ADDENDUM TO “PRINCIPAL BUNDLES ON PROJECTIVE VARIETIES AND THE DONALDSON-UHLENBECK COMPACTIFICATION”

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Abstract

In this addendum we complete the proof of Theorem 7.10 in [1].

In the paper [1], the proof of Theorem 7.10 is incomplete. This theorem proves the asymptotic non-emptiness of the moduli space of stable $G$–bundles on smooth projective surfaces, with structure group $G$ which is simple and simply connected. This was pointed to us by Prof. S. Ramanan. In this addendum we complete the proof of this theorem. We retain the notations of [1].

The results in [1] which need to be modified are Lemma 7.5 and the proof of Proposition 7.6. In the cases which were considered in the proof of [1, Proposition 7.6], the case when the monodromy group of a rank 2 stable bundle is the normalizer of a one-dimensional torus in $SL(2)$ was not taken into consideration. When this case is allowed, the loci of bundles in $M_C(SL(2))^a$, where the monodromy groups are proper subgroups of $SL(2)$, is no longer a countable set. In fact, it contains a subvariety whose dimension is $g(C) - 1$, $g(C)$ being the genus of $C$. We therefore cannot work with the restriction to general curves as was done earlier in [1]. Instead, we use a technique due to S. Donaldson (cf. [2]) which completes the argument without much difficulty.

Recall from [1, Page 393] that we wish to estimate the dimension of subset $Z_C$ of representations of $\pi_1(C)$ in $SU(2)$ which lie entirely in these families of groups up to conjugacy by the diagonal action of $SU(2)$.

Since all the cases except $M(V) = N(T)$ have been handled in [1] we need only take care of the locus of those bundles whose monodromy lies in the normalizer of the one dimensional torus. In this case it is easy to see that such a rank two bundle can be realised as a direct image of a line bundle from an unramified two sheeted cover $p : D \to C$ of $C$, and since we need the structure group to be $SL(2)$, the locus is precisely

$$\{p_*(L)|\det(p_*(L)) \simeq \mathcal{O}_C\}.$$
Consider the “det” map $\text{det} : \text{Pic}(D) \rightarrow \text{Pic}(C)$ given by $L \mapsto \text{det}(p_*(L))$. This map is surjective with kernel as the Prym variety $\text{Prym}(p)$ and $\dim(\text{Prym}(p)) = g(D) - g(C) = g(C) - 1$. These comments immediately imply the following lemma.

**Lemma 1.** (cf. [1, Lemma 7.5]) The locus of points $Z_C$, in the moduli space of stable vector bundles $M_{C, SL(2)}$ of rank 2 and trivial determinant on the curve $C$ whose monodromy is among the set of subgroups listed above has dimension $\leq g(C) - 1$. In particular, if $g \geq 2$, $Z_C$ is a proper subset of the moduli space $M^s_{C, SL(2)}$ which does not contain any open subset.

We now recall the following result, which is a consequence of Donaldson’s proof of the generic smoothness theorem of the moduli space of $SL(2)$–bundles on algebraic surfaces ([2, page 309]).

**Proposition 2.** Let $C$ be any smooth curve on $X$ of genus at least 2. Then for $c = c_2(V) \gg 0$, the restriction map $r_C : M_{SL(2)}(c)^s \rightarrow M^s_{C, SL(2)}$ is defined on a Zariski open subset $U \subset M_{SL(2)}(c)^s$ and furthermore, it is differentially surjective.

Using this we have the following cf ([1, Prop 7.6]):

**Proposition 3.** (cf. [1, Prop 7.6]) There exists a rank 2 stable bundle $E$ with $c_2(E) \gg 0$ and trivial determinant on the surface $X$ such that the restriction $E|_C$ to a curve $C \subset X$ has monodromy subgroup to be the whole of $SL(2)$ itself.

**Proof.** Consider the rational map defined by restriction to the curve $C$, $r_C : M_{SL(2)}(c)^s \rightarrow M^s_{C, SL(2)}$. If the curve $C$ is as in Proposition 2, it implies that $\text{Im}(r_C) \subset M^s_{C, SL(2)}$ contains an open subset and hence (by Lemma 1), it is not completely contained in the subset $Z_C$ of stable bundles with finite monodromy groups or with monodromy being the normalizer of $T$. Therefore there exists at least one stable bundle with the whole of $SL(2)$ as its monodromy subgroup. This proves the proposition. q.e.d.

Now to complete the proof of Theorem 7.1 in [1], choose $C$ to be a high degree curve in $X$ and $E$ as in Proposition 3. Extending structure group of this bundle to $G$ as in [1], it is easy to see that $E(G)$ is stable on $X$ since $E(G)|_C$ is so (see [1, Lemma 4.5]).

**References**


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