H)) = W_j|_C$ is a non-trivial stable subbundle and by the choice of C, we know that $W_j|_C$ is also stable. This implies that $E|_C(\mathrm{Lie}(H_C)) = E|_C(\mathrm{Lie}(H))$, that is, $\mathrm{Lie}(H_C) = \mathrm{Lie}(H)$. Hence $H^o = H_C^o$.

Now by considering the associated H/H^o-bundle $E_H(H/H^o)$ (a finite bundle), by Lemma 5.4, we get that $H = H_C$. □

We can summarize the above results in the following effective result. Define

$$\ell(r,c) = \left\lfloor \frac{t-1}{r}\Delta \cdot \Theta^{d-1} + \frac{1}{mt(t-1)} + \frac{(t-1)\beta_t}{mt} \right\rfloor$$

with $\Delta = 2tc$ and $t = \mathrm{rank}(\mathrm{Sym}^{J(r)}(E))$, and $J(r)$ is as in (5.1) above.

Theorem 5.8. *Let $S(r,c)$ be the set of isomorphism classes of polystable bundles E with* $c_1(E) = 0$ *and such that* $\mathrm{rank}(E) \leqslant r$ *and* $c_2(E) \leqslant c$*. Let* $\ell = \ell(r,c)$*. Then, for any* $m > \ell$ *and any smooth curve* $C \in |m\Theta|$, $\forall E \in S(r,c)$,

$$\mathcal{H}_x(E) = \mathcal{H}_x(E|_C).$$

Further, the groups $\mathcal{H}_x(E|_C)$ *are the Zariski closures of the Narasimhan–Seshadri representation* $\pi_1(C,x) \to \mathrm{GL}(E_x)$ *associated to the bundle E.*

REMARK 5.9. We observe that all the results proved above really do not use the ground field of complex numbers except in interpreting the holonomy group as Zariski closures of the Narasimhan–Seshadri representations (cf. [**33**]).

REMARK 5.10. One can interpret the above theorem as saying that any bounded collection of representations of $\varpi(X, x, \Theta)$ with reductive images can be realized as Zariski closures of the Narasimhan–Seshadri representations of $\pi_1(C, x)$.

5.2. *The Kobayashi–Hitchin correspondence*

Let E be a polystable bundle on X. By [**42**], E admits a unique Einstein–Hermitian connection. Assume for simplicity that $\det(E) = \mathcal{O}_X$. Let ∇ be the unique Einstein–Hermitian connection on E. Let $\mathrm{SU}(r) \subset \mathrm{SL}(r)$ be the unitary subgroup and let $E_{\mathrm{SU}} \subset E$ be a C^∞-reduction of the structure group of E to $\mathrm{SU}(r)$ so that the corresponding connection on E_{SU} gives ∇ on E. This reduction to $\mathrm{SU}(r)$ is also unique. Let the C^∞-connection on E_{SU} be denoted by ∇^{SU}. Using this connection, by parallel transport at $x \in X$, one can define a Lie subgroup $K \subset \mathrm{SU}(r)$, namely, the *smooth holonomy* group of E_{SU} together with a smooth reduction of the structure group of E_{SU} to K (see [**23**]).

We then have the following theorem (cf. [**7**]).

THEOREM 5.11 (Biswas). *Let E be stable and assume that $\det(E) \simeq \mathcal{O}_X$. Let $\overline{K} \subset \mathrm{SL}(r)$ be the Zariski closure, in $\mathrm{SL}(r)$, of $K \subset \mathrm{SU}(r)$. Then we have $\mathcal{H}_x(E) \simeq \overline{K}$. In other words, $\mathcal{H}_x(E)$ is indeed the complexification of the holonomy group associated to the Einstein–Hermitian connection on E.*

Proof. By [**6**, Proposition 3.5], we get a C^∞-reduction of the structure group $E_{\overline{K}} \subset E_{\mathrm{SU}}$. Further, $E_{\overline{K}}$ is preserved by the connection ∇^{SU}, which endows $E_{\overline{K}}$ with a unique connection inducing ∇^{SU} on E_{SU}. This shows that the C^∞-reduction of the structure group $E_{\overline{K}} \subset E$ is in fact a holomorphic reduction of the structure group. By the defining property of $\mathcal{H}_x(E)$, it follows that $\mathcal{H}_x(E) \subset \overline{K}$.

Further, it is shown in [**7**] that even though $H = \mathcal{H}_x(E)$ could be disconnected, it can still be shown that the holomorphic reduction E_H is preserved by the connection ∇. Finally, by the minimality property of $E_{\overline{K}}$, namely, $E_{\overline{K}}$ is the minimal reductive reduction preserved by ∇, one concludes that $\mathcal{H}_x(E) = \overline{K}$. □

5.3. *Semistable bundles*

We make a few remarks on holonomy group schemes of lf-graded bundles, which are not necessarily polystable.

Let C be a smooth projective curve over a field of characteristic 0 and fix a base point $x \in C$. By the considerations above and folklore, it is easy to see that the category of semistable bundles of degree 0 form a neutral Tannaka category and give a group scheme $\varpi(C, x)$, the representations of which give all semistable bundles of degree 0.

Let V be a semistable bundle of degree 0 and let $\mathcal{H}_x(V) \subset \mathrm{GL}(V_x)$ be the associated holonomy group. As we have remarked, when V is polystable of degree 0, this group can be realized as the image of the holonomy group scheme or as the Zariski closure of the Narasimhan–Seshadri representation.

By viewing V as a Higgs bundle with trivial Higgs structure, we see that by [**40**, Corollary 3.10, p. 40], there is a holomorphic (Weil) representation $\rho : \pi_1(C, x) \to \mathrm{GL}(V_x)$, which is compatible with the unitary Narasimhan–Seshadri representations giving the canonical polystable bundle $\mathrm{gr}(V)$. This is a special Weil representation giving the bundle V. The flat connection on the semistable bundle coming as extensions of the Yang–Mills connections on the stable bundles was given by Simpson. So we call this Weil representation the Weil–Simpson representation.

Abstractly, since V is semistable of degree 0, Weil's theorem applies since the indecomposable summands are of degree 0. Therefore there are holomorphic representations η, which realize V. Among these, the Weil–Simpson representation ρ is canonical to the extent that it induces the unitary ones on the terms of the associated graded bundles.

Now, viewing $(V, 0)$ as a semiharmonic Higgs bundle, and noting that all subbundles of $T^{a,b}V$ of degree 0 are simply the sublocal systems, we can talk of the *monodromy subgroup* $M(V, x) \subset \mathrm{GL}(V_x)$. This subgroup can be described as a minimal reduction subgroup of V. It is easy to see that $M(V, x)$ coincides with $\mathcal{H}_x(V)$.

Thus the representation $\rho : \pi_1(C, x) \to \mathrm{GL}(V_x)$ factors via $\rho : \pi_1(C, x) \to \mathcal{H}_x(V)$, and has the characterizing property that the quotient $\rho : \pi_1(C, x) \to \mathcal{H}_x(V)/R_u(\mathcal{H}_x(V))$ is *unitary*.

The total holonomy group scheme $\varpi(C, x)$ can thus be described as the projective limit of the Weil–Simpson-type representations of $\pi_1(C, x)$.

5.4. *The Simpson connection*

Recall the following result of Simpson [**40**, Corollary 3.10]: there exists an equivalence of categories between holomorphic flat bundles, which are extensions of unitary flat bundles, and semistable bundles with vanishing Chern classes in degree 1 and 2. Observe that when $c_1 = c_2 = 0$, then any semistable bundle is automatically lf-graded.

We then have the following relationship with $\mathcal{H}_{x,\Theta}(V)$.

THEOREM 5.12. *Let V be a semistable bundle on X with $c_1 = c_2 = 0$. Then the holonomy group $\mathcal{H}_{x,\Theta}(V)$ (which is now independent of the polarization) can be realized as the holonomy group of the canonical Simpson connection on V.*

Proof. The Simpson connection is realized as the canonical flat connection obtained as the extension of the unitary connection coming from the summands of the associated polystable bundle; viewing $(V, 0)$ as a Higgs bundle, its monodromy group coming from the canonical flat connection. The rest of the proof is as we have indicated for curves above. □

REMARK 5.13. In this connection and a generalization of Simpson's theorem, we refer the reader to the paper by Biswas and Subramanian [**10**].

5.5. *Some more remarks in characteristic* 0

(1) Let $\mathcal{C}^{\mathrm{ps}}$ be the full subcategory of $\mathcal{C}^{\ell f}$ consisting of *polystable bundles* of degree 0. It is easy to see that $\mathcal{C}^{\mathrm{ps}}$ is also a neutral Tannaka category. Moreover, it is a *semisimple* category (see [**14**]).

(2) Define the *pro-reductive quotient* $\varpi(X)_{\mathrm{ps}}$ of $\varpi(X)$ universally as follows: whenever $\rho : \varpi(X) \to G$ is *surjective* with G reductive, the representation ρ factors through a representation $\rho' : \varpi(X)_{\mathrm{ps}} \to G$ (which is automatically surjective).

(3) The Grothendieck–Tannaka group scheme $\mathrm{Aut}^{\otimes}(\mathcal{C}^{\mathrm{ps}})$ is isomorphic to $\varpi(X)_{\mathrm{ps}}$.

REMARK 5.14. The holonomy group $\varpi(X)$ by its Tannakian definition is an affine group scheme, which is realized as an inverse limit of algebraic groups. In particular, the topology that it gets is the *projective limit topology*.

With this topology, one can ask the question 'Can one compute $\pi_0(\varpi(X))$?' There are a few examples where we can say something. Let X be a smooth projective variety over characteristic 0, such that $\pi_1^{\text{ét}}(X,x) = 1$. In particular, X has no étale covering. Then it is immediate that, for every $E \in \text{obj}(\mathcal{C}^{\ell f})$, the holonomy group scheme $\mathcal{H}_{x,\Theta}(E)$ is connected.

Now to see that the entire group scheme $\varpi(X,x,\Theta)$ is connected, it is enough to check (see [**14**, Corollary 2.22]) that there are no non-trivial epimorphisms to any finite group. If $\phi : \varpi(X,x,\Theta) \to H$ is such a homomorphism, with H finite, then embed $H \hookrightarrow \text{GL}(V)$ and consider the composite $\varpi(X,x,\Theta) \to \text{GL}(V)$. This induces an lf-graded bundle with finite and hence disconnected holonomy, contradicting what we have mentioned above.

In fact (at least over characteristic 0), we can say more (cf. [**15**]).

Proposition 5.15. *We have an exact sequence of group schemes*:

$$1 \longrightarrow \varpi(X,x,\Theta)^o_{\text{ps}} \longrightarrow \varpi(X,x,\Theta)_{\text{ps}} \longrightarrow \pi_1^{\text{ét}}(X,x) \longrightarrow 1. \tag{5.1}$$

Proof. Observe that the quotient surjection is as in Proposition 3.10 together with the observation that *any bundle which is trivialized by a finite étale covering is actually polystable*.

Let $\phi : Y \to X$ be an étale Galois covering. This induces a functor $\phi^* : \mathcal{C}(X)^{\text{ps}} \to \mathcal{C}(Y)^{\text{ps}}$ by taking pull-backs and noting that polystable bundles pull-back to polystable bundles. Thus, by Proposition 6.16 we have an exact sequence

$$1 \longrightarrow \varpi(Y)_{\text{ps}} \longrightarrow \varpi(X)_{\text{ps}} \longrightarrow \text{Gal}(Y/X) \longrightarrow 1.$$

By taking inverse limit over Galois coverings of X, we get the following exact sequence:

$$1 \longrightarrow \varprojlim \varpi(Y)_{\text{ps}} \longrightarrow \varpi(X)_{\text{ps}} \longrightarrow \pi_1^{\text{ét}}(X,x) \longrightarrow 1.$$

Clearly, therefore, we have an inclusion $\varpi(X,x,\Theta)^o_{\text{ps}} \subset \varprojlim \varpi(Y)_{\text{ps}}$. Hence, to complete the proof we need only show that $\varprojlim \varpi(Y)_{\text{ps}}$ is connected.

Note that the category of finite-dimensional representations of the inverse limit group scheme, $\varprojlim \varpi(Y)_{\text{ps}}$, on k-vector spaces is the category $\varinjlim \mathcal{C}(Y)^{\text{ps}}$. Let W be an object in $\varinjlim \mathcal{C}(Y)^{\text{ps}}$. Then we need to show (by [**14**, Corollary 2.22]) that the strictly full subcategory with objects isomorphic to subquotients of nW, where $n \geqslant 0$, is *not stable* under $\otimes$ (where $nW \simeq \oplus W$ (n copies)).

Suppose that it is stable under $\otimes$. Then we will show that W is a trivial object in the direct limit category. Since our category is semisimple (because all the categories $\mathcal{C}(Y)^{\text{ps}}$ are semisimple), we can get a decomposition of W as

$$W = W_1 \oplus \ldots \oplus W_s,$$

where the W_i are simple objects. By assumption, for every $j \geqslant 1$, the object $W_1^{\otimes j}$ is isomorphic to a subquotient of nW for some n. Since we are in characteristic 0, this implies that $W_1^{\otimes j} = \bigoplus_{i=1}^s m_{ij} W_i$. Let $r > s$ and taking tensor powers $W_1^{\otimes j}$, with $j = 1, \ldots, r$, we get an integral dependence relation among the columns of the matrix (m_{ij}).

Now following the argument in [**15**, p. 14], we get an integral polynomial relation

$$P(W_1) = Q(W_1).$$

Associated to the object W_1, we have a chain of bundles V_i on the inverse system of Galois covers Y_i/X. Then, by the isomorphism above for the polynomials in W_1, we get an l_0 such that, for $l \geqslant l_0$,

$$P(V_l) = Q(V_l) \quad \text{on } Y_l.$$

This implies by Weil's theorem that V_l are finite bundles and hence trivialized in an étale cover of Y_l. Hence, the class of W_1 in the category $\varinjlim \mathcal{C}(Y)^{\mathrm{ps}}$ is trivial. Similarly all the W_i are trivial and hence so is W. This proves the proposition. □

6. *Genuinely ramified maps, stability and the holonomy group scheme*

Let X be smooth and projective and k be an algebraically closed field of arbitrary characteristic (cf. [**35**]). In this section we study the behaviour of $\varpi(X, x)$ under coverings. This will play a key role in the proof of existence of strongly stable principal bundles.

Let E be a torsion-free sheaf on X. Then one has a *unique* filtration called the Harder–Narasimhan filtration, $E_\bullet := \{0 = E_0 \subset \ldots \subset E_l = E\}$ by non-zero subsheaves such that each $\mathrm{gr}_i = E_i/E_{i-1}$ is semistable torsion-free and $\mu_i := \mu(E_i/E_{i-1}) > \mu_{i+1} := \mu(E_{i+1}/E_i)$. The subsheaves E_i are defined inductively as the inverse image of the maximal subsheaf of maximal slope in E/E_{i-1}. The successive quotients E_i/E_{i-1} are termed the *Harder–Narasimhan factors* of the sheaf E. The subsheaf E_1 is called the *maximal subsheaf* of E. This subsheaf is *semistable* and is denoted by $E_{\max}$. Its slope $\mu(E_1) = \mu(E_{\max})$ is called the *maximal slope* of E and denoted by $\mu_{\max}(E)$. Note that one always has $\mu_{\max}(E/E_{\max}) < \mu_{\max}(E)$.

Lemma 6.1. *Let $f : X \to Z$ be a finite separable morphism of smooth projective varieties. Then, for any semistable vector bundle W on X, $f_*(W)$ is locally free and we have the inequality*

$$\mu_{\max}(f_*W) \leqslant \frac{\mu(W)}{\deg\ f}.$$

Proof. The sheaf f_*W is locally free for any locally free W, since f is flat and Z is smooth. The inequality follows from the fact that $\mathrm{Hom}_Z(F, f_*W) \cong \mathrm{Hom}_X(f^*F, W)$. Hence semistable bundles of slope greater than $\mu(W)/\deg\ f$ have no morphism to f_*W. □

We have the following lemma (cf. [**36**, Corollary 1.21]).

Lemma 6.2. *Let $f : X \to Z$ be a finite separable morphism of smooth projective varieties. Then $X \to Z$ unramified étale if and only if $f_*(\mathcal{O}_X)$ is semistable of degree* 0.

Proof. Assume that $f_*(\mathcal{O}_X)$ is of degree 0. Its semistability is a trivial consequence of Lemma 6.1, which implies that $\mu_{\max} f_*\mathcal{O}_X = 0$ and the equality is because $\mathcal{O}_Z \subset (f_*\mathcal{O}_X)_{\max}$. Hence $\mu(f_*\mathcal{O}_X) = 0$ if and only if $\mu(f_*\mathcal{O}_X) = \mu_{\max} f_*\mathcal{O}_X$ if and only if $f_*\mathcal{O}_X$ is semistable.

Let $R \subset X$ be the ramification locus and $B = f(R) \subset Z$ be the branch locus.

Let Θ be a very ample polarization on Z and C be a general smooth irreducible complete intersection curve with respect to Θ. Let $D := f^{-1}(C)$ be the inverse image of C. Being a general CI curve, C will meet the branch locus $B \subset Z$ and hence D will meet the ramification locus $R \subset X$.

By Bertini (applied to the sublinear system coming from the pull-back of the sections of $|\Theta|$ and choosing C to meet B transversally), we see that D is smooth, as is the curve C.

Thus, we see that $D \to C$ is étale if and only if $f : X \to Z$ is étale.

Let I_C denote the ideal sheaf of C. Then $I_D = f^*I_C$ is the ideal sheaf of D. Now taking the direct image of the exact sequence

$$0 \longrightarrow I_D \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0, \tag{6.1}$$

we get the exact sequence (since f is finite)

$$0 \longrightarrow f_*I_D \longrightarrow f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_D \longrightarrow 0. \tag{6.2}$$

By the projection formula $f_*I_D = I_C \otimes f_*\mathcal{O}_X$, hence we obtain

$$0 \longrightarrow I_C \otimes f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_D \longrightarrow 0. \tag{6.3}$$

Now tensor the exact sequence

$$0 \longrightarrow I_C \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_C \longrightarrow 0 \tag{6.4}$$

by $f_*\mathcal{O}_X$ and comparing with (6.3), it follows that $f_*\mathcal{O}_D \cong f_*\mathcal{O}_X\ |_C$.

Now observe that D is connected. We argue by induction on $\dim(X)$. Assume that $\dim(X) = 2$. Now observe that $H^1(X, I_D) = H^1(Z, f_*I_D) = H^1(Z, I_C \otimes f_*\mathcal{O}_X) = H^1(Z, (f_*\mathcal{O}_X)(-m)) = 0$, by the Enriques–Severi lemma [**19**, Chapter 3, Corollary 7.8], since $m \gg 0$, C being chosen a high-degree CI curve. Therefore, $H^0(\mathcal{O}_D) = k$.

Observe that, by Mehta–Ramanathan, the semistability of $f_*(\mathcal{O}_X)$ will imply that $f_*(\mathcal{O}_X)|_C$ is semistable of degree 0 on C. Thus, we see that $f_*\mathcal{O}_D$ is semistable of degree 0 and we are reduced to the case when $\dim(X) = \dim(Z) = 1$.

Let $\dim(X) = \dim(Z) = 1$ and $f_*(\mathcal{O}_X)$ be semistable of degree 0 on Z. By an application of Riemann–Hurwitz formula and [**19**, Chapter 4, Example 2.6], we see that $\deg(R) = 0$; and R being effective, it follows that f is unramified.

The converse is more or less obvious. □

Remark 6.3. We have in fact the following equivalence: $f_*\mathcal{O}_X$ is semistable $\Longleftrightarrow$ $\deg(f_*\mathcal{O}_X) = 0 \Longleftrightarrow f$ is étale.

Lemma 6.4. *Let $f : X \to Z$ be a finite separable morphism of smooth projective varieties. Then we have the following*: $(f_*\mathcal{O}_X)_{\max}$ *forms a sheaf of subalgebras of* $f_*\mathcal{O}_X$ *on* Z *and is also a locally free subsheaf of* $f_*(\mathcal{O}_X)$.

Proof. Lemma 6.1 shows that $\mu_{\max} f_*\mathcal{O}_X = 0$. Again since f is separable, the Harder–Narasimhan filtration of $f_*\mathcal{O}_X$ pulls back to the Harder–Narasimhan filtration and hence we have

$$\mu_{\max} f^*(f_*\mathcal{O}_X) = 0. \tag{6.5}$$

To prove that $(f_*\mathcal{O}_X)_{\max}$ forms a sheaf of subalgebras, we need to show that the algebra structure given by the multiplication

$$f_*\mathcal{O}_X \otimes_{\mathcal{O}_Z} f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_X \tag{6.6}$$

restricts to a multiplication on $(f_*\mathcal{O}_X)_{\max}$. In other words, we need to show that

$$(f_*\mathcal{O}_X)_{\max} \otimes_{\mathcal{O}_Z} (f_*\mathcal{O}_X)_{\max} \longrightarrow f_*\mathcal{O}_X \tag{6.7}$$

has image contained in $(f_*\mathcal{O}_X)_{\max}$.

Since $(f_*\mathcal{O}_X)_{\max}$ is semistable of degree 0 and $f_*\mathcal{O}_X/(f_*\mathcal{O}_X)_{\max}$ has maximal slope < 0, it suffices to prove that $f_*\mathcal{O}_{X\max} \otimes_{\mathcal{O}_Z} f_*\mathcal{O}_{X\max}$ is semistable of degree 0. Since each bundle has degree 0, it suffices to show that $(f_*\mathcal{O}_X)_{\max} \otimes_{\mathcal{O}_Z} (f_*\mathcal{O}_X)_{\max}$ has no subsheaf of positive slope.

Now note that, by the projection formula,

$$f_*\mathcal{O}_X \otimes_{\mathcal{O}_Z} f_*\mathcal{O}_X \cong f_*f^*(f_*\mathcal{O}_X).$$

Since we have already noted that $f^*(f_*\mathcal{O}_X)$ has maximal slope 0, it has no subsheaves of positive slope.

By Lemma 6.1 we see that $(f_*\mathcal{O}_X)_{\max}$ forms a sheaf of subalgebras of $(f_*\mathcal{O}_X)$. Since $(f_*\mathcal{O}_X)_{\max}$ is torsion-free, it is locally free on a big open subset U. Taking $Y = \mathcal{S}\mathrm{pec}\,(S(f_*\mathcal{O}_X)^*_{\max})$ and restricting it to U, we get an étale cover $T \to U$ (by Lemma 6.2).

Since $U \subset Z$ is a big open subset, it follows that $\pi_1^{\text{ét}}(U) = \pi_1^{\text{ét}}(Z)$ (by 'purity of branch locus'; see [**31**, Example 5.2(h), p. 42]).

Hence, the étale cover $T \to U$ extends uniquely to an étale cover $g : Y \to Z$. It is easy to see that this scheme Y is smooth and $g_*(\mathcal{O}_Y) = (f_*\mathcal{O}_X)_{\max}$, implying that $(f_*\mathcal{O}_X)_{\max}$ is locally free. □

DEFINITION 6.5. Let $f : X \to Z$ be a finite morphism of smooth varieties. Then f is said to be *genuinely ramified* if f is separable and does not factor through an étale cover of Z.

PROPOSITION 6.6. *Let X and Z be smooth projective varieties and let $f : X \to Z$ be a finite separable morphism. Then f is genuinely ramified if and only if $(f_*\mathcal{O}_X)_{\max} \cong \mathcal{O}_Z$.*

Proof. If $\mathrm{rank}((f_*\mathcal{O}_X)_{\max}) = r > 1$, then, by Lemma 6.4, f factors through Y, a non-trivial étale cover of rank r and hence it is not genuinely ramified.

Suppose that f is not genuinely ramified and that it factors through $g : Y \to Z$, which is unramified. Then $g_*\mathcal{O}_Y$ is semistable of degree 0 (by Lemma 6.2), which is also a subbundle of $f_*\mathcal{O}_X$. Hence $(f_*\mathcal{O}_X)_{\max}$ has rank greater than 1. □

COROLLARY 6.7. *Let X be a smooth projective variety of* $\dim(X) = d$. *Let $f : X \to \mathbb{P}^d$ be a finite separable morphism. Then $(f_*\mathcal{O}_X)_{\max} \cong \mathcal{O}_{\mathbb{P}^d}$, that is, f is genuinely ramified.*

PROPOSITION 6.8. *Let $f : X \to Z$ be a genuinely ramified morphism of smooth projective varieties.*

(i) *If V and W are two semistable bundles on Z of the same slope*, *then*

$$\mathrm{Hom}_Z(V, W) \cong \mathrm{Hom}_X(f^*V, f^*W).$$

(ii) *If V is a stable bundle on Z*, *then f^*V is stable on X.*

(iii) *If V is a semistable bundle and $W \subset f^*V$ is a subbundle of the same slope as f^*V*, *then W is isomorphic to the pull-back of a subbundle of V.*

Proof. (i) Given two semistable bundles V and W of the same slope on Z, we have

$$\mathrm{Hom}_X(f^*V, f^*W) \simeq \mathrm{Hom}_Z(V, f_*f^*W) \simeq \mathrm{Hom}_Z(V, W \otimes f_*\mathcal{O}_X).$$

Further, since f is genuinely ramified, it follows that $f_*\mathcal{O}_X/\mathcal{O}_Z$ has negative maximal slope (see Proposition 6.6) and we have:

$$\mathrm{Hom}_Z(V, W \otimes f_*\mathcal{O}_X) \simeq \mathrm{Hom}_Z(V, W). \tag{6.8}$$

(ii) Since the socle (maximal subbundle, that is, a direct sum stable bundle (see [**29**])) is unique, it follows that the socle of f^*V descends to the socle of V when f is separable. Since V is stable, this descended bundle has to be V itself. This shows that the pull-back of a stable bundle is polystable under any finite separable map. Now the stability of f^*V for genuinely ramified maps follows from (1) as $f^*(V)$ cannot have endomorphisms.

(iii) Let V be a semistable bundle over C. Let $W \subset f^*V$ be a subbundle of the same slope. Then the socle $\mathrm{Soc}(W)$ of W is contained in $\mathrm{Soc}(f^*V)$ of f^*V and hence a direct summand of $\mathrm{Soc}(f^*V)$. But by uniqueness of the socle, $\mathrm{Soc}(f^*V)$ is $f^*(\mathrm{Soc}(V))$. Since stable bundles pull back to stable bundles, $\mathrm{Soc}(W)$ coincides with some factors of $f^*(\mathrm{Soc}(V))$ and hence is a pull-back. Now the assertion follows by induction on the rank applied to the bundle $W/\mathrm{Soc}(W) \subset f^*V/\mathrm{Soc}(W)$. □

LEMMA 6.9. *Let $f : X \to Z$ be a finite separable morphism of smooth varieties. Let E be an lf-graded bundle on Z with respect to a fixed polarization on Z. Then $f^*(E)$ is lf-graded with respect to the pull-back polarization. In particular, this induces a homomorphism of group schemes*

$$f_* : \varpi(X, x, f^*(\Theta)) \longrightarrow \varpi(Z, f(x), \Theta).$$

Proof. Observe that if E is stable, then $f^*(E)$ is polystable and hence lf-graded. Further, extensions of lf-graded bundles pull back to extensions of bundles, each of which is lf-graded by induction on ranks. Hence by Lemma 2.6 the lemma follows. □

THEOREM 6.10. *A finite separable morphism $f : X \to Z$ is genuinely ramified if and only if the induced map of the holonomy group schemes $f_* : \varpi(X, x, f^*(\Theta)) \to \varpi(Z, f(x), \Theta)$ is surjective.*

Proof. By [**34**, Proposition 5, Appendix] (or [**14**, Proposition 2.21]), a map $f : G \to H$ of affine group schemes is *surjective* if and only if the natural induced functor of Tannaka categories, $f^* : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$, is *fully faithful* and further if any sequence of G-modules $0 \to W' \to f^*V \to W'' \to 0$ is obtained by pulling up a sequence $0 \to V' \to V \to V'' \to 0$.

Therefore we see that $f_* : \varpi(Y) \to \varpi(Z)$ is surjective if and only if, for every lf-graded bundle V on Z, any lf-graded subbundle $W' \subset f^*V$ is a pull-back from an lf-graded bundle on Z.

Assume that f is genuinely ramified. Then the condition follows from Proposition 6.8(iii).

Conversely, suppose that $f_* : \varpi(X) \to \varpi(Z)$ is surjective. This implies that the map at the level of categories is fully faithful.

Let the rank of the bundle $(f_*\mathcal{O}_X)_{\max}$ be r. This bundle gives a sheaf of algebras as in the proof of Proposition 6.6. This (by Lemma 6.2) gives rise to an étale cover $g : Y \to Z$, where $g_*\mathcal{O}_Y = (f_*\mathcal{O}_X)_{\max}$. Thus, if $s = H^0(\mathcal{O}_X, f^*f_*\mathcal{O}_X)$, then $s > 1$ if and only if $r > 1$.

By the full faithfulness, we have an identification of Hom and therefore we have

$$1 = \dim(\mathrm{Hom}(\mathcal{O}_Z, (f_*\mathcal{O}_X)_{\max}) = \dim(\mathrm{Hom}(\mathcal{O}_X, f^*(f_*\mathcal{O}_X)_{\max}))) = s$$

because $\dim(\mathrm{Hom}(\mathcal{O}_Z, (f_*\mathcal{O}_X)) = \dim(\mathrm{Hom}(\mathcal{O}_Z, (f_*\mathcal{O}_X)_{\max}) = 1$. Hence $r = 1$, which implies by Proposition 6.6 that the map $f : X \to Z$ is genuinely ramified. □

We have the following lemma, which plays a key role in later applications.

LEMMA 6.11. *Let $f : X \to Z$ be a genuinely ramified morphism of smooth projective varieties. Let E be a principal G-bundle on Z, which is strongly stable with full holonomy. Then, $f^*(E)$ is a strongly stable G-bundle with full holonomy.*

Proof. Since E is strongly stable with full holonomy, it follows by Lemma 4.8 that $E(V)$ is strongly stable for every *irreducible* G module V. We note that this property classifies G-bundles that are strongly stable with full holonomy. By Proposition 6.8(ii), $f^*E(V)$ is stable for all irreducible G-modules, consequently $f^*(E)$ is strongly stable with full holonomy. □

6.1. *Behaviour under étale maps*

Let $\phi : Y \to X$ be a finite morphism. Then, by Lemma 6.9, we have a homomorphism $\phi_* : \varpi(Y) \to \varpi(X)$.

LEMMA 6.12. *Let $\phi : Y \to X$ be an étale Galois cover. A bundle W on X is lf-graded if and only if $\phi^*(W)$ is lf-graded.*

Proof. Let $W \to F \to 0$ be a stable degree 0 torsion-free quotient. Then we *claim* that F is *locally free*. Pulling back to Y, we get

$$\phi^*(W) \longrightarrow \phi^*(F) \longrightarrow 0$$

and since F is torsion-free and stable, and ϕ is étale, it follows that $\phi^*(F)$ is torsion-free and polystable (to see this, note that the socle $\mathrm{Soc}(\phi^*(F)) \subset \phi^*(F)$ is Galois invariant and hence descends to a subsheaf F' of F. Since F is stable, it follows that $F' = F$ and hence $\mathrm{Soc}(\phi^*(F)) \simeq \phi^*(F)$).

Now since $\phi^*(W)$ is assumed to be lf-graded, it has WRP and hence by Proposition 2.12 we get that $\phi^*(F)$ is locally free. Therefore $\phi_*(\phi^*(F)) \simeq F \otimes \phi_*(\mathcal{O}_Y)$ is locally free. Hence, so is F, proving the claim. The converse is shown in Lemma 6.9. □

COROLLARY 6.13. *Let $\phi : Y \to X$ be a finite étale morphism. A bundle W on X is lf-graded if and only if $\phi^*(W)$ is lf-graded.*

Proof. Let $Z \to Y \to X$ be the Galois completion and let $\psi : Z \to Y$ and $f : Z \to X$ be the composite. (The existence of such a Galois completion is obvious from classical Galois theory since X is normal. For the general case, see [**32**, 4.4.1.8].)

Assume that $\phi^*(W)$ is lf-graded. Then, by Lemma 6.9, $\psi^*(\phi^*(W)) = f^*(W)$ is lf-graded. Hence, by Lemma 6.12, W is lf-graded. □

LEMMA 6.14. *Let $\phi : Y \to X$ be an étale Galois cover. Let W be an lf-graded bundle on Y. Then $\phi_*(W)$ is lf-graded.*

Proof. Consider the diagram (6.9), when the map ϕ is a Galois covering. Then, it can be regarded as a principal $\mathrm{Gal}(Y/X)$-bundle. The pull-back $p : Y \times_X Y \to Y$ always has a canonical section, but since it is a $\mathrm{Gal}(Y/X)$-bundle, it implies that $Y \times_X Y \simeq Y \times \mathrm{Gal}(Y/X)$.

Therefore, we see that $\phi^*(\phi_*(W)) \simeq \bigoplus_{g \in \mathrm{Gal}(Z/X)} g^*W$ and since W is lf-graded, so is $\phi^*(\phi_*(W))$. This implies, by Lemma 6.12, that $\phi_*(W)$ is lf-graded. □

COROLLARY 6.15. *Let $\phi : Y \to X$ be a finite étale morphism. Let W be an lf-graded bundle on Y. Then $\phi_*(W)$ is lf-graded.*

Proof. Again take $Z \to Y \to X$. Since W is lf-graded, so is $\psi^*(W)$, and hence $f_*(\psi^*(W)) \simeq \phi_*(\psi_*(\psi^*(W)))$ is lf-graded. Observe that W is a degree 0 subbundle of $\psi_*(\psi^*(W)$.

Observe that $\deg(\phi_*(W)) = 0$ since ϕ is étale. Hence, $\phi_*(W)$ is a degree 0 subbundle of the lf-graded bundle $\phi_*(\psi_*(\psi^*(W)))$.

This implies that $\phi_*(W)$ has WRP by Lemma 2.13 and hence is lf-graded. □

Proposition 6.16. *Let $\phi : Y \to X$ be an étale covering. Then the induced homomorphism $\phi_* : \varpi(Y) \to \varpi(X)$ is a closed immersion. Furthermore, if ϕ is also a Galois covering, then we have an exact sequence*

$$1 \longrightarrow \varpi(Y) \longrightarrow \varpi(X) \longrightarrow \mathrm{Gal}(Y/X) \longrightarrow 1.$$

Proof. By [**14**, Proposition 2.21(b)], we need to check that if W is an object in $\mathcal{C}(Y)$, then W is isomorphic to a subquotient of an object of the form $\phi^*(W')$ with W' in $\mathcal{C}(X)$.

Consider the fibre square

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow[p]{} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \phi} \\ Y & \xrightarrow[\phi]{} & X. \end{array} \tag{6.9}$$

Then, since $\phi : Y \to X$ is étale finite, the projection $p : Y \times_X Y \to Y$ is a degree d cover of Y, possibly disconnected. Since ϕ is flat, it follows that $\phi^*(\phi_*(W)) \simeq p_*(p^*(W))$. Note that W is a subbundle of $p_*(p^*(W))$ and hence, by Corollaries 6.13 and 6.15, it follows that W is a subbundle of the pull-back of an object in $\mathcal{C}(X)$, namely $\phi_*(W)$.

The second half of the proposition follows from the arguments in [**15**, Lemma 15]. □

Corollary 6.17. *Let $\phi : Y \to X$ be a finite étale morphism. A bundle W is an lf-graded bundle on Y if and only if $\phi_*(W)$ is lf-graded.*

Proof. One way is simply Corollary 6.15. The converse follows from the proof of Proposition 6.16 since W is realized as a degree 0 subbundle of $\phi^*(\phi_*(W))$, which is lf-graded. □

Remark 6.18. By Theorem 6.10 and Proposition 6.16, we get a complete factorization of the induced homomorphism under any finite separable morphism of smooth projective varieties.

7. *Existence of unobstructed stable bundles on a surface*

For this section we assume that the ground field k is an algebraically closed field of $\mathrm{char}(k) \geqslant 3$ and let $\dim(X) = 2$. This section is inspired by some results from Donaldson's paper [**16**], where he proves generic smoothness of the moduli space (see also [**25**]).

Proposition 7.1. *Let M be a line bundle on X. Then there exists a constant $\alpha(M)$ such that, for all $c > \alpha(M)$, there exists a stable E of rank 2 with $\det(E) \simeq \mathcal{O}_X$ and such that $c_2(E) = c$ with the following vanishing property*:

$$h^0(\mathrm{ad}(E) \otimes M) = 0.$$

Proof. We note that we can assume, to start with, $\det(E) \simeq Q$ with $Q \simeq 2n\Theta$, where Θ is the hyperplane line bundle on X. For then, we take $V = E \otimes (-n\Theta)$. Then $\det(V) \simeq \mathcal{O}_X$ and V is also stable, and furthermore $\mathrm{ad}(V) \simeq \mathrm{ad}(E)$. □

Assumption 7.2. We now choose Q and Z as follows.

(1) Choose $Q = 2n\Theta$ so that $h^0(Q) > 0$ and $\deg(Q) > \deg(M)$ so that $h^0(Q^* \otimes M) = 0$.

(2) Choose Z so that $H^0(Q \otimes M \otimes I_Z) = 0$. This can, for example, be made by choosing Z general with $\ell(Z) > h^0(Q \otimes M)$. This therefore also implies $h^0(M \otimes I_Z) = 0$ since Q has sections.

(3) Choose the length of the cycle $\ell(Z)$ as well as the degree of Q (with respect to Θ) to be large so that we have stable bundles in the Serre construction (see [**20**, Chapter 5]), that is,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow Q \otimes I_Z \longrightarrow 0. \tag{7.1}$$

We then do the following. Tensor the exact sequence above with the line bundle M:

$$0 \longrightarrow M \longrightarrow E \otimes M \longrightarrow Q \otimes M \otimes I_Z \longrightarrow 0. \tag{7.2}$$

We need to prove that $H^0(\mathrm{ad}(E) \otimes M) = 0$, which follows from the following lemma applied to the exact sequence

$$0 \longrightarrow \mathrm{ad}(E) \longrightarrow \mathcal{E}\mathrm{nd}(E) \longrightarrow \mathcal{O} \longrightarrow 0,$$

where the map $\mathcal{E}\mathrm{nd}(E) \longrightarrow \mathcal{O}$ is the 'Trace' map.

LEMMA 7.3. *Given M, choose E as in Assumption 7.2. Then $H^0(\mathcal{E}\mathrm{nd}(E) \otimes M) \simeq H^0(M)$. Moreover, any $\phi \in H^0(\mathcal{E}\mathrm{nd}(E) \otimes M)$ can be expressed uniquely as $\mathrm{id}_E \otimes \psi$, where $\psi \in H^0(M)$. Furthermore, one can identify $\psi = \mathrm{trace}(\phi)$. In particular, if $\phi \in H^0(\mathrm{ad}(E) \otimes M)$, then $\phi = 0$.*

Proof. We first *claim* that if $\phi \in H^0(E \otimes M)$ is such that $\phi \circ \theta = 0$, then $\phi = 0$.

To see this, apply the functor $\mathrm{Hom}(-, E \otimes M)$ to the exact sequence (7.1). Then we get

$$0 \longrightarrow \mathrm{Hom}(Q \otimes I_Z, E \otimes M) \longrightarrow \mathrm{Hom}(E, E \otimes M) \longrightarrow \mathrm{Hom}(\mathcal{O}_X, E \otimes M),$$

where the last map is $\phi \to \phi \circ \theta$. Since $\phi \circ \theta = 0$, it implies that there is a $\gamma \in \mathrm{Hom}(Q \otimes I_Z, E \otimes M)$, which maps to ϕ. Now any map

$$\gamma : Q \otimes I_Z \longrightarrow E \otimes M$$

factors via a map $\delta \in \mathrm{Hom}(Q, E \otimes M)$ since $E \otimes M$ is locally free and we have a commutative diagram as follows:

$$\begin{array}{ccc} Q \otimes I_Z & \xrightarrow{\gamma} & E \otimes M \\ {\scriptstyle i}\downarrow & \nearrow_{\delta} & \\ Q & & \end{array}$$

where $i : Q \otimes I_Z \hookrightarrow Q$ is the canonical inclusion. Now, by Assumption 7.2, we have $h^0(E \otimes M \otimes Q^*) = 0$ (by tensoring the exact sequence (7.2) by Q^*). Hence $\delta = 0$, implying $\gamma = 0 = \phi$. This proves our claim.

Let $\phi : E \to E \otimes M$ and consider the composite

$$\phi \circ \theta : \mathcal{O}_X \longrightarrow E \otimes M$$

By Assumption 7.2, since $h^0(Q \otimes M \otimes I_Z) = 0$, we have a $\psi : \mathcal{O}_X \to M$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{\theta} & E & \longrightarrow & Q \otimes I_Z & \longrightarrow & 0 \\ & & {\scriptstyle \psi}\downarrow & & {\scriptstyle \phi}\downarrow & & & & \\ 0 & \longrightarrow & M & \longrightarrow & E \otimes M & \longrightarrow & Q \otimes I_Z \otimes M & \longrightarrow & 0. \end{array} \tag{7.3}$$

Observe that if we tensor $\psi : \mathcal{O}_X \to M$ by E, we again have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{\theta} & E & \longrightarrow & Q \otimes I_Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{id}_E \otimes \psi} & & & & \\ 0 & \longrightarrow & M & \longrightarrow & E \otimes M & \longrightarrow & Q \otimes I_Z \otimes M & \longrightarrow & 0, \end{array} \tag{7.4}$$

where we have $\psi = \mathrm{Trace}(\mathrm{id}_E \otimes \psi)$. Since $(\theta \otimes \mathrm{id}_M) \circ \psi = \phi \circ \theta = (\mathrm{id}_E \otimes \psi) \circ \theta$ by the commutativity of the two diagrams, we conclude that

$$(\phi - \mathrm{id}_E \otimes \psi) \circ \theta = 0.$$

By the claim made above and for suitable choices of Z and M, we get

$$\phi = \mathrm{id}_E \otimes \psi.$$

Hence $\psi = \mathrm{Trace}(\phi)$. This proves the first part of the lemma. Further, if $\phi \in H^0(\mathrm{ad}(E) \otimes M)$, it follows that $\psi = 0$ and hence $\phi = 0$. □

8. *Surjectivity of the restriction map*

In this section the assumptions on the field are as in the previous section, namely, $\mathrm{char}(k) \geqslant 3$ with $\dim(X) = 2$.

THEOREM 8.1. *Given a curve $C \subset X$ of genus $g(C) \geqslant 2$, there exists a constant $\alpha(C)$ such that, whenever $c_2 \geqslant \alpha(C)$, there exists a stable E with $c_2 = c_2(E)$ and such that the restriction map*

$$H^1(\mathrm{ad}\, E)) \longrightarrow H^1(\mathrm{ad}(E|_C)$$

is surjective. In particular, there exists a Zariski open subset $U \subset M_X(\mathrm{SL}(2))^s$ such that, for $E \in U$, the bundle $E|_C$ is stable and the restriction map is differentially surjective.

Proof. Observe that, by Serre duality $H^2(\mathrm{ad}(E) \otimes \mathcal{O}_X(-C)) = H^0(\mathrm{ad}(E) \otimes \mathcal{O}(C) \otimes K_X)^*$ (here we use the self-duality of $\mathrm{ad}(E)$ since $\mathrm{char}(k) > 2$). Therefore, taking $M = \mathcal{O}_X(C) \otimes K_X$ and choosing E as in Proposition 7.1, we get the required vanishing of H^2 and the surjectivity. This implies that at the level of infinitesimal deformations, we have the *surjectivity* of the restriction map of formal schemes:

$$\mathrm{Def}(E) \longrightarrow \mathrm{Def}(E|_C).$$

Now note that, for any bundle on C, there are stable bundles in its neighbourhood since the moduli space on curves is irreducible and the stable bundles are dense. Hence we have shown that a general stable bundle on C lifts to a stable bundle on X. This proves the theorem. □

REMARK 8.2. We can see the above deformation argument more transparently using stacks as follows. Let $\mathfrak{M}_X(\mathrm{SL}(2))$ or $\mathfrak{M}_C(\mathrm{SL}(2))$ be the moduli stack of SL(2)-bundle on X or C, respectively. The restriction map gives a morphism of stacks from the open substack $\mathfrak{M}_X(\mathrm{SL}(2))^s$ of stable bundles to $\mathfrak{M}_C(\mathrm{SL}(2))$. The differential of this map at $E \in \mathfrak{M}_X(\mathrm{SL}(2))^s(k)$ is the map $H^1(\mathrm{ad}\, E) \to H^1(\mathrm{ad}(E|_C))$, which we have shown to be surjective. Hence the image contains a stable bundle on C and the differential is surjective at this point too. This proves the required surjectivity.

LEMMA 8.3 (Converse to Mehta–Ramanathan's restriction theorem for strongly stable bundles). *Let P be a G-bundle on X and let $C \subset X$ be a curve belonging to the polarization $m\Theta$ such that $P|_C$ is strongly stable. Then P is strongly stable with respect to Θ.*

Proof. We first claim that P is itself a stable principal H-bundle. For, if $Q \subset G$ is a parabolic subgroup, χ a dominant character of Q, and P_Q a Q-bundle obtained from a reduction of the structure group to Q, then note that

$$\deg P_Q(\chi) \cdot m \cdot a = \deg P_Q(\chi)_{|C},$$

where $P_Q(\chi)$ denotes the line bundle associated to the character χ. Since $a > 0$, it follows by the stability of $P_{|C}$ that $\deg P_Q(\chi)_{|C} > 0$ and hence $\deg P_Q(\chi) > 0$, that is, P is stable. The Frobenius pull-backs behave similarly. To see this, observe that the Frobenius pull-back $F^r(P)$ can be realized as the associated bundle $P(F^r_*(G))$. Hence if we know that strong stability holds on C, by observing that taking associated constructions commutes with the restriction map, we get strong stability of P as well. □

9. *Holonomy groups of* SL(2)*-bundles on a general plane curve*

Towards constructing bundles on surfaces with full holonomy, we rely on restricting bundles to curves and then lifting back. In this section we construct bundles on plane curves with full holonomy essentially following [**9**]. The ground field k is an *uncountable algebraically closed field of characteristic* $p > 0$ in what follows.

REMARK 9.1. Let $G_q := \mathrm{SL}(2, \mathbb{F}_q) \subset \mathrm{SL}(2, k)$. It is well known that G_q is generated by the elements $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

Further, observe that $G_q \subset \mathrm{SL}(2, k)$ is an *irreducible* subgroup.

PROPOSITION 9.2. *Let $C \subset \mathbb{P}^2$ be a general plane curve of genus at least* 2*. Then there exists a strongly stable vector bundle of rank* 2 *on C with trivial determinant.*

Proof. Choose a nodal plane curve C_0 of arithmetic genus $g \geqslant 2$, the irreducible components of which are lines in $\mathbb{P}^2$. Observe that $g = (k-1)(k-2)/2$, where $C_1, \ldots, C_k$ are the irreducible components of C_0. Note that the étale fundamental group of C_0 is the pro-finite completion of the free group on g generators.

Consider the space S_d of degree d curves in $\mathbb{P}^2$. Then it is well known that $S_d \simeq \mathbb{P}^n$, where $n = \binom{d+3}{3}$. Hence S_d is *irreducible*.

Assume that $q \geqslant 4$. In Remark 9.1 we noted that the subgroup G_q of G is generated by two elements.

Since the étale fundamental group of C_0 is the pro-finite completion of the free group on g generators, with $g \geqslant 2$, by mapping any two generators to the generators of G_q, we have a surjective homomorphism from the étale fundamental group of C_0 onto G_q. Hence there is an étale Galois covering of the nodal curve C_0 with Galois group G_q.

We will prove that there is a neighbourhood U of C_0 in S_d such that every curve C in U has a Galois étale covering $D \to C$ with Galois group G_q.

Choose a curve $\mathrm{Spec}(R) \to S_d$ such that the special fibre over $\mathrm{Spec}(k) \to S_d$ is C_0 and with general fibre smooth. We thus get a family of curves

$$C_R \longrightarrow \mathrm{Spec}\, R,$$

which we may assume is a proper separable morphism of algebraic varieties. Let K denote the quotient field of R and let C_K denote the generic fibre of C_R. Then we have a *specialization* homomorphism

$$\pi_1^{\text{ét}}(C_K) \longrightarrow \pi_1^{\text{ét}}(C_0),$$

where $\pi_1^{\text{ét}}$ is the étale fundamental group. This specialization homomorphism is surjective if the residue field is algebraically closed and if C_0 is connected (see [**32**, 9.2, 9.3] for a proof). Now, giving a finite Galois étale cover of C_K with Galois group Γ is equivalent to giving a surjective group homomorphism

$$\pi_1^{\text{ét}}(C_K) \longrightarrow \Gamma \longrightarrow 1.$$

If C_0 has a finite Galois étale cover with Galois group Γ, then we obtain a surjective group homomorphism

$$\pi_1^{\text{ét}}(C_0) \longrightarrow \Gamma \longrightarrow 1.$$

Hence if the specialization homomorphism $\pi_1^{\text{ét}}(C_K) \to \pi_1^{\text{ét}}(C_0)$ is surjective, then Γ is also a quotient of $\pi_1^{\text{ét}}(C_K)$. In particular, in our case the étale cover of the special fibre generalizes.

This proves that there is a neighbourhood $U \subset S_d$ of the nodal curve C_0 with the property that every curve in the family over $U \subset S_d$ has a Galois étale covering with Galois group G_q.

The covering $D \to C$ is a Galois étale cover with Galois group G_q, which is therefore a principal G_q-bundle. Denote this by E_D. Let E be the $\mathrm{SL}(2,k)$-bundle obtained from E_D by extension of the structure group from the inclusion $G_q \subset \mathrm{SL}(2,k)$. Then E is a *finite vector bundle* in the language of [**34**]. Furthermore, the Frobenius pull-back $F^*(E)$ is obtained by extension of the structure group via the composition $G_q \to \mathrm{SL}(2) \to \mathrm{SL}(2)$, where the last map is the Frobenius morphism on the group $\mathrm{SL}(2)$. It follows that $F^*(E)$ is also a finite bundle. This implies by [**34**, Proposition 3.4] that E is *strongly semistable*.

Moreover, the holonomy of E is precisely G_q, which is a reduced irreducible subgroup of $\mathrm{SL}(2)$, and hence it follows that E is *strongly stable*. □

9.1. *Strongly stable* $\mathrm{SL}(2,k)$*-bundles with full holonomy*

With this proposition in place, we now run through some arguments from [**9**], which will ensure that there are curves $C \in S_d$ that support strongly stable bundles E, such that the holonomy subgroup scheme $\mathcal{H}_{x,\Theta}(E)$ is reduced and in fact coincides with $\mathrm{SL}(2,k)$.

THEOREM 9.3 (see [**9**]). *There exists an* $\mathrm{SL}(2,k)$*-bundle* E *on a general curve* $C \subset \mathbb{P}^2$*, which is strongly stable with full holonomy group, that is,* $\mathcal{H}_{x,\Theta}(E) = \mathrm{SL}(2,k)$.

Proof. We outline the proof in the following steps (see [**9**, Section 6]).

Step 1. By Proposition 9.2 we have a general plane curve C and a strongly stable bundle E on it with holonomy group $\mathcal{H}_{x,\Theta}(E) = \mathcal{H}_{x,\Theta}(E)_{\text{red}} = G_q$. Let H be a reductive subgroup of $\mathrm{SL}(2,k)$ defined over $\overline{\mathbb{F}}_p$. Then, for $q \gg 0$, $G_q \not\subseteq H$. Furthermore, none of the conjugates of H contain G_q. Hence, for a fixed $H \subset \mathrm{SL}(2,k)$ and its conjugates, we can get an E such that $\mathcal{H}_{x,\Theta}(E) = G_q$, and this E will have no reduction to any conjugate of H.

Step 2. We need an E that is strongly stable with reduced holonomy such that $\mathcal{H}_{x,\Theta}(E)$ is not contained in any reductive $H \subset \mathrm{SL}(2,k)$ over k. To this end, define the subset

$$S_{H,n} := \{F^n(E) \text{ is stable and does not admit a degree 0 reduction to } H\}.$$

Then $S_{H,n} \neq \emptyset$. For example, the E obtained above for a large q will lie in $S_{H,n}$. To see this, suppose that some Frobenius power $F^n(E)$ has a degree 0 reduction to a H. Then, by its definition, there exists an m such that $F^{n+m}(E) \simeq E$. Now if $E_H \subset F^n(E)$ is the degree 0 reduction to H, then $F^m(E_H) \subset F^{n+m}(E) = E$ will give a reduction of the structure group of E, which is a contradiction. Indeed, it is a countable intersection of non-empty dense open subsets.

Step 3. Define

$$S := \bigcap_{H,n} \{S_{H,n} | H \text{ reductive defined over } \overline{\mathbb{F}_p}\}$$

and all $n > 0$. By what has been remarked above, $S \neq \emptyset$. Let $E \in S$. There is no reason as to why $\mathcal{H}_{x,\Theta}(E)$ is reduced. But we observe that there is always an r such that if $P = F^r(E)$, then the holonomy group of P, namely $\mathcal{H}_{x,\Theta}(P)$, is *reduced*.

Step 4. Now observe that by the choice of E and P, the subgroup $\mathcal{H}_{x,\Theta}(P) \subset \mathrm{SL}(2,k)$ is *irreducible* and so is reductive. By the choice of P, it follows that $\mathcal{H}_{x,\Theta}(P) = \mathrm{SL}(2,k)$ and we are done. □

10. *Existence of stable principal bundles on surfaces*

In this section the ground field k is an uncountable algebraically closed field of positive characteristic $p \geqslant 3$. In this section G is a *simple simply connected algebraic group*. Recall the notion of a principal three-dimensional subgroup of G. In characteristic p for $p > h_G$, where h_G is the Coxeter number of G (defined as $((\dim(G)/\mathrm{rank}(G)) - 1)$, there always exists an irreducible subgroup of G (in the sense that it is not contained in any proper parabolic subgroup of G), which is the image of the principal homomorphism $\rho : \mathrm{SL}(2,k) \to G$ (see [**39**]).

THEOREM 10.1. *There exists a strongly stable* $\mathrm{SL}(2,k)$*-bundle* E *on* $\mathbb{P}^2$*, such that* $E|_C$ *is a strongly stable bundle with full holonomy. Furthermore, if* $p > h(G)$*, then the principal* G*-bundle* $E(\rho)$ *is also strongly stable.*

Proof. Choose a general plane curve C that supports a strongly stable $\mathrm{SL}(2,k)$ bundle with full holonomy, which exists by Theorem 9.3. By Theorem 8.1, we can lift a *general* bundle on C with full holonomy to a bundle E on $\mathbb{P}^2$. Then, by Lemma 8.3, the bundle E will be strongly stable. Moreover, since $E|_C$ has full holonomy, this implies that every associated bundle $E(V)|_C$ is strongly stable for all irreducible $\mathrm{SL}(2,k)$-modules V by Lemma 4.8. By Lemma 8.3, it implies that $E(V)$ is a strongly stable bundle on $\mathbb{P}^2$. Hence, by Lemma 4.8, the bundle E has full holonomy.

Since $E|_C$ is strongly stable with full holonomy, and since ρ is irreducible, it implies that $E(G)|_C$ is stable. The same argument applies for the Frobenius pull-backs proving that $E(G)|_C$ is strongly stable. This implies, by Lemma 8.3, that $E(G)$ is strongly stable on $\mathbb{P}^2$. □

LEMMA 10.2. *Let* X *be a smooth projective surface and* Θ *be an ample line bundle on* X*. Then there exists a genuinely ramified morphism* $f : X \to \mathbb{P}^2$ *such that* $f^*(\mathcal{O}(1))$ *is some power of* Θ*.*

Proof. Embed $X \subset \mathbb{P}^n$ using the very ample line bundle $m\Theta$. Now choose a point in $\mathbb{P}^n$ away from X and project from it. With a little care, one can choose projections successively so that the maps are *separable*. This can be seen as follows. Choose a generic codimension 2 subspace $\mathbb{P}^{n-2} \subset \mathbb{P}^n$. This will meet X in a finite set of points each with multiplicity of 1. One can now choose any hyperplane in this $\mathbb{P}^{n-2}$ that avoids these points. Then projecting from this hyperplane, we get a map to $\mathbb{P}^2$, which is separable.

Consider the resulting map $f : X \to \mathbb{P}^2$, which is chosen to be separable and finite. Since f is obtained by projection, it has the added property that $f^*\mathcal{O}_{\mathbb{P}^2}(1) = m\Theta$. □

THEOREM 10.3. *There exists an* $\mathrm{SL}(2,k)$*-bundle* P *on* X *that is strongly stable with full holonomy group* $\mathrm{SL}(2,k)$*. Furthermore, if* $\mathrm{SL}(2,k) \subset G$ *is a principal homomorphism (which exists if* $p > h_G$*), then the associated* G*-bundle* $P(G)$ *is strongly stable with respect to* Θ*.*

Proof. Let $f : X \to \mathbb{P}^2$ be a genuinely ramified morphism (obtained by Lemma 10.2) and E on $\mathbb{P}^2$ be the vector bundle on $\mathbb{P}^2$ constructed by Theorem 10.1, and let $P = f^*(E)$. By Lemma 4.8, P is a strongly stable $\mathrm{SL}(2,k)$-bundle with full holonomy. By Lemma 4.8, since $\rho : \mathrm{SL}(2,k) \to G$ is a principal homomorphism, its image is an irreducible subgroup of G. Hence it follows that $P(G)$ is a strongly stable G-bundle. This completes the proof of the theorem. □

10.1. *Existence of strongly stable G-bundles with full holonomy*

Theorem 10.3 gives the existence of stable G-bundles on an arbitrary smooth projective surface. This shows that the moduli space of stable bundles is non-empty. But by this construction, the stable G-bundle constructed has its holonomy inside the principal $\mathrm{SL}(2,k)$. It is of interest, especially for the geometry of the moduli space of G-bundles, to show that the existence of stable G-bundles with holonomy group the whole of G, and Theorem 10.6 below proves this result drawing on the techniques developed so far. Even over fields of characteristic 0, such existence results are not known; these correspond to the *irreducible* anti-self-dual G-connections (cf. [**1**, p. 443]).

PROPOSITION 10.4. *Let* E *be a strongly stable* G*-bundle on* $\mathbb{P}^2$*. Then the holonomy group scheme* $\mathcal{H}_x(E)$ *is reduced and connected.*

Proof. By Remark 4.6, and the comments following it, we may choose a faithful G-module W and work with the strongly semistable vector bundle $V = E(W)$ since $\mathcal{H}_x(E) = \mathcal{H}_x(V)$. Again, by Remark 4.6, the Frobenius morphism induces a functor at the level of categories, $F^* : \mathcal{C}(V) \to \mathcal{C}(F^*(V))$ given by $A \mapsto F_X^*(A)$ and this gives the map $F_{\mathbb{P}^2}^* : \mathcal{H}_x(F^*(V)) \to \mathcal{H}_x(V)$. We now claim that

$$\mathcal{H}_x(F^*(V)) = \mathcal{H}_x(V). \tag{10.1}$$

To see this, observe firstly that the map $F_{\mathbb{P}^2}^*$ is an inclusion $\mathcal{H}_x(F^*(V)) \hookrightarrow \mathcal{H}_x(V)$. To show that this map is surjective, we appeal to [**14**, Proposition 2.21(a)]. The induced functor $F^* : \mathcal{C}(V) \to \mathcal{C}(F^*(V))$ is fully faithful:

$$\mathrm{Hom}_{\mathbb{P}^2}(F^*U_1, F^*U_2) \simeq \mathrm{Hom}_{\mathbb{P}^2}(U_1, F_*F^*U_2) \simeq \mathrm{Hom}_{\mathbb{P}^2}(U_1, U_2 \otimes F_*\mathcal{O}_{\mathbb{P}^2}).$$

On the other hand, $F_*\mathcal{O}_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2} \oplus$ terms with negative degree (cf., for example, [**17**, Lemma 3.7, p. 64]).

Since U_i are strongly semistable of degree 0, it follows easily that $\mathrm{Hom}_{\mathbb{P}^2}(U_1, U_2 \otimes F_*\mathcal{O}_{\mathbb{P}^2}) = \mathrm{Hom}_{\mathbb{P}^2}(U_1, U_2)$. Hence, $\mathrm{Hom}_{\mathbb{P}^2}(F^*U_1, F^*U_2) \simeq \mathrm{Hom}_{\mathbb{P}^2}(U_1, U_2)$.

Let $U \in \mathcal{C}(V)$ and let $A \subset F^*(U)$ be a subobject in $\mathcal{C}(F^*(V))$. So A is a degree 0 subbundle. In particular, A is μ-semistable. Let $T_{\mathbb{P}^2}$ be the tangent bundle of $\mathbb{P}^2$. We get an $\mathcal{O}_{\mathbb{P}^2}$-linear homomorphism $f : T_{\mathbb{P}^2} \to \mathcal{H}\mathrm{om}(A, F^*(U)/A)$. Since we are on $\mathbb{P}^2$, it follows that μ-semistability is strong semistability and hence μ-semistable bundles are closed under tensor products. Therefore, $\mathcal{H}\mathrm{om}(A, F^*(U)/A)$ is μ-semistable of degree 0. Note that $T_{\mathbb{P}^2}$ is semistable with $\mu(T_{\mathbb{P}^2}) > 0$. This implies that $f = 0$. Hence, by standard descent theory for purely inseparable extensions, A descends to a subbundle $A_1 \subset U$ (cf. [**28**, Proposition 1.7]). This proves the conditions in [**14**, Proposition 2.21(a)] and hence the claim (10.1). □

As we have already observed (Remark 4.6), by a sufficiently large number of Frobenius pull-backs, we see that $\mathcal{H}_x(F^n(V))$ is *reduced*. Hence, by a repeated application of (10.1), we get that $\mathcal{H}_x(V)$ is *reduced*. The simple connectedness of $\mathbb{P}^2$ now forces that $\mathcal{H}_x(V)$ is also *connected*.

Remark 10.5. It is a fact shown in [**28**, Proposition 1.7] and more generally in [**30**] that on smooth projective X such that $\mu(T_X) > 0$, μ-(semi) stability is equivalent to strong (semi) stability.

Theorem 10.6. *Let X be a smooth projective surface and let* $\mathrm{char}(k) = p$ *with $p > h_G$. If $c_2 \gg 0$, then there exist stable principal G-bundles on X with holonomy group the whole of G.*

Proof. Observe that, by Lemma 6.11, if we construct a stable G-bundle on $\mathbb{P}^2$ with full holonomy, in other words with holonomy group the whole of G, then, by pulling back by a genuinely ramified cover $f : X \to \mathbb{P}^2$, we get the required bundle on an arbitrary X.

By Theorem 10.3, the moduli space $M_{\mathbb{P}^2}(G, c_2)^s \neq \emptyset$ for large c_2 and $p > h_G$. Let E be a stable G-bundle on $\mathbb{P}^2$. Deformation theory shows that if $H^2(E(\mathrm{ad}G)) = 0$, then the point E is a smooth point in $M_{\mathbb{P}^2}(G, c_2)^s$ and, further, the component containing E has the expected dimension. By Serre duality, $H^2(E(\mathrm{ad}G)) = H^0(E(\mathrm{ad}G) \otimes \mathcal{O}(-3))$. Since E is stable, being on $\mathbb{P}^2$ it is strongly stable and hence the bundle $E(\mathrm{ad}G)$ is semistable of degree 0. Therefore $H^0(E(\mathrm{ad}G) \otimes \mathcal{O}(-3)) = 0$.

By Proposition 10.4, the holonomy groups of strongly stable bundles on $\mathbb{P}^2$ are reduced and connected. Suppose that all the stable G-bundles in this smooth component have holonomy inside a smaller reductive group $H \subsetneq G$. Now up to conjugacy, there are only countably many reductive subgroups of a semisimple algebraic group G. Observe also that conjugate subgroups give isomorphic G-bundles in $M_{\mathbb{P}^2}(G, c_2)^s$. By a simple dimension count and an application of the Baire category theorem, we see that the moduli space $M_{\mathbb{P}^2}(G, c_2)^s$ cannot be covered by stable bundles with structure groups in $H \subsetneq G$. Hence, there is a stable G-bundle on X with holonomy group the whole of G. □

Remark 10.7. Existence of a smooth point on the moduli space of G-bundles on an arbitrary surface is still unknown. This is true on $\mathbb{P}^2$, which is what we use.

Remark 10.8. In the case when $G = \mathrm{GL}(n)$, such non-emptiness results are shown over fields of arbitrary characteristics in [**25**].

11. *Miscellaneous remarks*

Remark 11.1. All the Tannakian constructions in Sections 1–4 go through for big open subsets (complement codimension at least 2) $U \subset X$. Furthermore, $\varpi(U, x, \Theta)$ is well defined and there is a natural map $(\mathcal{C}_X^{\ell f}, x) \hookrightarrow (\mathcal{C}_U^{\ell f}, x)$ inducing a homomorphism $\varpi(U, x) \to \varpi(X, x)$. Since morphisms of vector bundles extend across codimension at least 2, it follows by the surjectivity criterion on Tannaka categories that this homomorphism is surjective.

REMARK 11.2. Note that $\varpi(\mathbb{P}^1)$ is *trivial*. On the other hand, $\varpi(\mathbb{P}^1 \times \mathbb{P}^1)$ is non-trivial by Theorem 10.3.

REMARK 11.3. The group scheme $\varpi(\mathbb{P}^2)$ has no characters since degree 0 line bundles on $\mathbb{P}^2$ are trivial. If $Z \to \mathbb{P}^2$ is the blow-up at a point, then Z has the rank of the Neron–Severi group $\mathrm{NS}(Z)$, that is 2, implying the existence of degree 0 line bundles. Thus $\varpi(Z)$ has non-trivial characters. This implies that even as abstract groups $\varpi(Z) \neq \varpi(\mathbb{P}^2)$. This shows that ϖ is not a birational invariant.

REMARK 11.4. We have remarked that $\varpi(X)$ depends on the choice of the polarization. It will be interesting to see how the group scheme changes with the polarization and what happens with the wall phenomenon.

REMARK 11.5. Throughout this paper, we assume that the degree of all our bundles is 0. In the light of [**8**], one should be able to define the holonomy group scheme for strongly lf-graded bundles with arbitrary μ (cf. also [**4**, Section 3]).

REMARK 11.6. After an earlier version of this paper was posted in the archives, we received a very interesting paper from Langer [**26**], where he studies the Tannaka category of strongly semistable bundles with all Chern classes zero (analogous to the semiharmonic Higgs bundles). Langer's construction can be seen as a special case when we assume that the second Chern classes of the bundles vanish. The S-fundamental group scheme can be seen easily enough to be a quotient of the holonomy group scheme $\varpi(X, x, \Theta)$.

11.1. *Some remarks on the secondary slope*

These remarks follow from a suggestion by Nori. Let $\mathcal{C}_o^{\ell f}$ be the subcategory of $\mathcal{C}^{\ell f}$ consisting of bundles with $c_1 = 0$. This allows us to define the following *secondary slope*:

$$\mu_2(E) := \frac{c_2(E) \cdot \Theta^{d-2}}{\mathrm{rank}(E)}.$$

LEMMA 11.7. *Let V and W be bundles with $c_1 = 0$. Then*

$$\mu_2(V \otimes W) = \mu_2(V) + \mu_2(W).$$

Proof. We see this easily as follows: the Chern character is given by

$$\mathrm{ch}(E) = r + c_1(E) + \tfrac{1}{2}(c_1^2 - 2c_2) + \dots .$$

It satisfies $\mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\mathrm{ch}(W)$. Since $c_1 = 0$, we have the equation

$$\frac{\mathrm{ch}(V \otimes W)}{rs} = \frac{\mathrm{ch}(V)}{r}\frac{\mathrm{ch}(W)}{s}$$

implying

$$1 - \frac{c_2(V \otimes W)}{rs} + \dots = \left(1 - \frac{c_2(V)}{r} + \dots\right)\left(1 - \frac{c_2(W)}{s} + \dots\right).$$

Hence, comparing terms degreewise, the formula for μ_2 follows. □

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References

1. M. Atiyah, N. Hitchin and I.M. Singer, 'Self-duality in four dimensional Riemannian geometry', *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 362 (1978) 425–461.
2. V. Balaji, 'Principal bundles on projective varieties and the Donaldson–Uhlenbeck compactification', *J. Differential Geom.* 76 (2007) 351–398, arXiv:math.AG/0505106.
3. V. Balaji, 'Addendum to Principal bundles on projective varieties and the Donaldson–Uhlenbeck compactification', *J. Differential Geom.* 83 (2009) 461–463.
4. V. Balaji and J. Kollár, 'Holonomy groups of stable vector bundles', *Publ. RIMS Kyoto Univ.* 44 (2008) 183–211, archiv:math.AG/06001120.
5. V. Balaji and A. J. Parameswaran, 'Semistable principal bundles-II (in positive characteristics)', *Transform. Groups* l8 (2003) 3–36.
6. I. Biswas, 'Stable bundles and extension of structure group', *Differential Geom. Appl.* 23 (2005) 67–78.
7. I. Biswas, 'On the algebraic holonomy of stable principal bundles', Preprint, 2010, arXiv:math/0703104v2.
8. I. Biswas and A. J. Parameswaran, 'Monodromy group for a strongly semistable principal bundle over a curve, II', *K-Theory* 1 (2008) 583–607.
9. I. Biswas, A. J. Parameswaran and S. Subramanian, 'Monodromy groups of strongly semistable bundles on curves', *Duke Math. J.* 132 (2006) 1–48.
10. I. Biswas and S. Subramanian, 'Flat holomorphic connections on principal bundles over a projective manifold', *Trans. Amer. Math. Soc.* 356 (2004) 3995–4018.
11. W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics (Cambridge University Press, Cambridge, 1993).
12. F. Coiai and Y. Holla, 'Extension of structure groups of principal bundles in positive characteristics', *J. reine angew. Math.* 595 (2006) 1–24.
13. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras* (Wiley-Interscience, New York, 1988).
14. P. Deligne and J. Milne, *Tannaka categories*, Lecture Notes in Mathematics 900 (Springer, New York, 1982).
15. C. Deninger and A. Werner, 'On Tannaka duality for vector bundles on p-adic curves', Preprint, 2005, arXiv:math.AG/0505263.
16. S. K. Donaldson, 'Polynomial invariants for smooth four manifolds', *Topology* 29 (1990) 257–315.
17. L. Ein, 'Stable vector bundles on projective spaces in char $p > 0$', *Math. Ann.* 254 (1980) 53–72.
18. T. L. G'omez, A. Langer, A. H. W. Schmitt and I. Sols, 'Moduli spaces for principal bundles in arbitrary characteristic', *Adv. Math.* 219 (2008) 1177–1245, math.AG/0506511.
19. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics 52 (Springer, London, 1977).
20. D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics E31 (Vieweg, Braunschweig, 1997).
21. S. Ilangovan, V. B. Mehta and A. J. Parameswaran, 'Semistability and semisimplicity in representations of low height in positive characteristic', *A tribute to C. S. Seshadri*, Trends in Mathematics (Birkhäuser Boston, MA, 2004).
22. C. Jordan, 'Mémoire sur les équations differentielles linéaires á intégrale algébrique', *J. Math.* 84 (1878) 89–215.
23. J. L. Koszul, *Lectures on fibre bundles and differential geometry*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics 20 (Springer, Berlin, 1960).
24. A. Langer, 'Semistable bundles in positive characteristics', *Ann. of Math.* 159 (2004) 251–276.
25. A. Langer, 'Moduli spaces and Castelnuovo–Mumford regularity of sheaves on surfaces', *Amer. J. Math.* 128 (2006) 373–417.
26. A. Langer, 'On the S-fundamental group scheme', *Ann. Inst. Fourier* (*Grenoble*) (2011), to appear.
27. M. Larsen and R. Pink, 'Finite subgroups of algebraic groups', Preprint, 1998, http://www.math.ethz.ch/∼pink/ftp/LP5.pdf.
28. V. B. Mehta and M. V. Nori, 'Semistable sheaves on homogeneous spaces and abelian varieties', *Proc. Indian Acad. Sci. Math. Sci.* 93 (1984) 1–12.
29. V. B. Mehta and A. Ramanathan, 'Restriction of stable sheaves and representations of the fundamental group', *Invent. Math.* 77 (1984) 163–172.
30. V. B. Mehta and A. Ramanathan, 'Homogeneous bundles in characteristic p', *Algebraic geometry—open problems* (eds C. Ciliberto *et al.*), Lecture Notes in Mathematics 997 (Springer, Berlin, 1982) 315–320.
31. J. S. Milne, *Étale cohomology* (Princeton University Press, Princeton, 1980).

32. J. P. Murre, *Lectures on an introduction to Grothendieck's theory of the fundamental group* (Tata Institute of Fundamental Research, Bombay, 1967).
33. M. S. Narasimhan and C. S. Seshadri, 'Stable and unitary vector bundles on a compact Riemann surface', *Ann. of Math.* 82 (1965) 540–567.
34. M. V. Nori, 'The fundamental group scheme', *Proc. Indian Acad. Sci. Math. Sci.* 91 (1982) 73–122.
35. A. J. Parameswaran and S. Subramanian, 'On the spectrum of asymptotic slopes', *Teichmüller theory and moduli problems*, Ramanujan Mathematical Society Lecture Notes Series 10 (Ramanujan Mathematical Society, Mysore, 2010) 519–528, arXiv:math.AG/0804.1626.
36. T. Peternell and A. J. Sommese, 'Ample vector bundles and branched coverings', *Comm. Algebra* 28 (2000) 5573–5599, arXiv:math.AG/9907081.
37. S. Ramanan and A. Ramanathan, 'Some remarks on the instability flag', *Tohôku Math. J.* 36 (1984) 269–291.
38. J. P. Serre, 'Sur la semi-simplicité des produits tensoriels de représentations de groupes', *Invent. Math.* 116 (1994) 513–530.
39. J. P. Serre, 'Moursund lectures', Mathematics Department, University of Oregon, 1998.
40. C. Simpson, 'Higgs bundles and local systems', *Publ. Math. Inst. Hautes Études Sci.* 75 (1992) 5–95.
41. C. H. Taubes, 'Self-dual connections on 4-manifolds with indefinite intersection matrix', *J. Differential Geom.* 19 (1984) 517–560.
42. K. Uhlenbeck and S.-T. Yau, 'On the existence of Hermitian Yang–Mills connections in stable bundles', *Comm. Pure Appl. Math.* 39 (1986) 257–297.
43. B. Weisfeiler, 'Post-classification version of Jordan's theorem on finite linear groups', *Proc. Natl. Acad. Sci. USA* 81 (1984) 5278–5279.

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