# An Introduction to the Mod $p$ Local Langlands Correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$

*A report submitted by*


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*as part of*

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## Preface

This document is a record of everything I read and learnt during my VSRP project under the guidance of Professor **Eknath Ghate**. It is divided into four chapters–

- Chapter 1: premise of the project and a rough description of the Langlands program.
- Chapter 2: introduction to mod $p$ smooth representation theory and classification of irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over an algebraically closed field of characteristic $p$. References: F. Herzig's course notes [**Herz**, **Herz+**] and C. Breuil's course notes [**Br07**].
- Chapter 3: background on the structure theory of extensions of local fields, classification of two-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over an algebraically closed field of characteristic $p$, a description of the mod $p$ local Langlands correspondence. References: Serre's Local Fields [**Ser80**] and C. Breuil's course notes [**Br07**].
- Chapter 4: a brief discussion on passing from Galois representations to representations of the Borel subgroup via Fontaine's $(\varphi, \Gamma)$-modules. Reference: L. Berger's course notes [**Ber10**], and [**Ber14**].

I am immensely grateful to my project advisor, Professor **Eknath Ghate**, for his exceptional dedication and guidance. Through online live sessions spanning multiple hours, he generously devoted his time to teach and mentor us. Additionally, his availability and responsiveness through email allowed for continuous and fruitful exchanges. This project has sparked my interest in the Langlands programme, and I am motivated to explore this topic further in my future studies due to the profound influence of his guidance.

I would also like to thank the organizers of VSRP for this wonderful program and my team members Atharva and Sudharshan for their contributions and discussions.

Undertaking this project has been a challenging and enriching experience for me. I eagerly intend to pursue related topics in in my future research endeavors.

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June 2023

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*Notations.*

- $p$ is a fixed prime number,
- $k$ is an algebraically closed field of characteristic $p > 0$,
- $G = \mathrm{GL}_n(\mathbb{Q}_p)$, $K = \mathrm{GL}_n(\mathbb{Z}_p)$, $K(r) = 1 + p^r \,\mathrm{Mat}_{n\times n}(\mathbb{Z}_p)$, $r \geqslant 1$.
- For a group $\Gamma$, $\mathsf{Mod}_\Gamma$ is the category of $\Gamma$-representations over $k$, and $\mathsf{Mod}^{\mathsf{sm}}{}_\Gamma$ is the category of smooth $\Gamma$-representations over $k$. $\mathsf{Vec}_k$ denotes the category of $k$-vector spaces.
- For a field $K$, we set $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$.
- $\mathbb{Q}_{p^n}$ denotes the unique (after fixing $\overline{\mathbb{Q}}_p$) unramified $n$-degree extension of $\mathbb{Q}_p$.

# Chapter 1

# Introduction

**0.1. Weil-Deligne representations.** There is a natural surjection $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ which comes from the residue field extension $\overline{\mathbb{F}}_p/\mathbb{F}_p$. The kernel of this map is called the **inertia group**. We remark that

$$\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \lim_n \mathrm{Gal}(\overline{\mathbb{F}}_{p^n}/\mathbb{F}_p) \cong \lim_n \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}},$$

the profinite completion of $\mathbb{Z}$. There is a special element, called the **Frobenius** in $\mathrm{Gal}(\overline{\mathbb{F}}_p, \mathbb{F}_p)$ which acts as raising to the $p$th power. This generates a cyclic group isomorphic to $\mathbb{Z}$ inside $\mathrm{Gal}(\overline{\mathbb{F}}_p, \mathbb{F}_p)$. The preimage of this subgroup along the natural map $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is called the **Weil group** of $\mathbb{Q}_p$, which we denote by $W_{\mathbb{Q}_p}$. A **Weil-Deligne representation** is a pair $(\rho_0, N)$ consisting of a representation $\rho_0$ of the Weil group $W_{\mathbb{Q}_p}$, along with a nilpotent operator $N$ called the **monodromy operator**, such that

$$\rho_0(\sigma) N \rho_0(\sigma)^{-1} = \|\sigma\| N$$

for all $\sigma \in W_{\mathbb{Q}_p}$, where $\|\sigma\|$ is the valuation of the element of $\mathbb{Q}_p^\times$ corresponding to $\sigma$ under the isomorphism $\mathbb{Q}_p^\times \xrightarrow{\sim} W_{\mathbb{Q}_p}^{\mathrm{ab}}$ given by local class field theory.

**0.2. $p$-adic local Langlands.** Fix a prime $\ell \neq p$. The classical local Langlands correspondence states roughly that there is an injective map

$$\left\{ \begin{array}{c} \text{continuous representations of} \\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \text{ on } n\text{-dimensional} \\ \overline{\mathbb{Q}}_\ell\text{-vector spaces, up to} \\ \text{isomorphism. Additionally,} \\ \text{Frobenius must act semisimply.} \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{irreducible, admissible} \\ \text{representations of } \mathrm{GL}_n(\mathbb{Q}_p) \\ \text{on } \overline{\mathbb{Q}}_\ell\text{-vector spaces,} \\ \text{up to isomorphism.} \end{array} \right\}.$$

A representation $G \to \mathrm{Aut}\, V$ called **admissible** if $V^K$ is finite dimensional for any compact open subgroup $K$ of $G$. To obtain a bijection, the left-hand side is enlarged by replacing it by the set of Frobenius-semisimple Weil-Deligne representations of the Weil group $W_{\mathbb{Q}_p}$. That is,

$$\left\{\begin{array}{c}\text{continuous Weil-Deligne}\\ \text{representations of } W_{\mathbb{Q}_p} \text{ on}\\ n\text{-dimensional } \overline{\mathbb{Q}}_\ell\text{-vector spaces,}\\ \text{up to isomorphism. Additionally,}\\ \text{Frobenius must act semisimply.}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{irreducible, admissible}\\ \text{representations of} \mathrm{GL}_n(\mathbb{Q}_p)\\ \text{on } \overline{\mathbb{Q}}_\ell\text{-vector spaces,}\\ \text{up to isomorphism.}\end{array}\right\}.$$

When $\ell = p$, we have "more" Galois representations due to the fact that $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ have compatible topologies.

For $n = 1$, this correspondence reduces to local class field theory. Indeed, for $n = 1$, the left hand side is just continuous homomorphisms $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Q}_\ell^\times$. Such maps factor as $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}} \to \mathbb{Q}_\ell^\times$. The local reciprocity map from local class field theory tells us that there is an "almost isomorphism"

$$\theta_p \colon \mathbb{Q}_p^\times \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}}.$$

This is an almost isomorphism in the sense that it factors as an isomorphism through the profinite completion of $\mathbb{Q}_p^\times$.

**0.3. Mod $p$ local Langlands.** We hope to prove something similar in the positive characteteristic case, which we state below–

*Let $k$ be any algebraically closed field of characteristic $p$. There is a canonical bijection between isomorphism classes of smooth irreducible 2-dimensional representations of $W_{\mathbb{Q}_p}$ over $k$ and isomorphism classes of smooth admissible irreducible supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $k$.*

We explain what supercuspidal means. Let $B \subset \mathrm{GL}_2(\mathbb{Q}_p)$ be the subgroup of all upper triangular invertible matrices. A representation $G \to \mathrm{Aut}\, V$ of a topological group $G$ is called **smooth** if the action map $G \times V \to V$ is continuous with the discrete topology on $V$. Among continuous admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, there are a few called **parabolic inductions**. They are of the form:

$$\text{c-Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi := \{f \colon \mathrm{GL}_2(\mathbb{Q}_p) \to k \text{ locally constant, } f(bg) = \chi(b) f(g) \text{ for all } b \in B, g \in \mathrm{GL}_2(\mathbb{Q}_p)\}$$

for a smooth character $\chi \colon B \to k^\times$, with left action of $\mathrm{GL}_2(\mathbb{Q}_p)$ given by $(g \cdot f)(h) := f(hg')$. Such representations are always smooth admissible and irreducible for "most" $\chi$. A smooth irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $k$ is called **supercuspidal** if it is not a subquotient of a parabolic induction.

# Chapter 2

# Mod $p$ representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

## 1. $p$-adic Groups

Consider $G := \mathrm{GL}_n(\mathbb{Q}_p)$. This is naturally a topological group with the topology coming from the complete topology of $\mathbb{Q}_p$. It is precisely the subspace topology of the product topology given by the inclusion $\mathrm{GL}_n(\mathbb{Q}_p) \hookrightarrow \mathbb{Q}_p^{\oplus n}$. A fundamental system of open neighborhoods of the identity $1 \in \mathrm{GL}_n(\mathbb{Q}_p)$ is given by

$$\mathrm{GL}_n(\mathbb{Z}_p) \supset 1 + p\,\mathrm{Mat}_n(\mathbb{Z}_p) \supset 1 + p^2\,\mathrm{Mat}_n(\mathbb{Z}_p) \supset \cdots \supset 1 + p^r\,\mathrm{Mat}_n(\mathbb{Z}_p) \supset \cdots$$

For brevity, denote $K = \mathrm{GL}_n(\mathbb{Z}_p)$ and $K(r) := 1 + p^r\,\mathrm{Mat}_n(\mathbb{Z}_p)$, $r \geqslant 1$.

**1.1. Proposition. —** $\mathrm{GL}_n(\mathbb{Z}_p)$ *is a maximal compact subgroup of* $\mathrm{GL}_n(\mathbb{Q}_p)$.

*Proof.* First of all, $\mathrm{GL}_n(\mathbb{Z}_p)$ is compact because it is the preimage of the closed set $\mathbb{Z}_p^\times$ under $\det\colon \mathrm{Mat}_n\,\mathbb{Z}_p \to \mathbb{Z}_p$, so its closed, and $\mathrm{Mat}_n\,\mathbb{Z}_p$ is compact because its homeomorphic to $\mathbb{Z}_p^{\oplus n^2}$. By the theory of Smith normal form, any $A \in \mathrm{GL}_n(\mathbb{Q}_p)$ can be written as $A = PDQ$ where $P, Q \in \mathrm{GL}_n(\mathbb{Z}_p)$ and $D$ is a diagonal matrix. We can also ensure that the diagonal of $D$ consists of powers of $p$ because we can "absorb" units (of $\mathbb{Z}_p$) into $P$ or $Q$. So any subgroup strictly containing $\mathrm{GL}_n(\mathbb{Z}_p)$ must contain a diagonal matrix with at least one of its diagonal entries having negative $p$-adic valuation because the product of diagonal entries, which is the determinant, must be a unit. The subgroup generated by such a diagonal matrix cannot be compact because of unbounded negative $p$-adic valuation. $\square$

For $n = 2$, we also define the following *closed* subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$– the **Borel subgroup** $B$ of all upper triangular matrices, the **maximal torus** $T$ consisting of diagonal matrices, the **unipotent radical** $U = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{Q}_p \right\}$. Symmetrically, $\overline{B}$ is the set of all lower triangular matrices and $\overline{U} = \left\{ \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} : t \in \mathbb{Q}_p \right\}$. There is an exact sequence

$$1 \to U \to B \xrightarrow{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}} T \to 1.$$

The above exact sequence is split due to the natural inclusion $T \hookrightarrow B$. Thus, $B = T \ltimes U$ and analogously $\overline{B} = T \ltimes \overline{U}$.

**1.2. Iwasawa decomposition. —** $G = \overline{B}K$.

*Proof.* We perform "integral" column operations to reduce a matrix to lower triangular form. Do the following:

- Permute the columns so that the top left entry has minimal $p$-adic valuation among entries in the first row.
- Now add suitable *integral* multiples of the first column to the others so that all the entries except the top left in the first row is 0.
- Repeat this on the bottom right minor with induction on size of the matrix. □

## 2. Smooth representations

Fix an algebraically closed field $k$ of characteristic $p > 0$ and $\Gamma$ a closed subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$, or a finite group with the discrete topology. When we say $\pi$ is a representation, we denote the vector space as $\pi$ itself. All our representations are going to be over $k$.

**2.1. Definition.** A representation $\pi$ of $\Gamma$ is said to be **smooth** if the action map $\Gamma \times \pi \to \pi$ is continuous where $\pi$ is given the discrete topology. This is equivalent to saying

$$\pi = \bigcup_{\text{open subgroups } W \subseteq \Gamma} \pi^W$$

as the following proposition shows.

**2.2. Proposition. —** *$\Gamma \times \pi \to \pi$ is continuous if and only if $\pi = \bigcup_{W \subseteq \Gamma, open} \pi^W$.*

*Proof.* If $\Gamma \times \pi \to \pi$ is continuous then the preimage of any vector $v \in \pi$ is certainly open as $\pi$ has the discrete topology. We can then intersect the preimage with the open set $\Gamma \times \{v\}$ to conclude that the stabilizer of $v$ is open. For the other direction, assume $\pi = \bigcup_{W \subset \Gamma,\text{ open}} \pi^W$. Take any vector $v \in \pi$ and look at its preimage, say $A$, along $\Gamma \times \pi \to \pi$. Pick any vector $x \in \pi$ such that $A \cap (\Gamma \times \{x\}) \neq \varnothing$. Then $A \cap (\Gamma \times \{x\})$ must be homeomorphic to the stabilizer of $v$. It follows that any point in $\Gamma \times \pi$, say $(\gamma, x)$, has a neighborhood contained in $\Gamma \times \pi$ which is homeomorphic to the stabilizer of $v$. Since the stabilizer of $v$ contains an open set, it follows that $\Gamma \times \pi$ is open. □

For any (not necessarily smooth) representation $\Gamma$, we define the **smoothening**

$$\pi^\infty := \bigcup_{\text{open subgroups } W} \pi^W.$$

Note that this is the largest smooth subrepresentation of $\pi$ and it is much stonger than just being a continuous representation. Smoothness implies that the representation is continuous with respect to any topology on $\pi$. From now onwards, $\mathsf{Mod}_\Gamma$ be the category of $\Gamma$-representations over $k$ and $\mathsf{Mod^{sm}}_\Gamma$ be the category of smooth $\Gamma$-representations over $k$. We remark that $(-)^\infty\colon \mathsf{Mod}_\Gamma \to \mathsf{Mod^{sm}}_\Gamma$ is right adjoint to the forgetful functor $\mathsf{Fgt}\colon \mathsf{Mod^{sm}}_\Gamma \to \mathsf{Mod}_\Gamma$. In terms of universal properties, any map to $W$, a (not necessarily smooth) $G$-representation, from a smooth $G$-representation factors through $W^\infty$.

**2.3. Induced representations.** If $H \subset G$ are arbitrary groups and $M$ a $H$-module, the canonical way to upgrade it to a $G$-module is to consider $M \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G]$. In a similar vein, we define two induction functors. The goal is to find both left and right adjoints to the restriction functor $(-)|_H\colon \mathsf{Mod^{sm}}_G \to \mathsf{Mod^{sm}}_H$. Assume $H$ is a closed subgroup of $\Gamma$ and $\sigma$ a smooth representation of $H$. Naively, one may define induction as $\mathrm{Hom}_{H\text{-set}}(\Gamma, \sigma)$. However, this is not necessarily smooth.

**2.4.** *Ordinary induction is not always smooth.* Take $H = \{1\}, \Gamma = \mathbb{Q}_p^\times$, and $\sigma = \Gamma$ set-theoretically. Then $\mathrm{Hom}_{H\text{-set}}(\Gamma, \sigma) = \mathrm{Mor}_{\mathrm{Set}}(\Gamma, \Gamma)$. Consider the stabilizer of $\mathrm{id}\colon \Gamma \to \Gamma$. If it is fixed by some $g \in \Gamma$ then it's clear that $wg = w$ for each $w \in \Gamma$. Hence, $\mathrm{Stab}_\Gamma \mathrm{id} = \{1\}$, which doesn't contain any open sets. So id is not a smooth vector.

Define the **(smooth) induction** as

$$\mathrm{Ind}_H^\Gamma \sigma := \mathrm{Hom}_{H\text{-set}}(\Gamma, \sigma)^\infty = \{f\colon \Gamma \to \sigma\colon f(h\gamma) = h \cdot f(\gamma) \text{ for all } h \in H, \gamma \in \Gamma\}^\infty.$$

The action of $\Gamma$ is given by $(g \cdot f)(\gamma) := f(\gamma g)$. This procedure gives a smooth representation of $\Gamma$ from a smooth representation of $H$. Note that the support of any $f \in \mathrm{Ind}_H^\Gamma \sigma$ is clopen in $H\backslash\Gamma$. We may define another type of induction functor for the case when $H$ is an open (hence closed) subgroup. Define the **compact induction** as

$$\text{c-}\mathrm{Ind}_H^\Gamma \sigma := \{f\colon \Gamma \to \sigma\colon f(h\gamma) = h\cdot f(\gamma) \text{ for all } h \in H, \gamma \in \Gamma,\ H\backslash \mathrm{Supp} f \text{ is compact (i.e., finite)}\},$$

where $\mathrm{Supp} f := f^{-1}(\sigma \setminus \{0\})$. It is easy to observe that this is a union of $W$-cosets so the (topological) quotient $\mathrm{Supp} f/W$ makes sense. This is another procedure which gives a smooth representation of $\Gamma$.

**2.5. Proposition. —** $\text{c-}\mathrm{Ind}_W^\Gamma \tau$ *is smooth. In particular,* $\text{c-}\mathrm{Ind}_W^\Gamma \tau \subseteq \mathrm{Ind}_W^\Gamma \tau$.

*Proof.* We have seen that $\text{c-}\mathrm{Ind}_W^\Gamma \tau$ is generated by $[1, x]$, $x \in \tau$, as a $\Gamma$-representation. So, it suffices to show that $[1, x]$ is a smooth vector for each $x \in \tau$. Denoting, $f = [1, x]\colon \Gamma \to \tau$, it is defined as $f(w) = wx$ for all $w \in W$ and $f(w) = 0$ otherwise. Then, it is easy to see that $\mathrm{Stab}_W x$ fixes $f$, which contains an open subgroup. Since $W$ is open in $\Gamma$, it follows that the same open subgroup is also open in $\Gamma$. $\square$

For $\gamma \in \Gamma$ and $x \in \tau$, denote by $[\gamma, x] \in \text{c-}\mathrm{Ind}_W^\Gamma \tau$ the function supported at $W\gamma^{-1}$ and defined by $[\gamma, x](\gamma^{-1}) = x$. It is easily seen that we have the following relations

$$[\gamma w, x] = [\gamma, w \cdot x] \text{ for } w \in W, \qquad g \cdot [\gamma, x] = [g\gamma, x] \text{ for } g \in \Gamma.$$

Also, any element of $\text{c-Ind}_W^\Gamma \tau$ can be written as

$$\sum_i [\gamma_i, x_i] = \sum_i \gamma_i \cdot [1, x_i].$$

Hence, $[1, x_i]$ generates $\text{c-Ind}_W^\Gamma \tau$ as a $\Gamma$-representation. Another way to define compact induction is $\text{c-Ind}_H^\Gamma \sigma = \sigma \otimes_{k[H]} k[G]$ as the following result shows.

**2.6. Proposition. —** $\text{c-Ind}_W^\Gamma \cong k[\Gamma] \otimes_{k[W]} \tau$ *as* $\Gamma$*-representations. The isomorphism is given by* $[\gamma, x] \to \gamma \otimes x$.

*Proof.* Consider the $k[W]$-bilinear map $k[\Gamma] \times \tau \to \text{c-Ind}_W^\Gamma$ given by $(\gamma, x) \mapsto [\gamma, x]$. Here, we assume left-action of $k[W]$ on $k[\Gamma]$. By the universal property of tensor products, we get a map of $k[W]$-modules $f\colon k[\Gamma] \otimes_{k[W]} \tau \to \text{c-Ind}_W^\Gamma$, $\gamma \otimes x \mapsto [\gamma, x]$. This upgrades to a map of $k[\Gamma]$ modules simply because $f(g\gamma \otimes x) = [g\gamma, x] = g \cdot [\gamma, x] = g \cdot f(\gamma \otimes x)$ for all $g \in \Gamma$. We claim that this is the required isomorphism. Indeed, it's clear that $f$ is surjective since functions of the form $[1, x]$ generate $\text{c-Ind}_W^\Gamma \tau$. For injectivity, let's assume that $\gamma_1 \otimes x_1 + \gamma_2 \otimes x_2 + \cdots + \gamma_n \otimes x_n \mapsto 0$. This means that $\sum_i [\gamma_i, x_i] = 0$. We know that $\mathrm{Supp}[\gamma_i, x_i] = W\gamma_i^{-1}$. Suppose $\gamma_1$ is in the coset $Wu$. In the sum $\sum_i [\gamma_i, x_i] = 0$, we can delete all the $[\gamma_i, x_i]$ with $\gamma_i \notin Wu$. Using $[\gamma w, x] = [\gamma, w \cdot x]$, we can assume $\gamma_i = \gamma_j := \gamma$ for $i \neq j$ while possibly changing the $x_i$'s. This means that $[\gamma, \sum_i x_i] = 0$. Hence, $\sum_i x_i = 0$. Thus, $\sum_i \gamma_i \otimes x_i = \gamma \otimes (\sum_i x_i) = 0$. □

**2.7. Frobenius Reciprocity. —** *Let* $H \subset \Gamma$ *and* $W \subset \Gamma$ *be closed and open subgroups respectively. Let* $\pi$ *be a smooth* $\Gamma$*-representation,* $\sigma$ *a smooth* $H$*-representation, and* $\tau$ *a smooth* $W$*-representation. We have canonical isomorphisms*

- $\mathrm{Hom}_\Gamma(\pi, \mathrm{Ind}_H^\Gamma \sigma) \cong \mathrm{Hom}_H(\pi|_H, \sigma)$.
- $\mathrm{Hom}_\Gamma(\text{c-Ind}_W^\Gamma \tau, \pi) \cong \mathrm{Hom}_W(\tau, \pi|_W)$.

*Proof.* For the first part, we have

$$\begin{aligned}
\mathrm{Hom}_\Gamma(\pi, \mathrm{Ind}_H^\Gamma \sigma) &= \mathrm{Hom}_\Gamma(\pi, \mathrm{Hom}_H(\Gamma, \sigma)^\infty) \\
&\cong \mathrm{Hom}_\Gamma(\pi, \mathrm{Hom}_H(\Gamma, \sigma)) && ((-)^\infty \text{ is adjoint to } \mathsf{Fgt}) \\
&\cong \mathrm{Hom}_\Gamma(\Gamma, \mathrm{Hom}_H(\pi, \sigma)) \\
&\cong \mathrm{Hom}_H(\pi, \sigma) \\
&= \mathrm{Hom}_H(\pi|_H, \sigma).
\end{aligned}$$

The second part is just Hom-tensor adjunction. To write it explicitly, we have the isomorphism

$$\mathrm{Hom}_\Gamma(k[\Gamma] \otimes_{k[W]} \tau, \pi) \cong \mathrm{Hom}_W(\tau, \pi)$$

given by $\varphi \longmapsto (t \mapsto \varphi(1 \otimes t))$ and its inverse $(\gamma \otimes t \mapsto \gamma \cdot \alpha(t)) \longleftarrow\!\!\shortmid \alpha$. □

In other words, $\mathrm{Ind}_H^\Gamma\colon \mathsf{Mod^{sm}}_H \to \mathsf{Mod^{sm}}_\Gamma$ is a right adjoint to the restriction functor $(-)|_H\colon \mathsf{Mod^{sm}}_\Gamma \to \mathsf{Mod^{sm}}_H$. Similarly, $\text{c-Ind}_W^\Gamma\colon \mathsf{Mod^{sm}}_H \to \mathsf{Mod^{sm}}_\Gamma$ is a left adjoint to the same. As a corollary, we obtain that $\mathrm{Ind}_H^\Gamma$ preserves limits, in particular left exact, and c-$\mathrm{Ind}_H^\Gamma$ preserves colimits, in particular right exact.

**2.8. Corollary (Ind behaves well with tensoring). —** *Let $H$ be a closed subgroup of $G$. Given a representations $W$ and $V$ of $G$ and $H$, respectively, we have* $(\mathrm{Ind}_H^G V) \otimes W \cong \mathrm{Ind}_H^G(V \otimes W|_H)$.

*Proof.* Immediate from Frobenius reciprocity and universal properties. $\square$

**2.9. Pro-$p$ groups.** A **pro-$p$ group** is a topological group which is the inverse limit of an inverse system of $p$-groups. Canonical example: $\mathbb{Z}_p$.

**2.10. Lemma. —** *$K(1)$ is a pro-$p$ group.*

*Proof.* Indeed, $K(1) \cong \lim(1 + \mathrm{Mat}_n(\mathbb{Z}/p^r\mathbb{Z}))$. $\square$

**2.11.** *Example.* Let $I(1)$ denote the preimage in $K$ of the upper unipotent

$$\begin{bmatrix} 1 & \star \\ 0 & 1 \end{bmatrix} \subset \mathrm{GL}_2(\mathbb{F}_p).$$

This subgroup is pro-$p$ by an argument similar to the previous lemma. It, and conjugates thereof, are called **pro-$p$ Iwahori subgroups** of $K$.

**2.12. Lemma. —** *Any nonzero smooth representation $\tau$ of a pro-$p$ group $H$ has $H$-invariant vectors, i.e., $\tau^H \neq 0$.*

*Proof.* Without loss of generality, assume $k = \mathbb{F}_p$. Fix a nonzero $v \in \tau \setminus \{0\}$. By continuity, there exists an open normal subgroup $U$ such that $x$ is fixed by $U$. Then $H/U$ is a $p$-group with a nonzero representation $\tau^U$. We may now replace $H$ by $H/U$ and assume $H$ is a finite $p$-group. The $H$-orbit of $x$ is finite and consequently there is a finite-dimensional subspace, say $\mathbb{F}_p^m$, which is stable under $H$. We may assume $m \geqslant 2$ because otherwise $H$ is trivial and there is nothing to do. Here, $\mathrm{GL}_m(\mathbb{F}_p)$ is finite as a set whose cardinality is divisible by $p$. Write $\mathrm{GL}_m(\mathbb{F}_p)$ as a disjoint union of $H$-orbits. By orbit-stabilizer theorem, all orbits have size a power of $p$. There is a singleton orbit $\{0\}$. Thus, there are other orbits of size 1. $\square$

## 3. Weights

**3.1. Proposition.** *Any irreducible smooth representation of* $\mathrm{GL}_n(\mathbb{Z}_p)$ *over $k$ factors through* $\mathrm{GL}_n(\mathbb{F}_p)$ *via the natural surjection* $\mathrm{GL}_n(\mathbb{Z}_p) \to \mathrm{GL}_n(\mathbb{Z}_p)/K(1) \cong \mathrm{GL}_n(\mathbb{F}_p)$. *Further, this induces* *a bijective correspondence*

$$\left\{\begin{matrix}\text{irreducible smooth representations}\\ \text{of } \mathrm{GL}_n(\mathbb{Z}_p) \text{ over } k\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{irreducible representations}\\ \text{of } \mathrm{GL}_n(\mathbb{F}_p) \text{ over } k\end{matrix}\right\}.$$

*Proof.* Since $\mathrm{GL}_n(\mathbb{Z}_p)/K(1) \cong \mathrm{GL}_n(\mathbb{F}_p)$, it's clear that there is a map from the right hand side to the left hand side. For the reverse direction, let $\pi$ be a smooth $\mathrm{GL}_n(\mathbb{Z}_p)$-representation. By Lemmas 2.10 and 2.12, $\pi^{K(1)}$ is nonzero and $\mathrm{GL}_n(\mathbb{Z}_p)$-stable because $K(1)$ is normal in $\mathrm{GL}_n(\mathbb{Z}_p)$. Because of irreducibility, we have $\pi = \pi^{K(1)}$. Thus, $\pi$ is also a representation of $\mathrm{GL}_n(\mathbb{F}_p) \cong \mathrm{GL}_n(\mathbb{Z}_p)/K(1)$. □

**3.2. Definition.** A smooth irreducible representation of $\mathrm{GL}_n(\mathbb{Z}_p)$, or equivalently $\mathrm{GL}_n(\mathbb{F}_p)$ by Proposition 3.1, is called a **weight**.

We remark that there are exactly $p-1$ weights, all one-dimensional, of $\mathrm{GL}_1(\mathbb{F}_p) \cong \mathbb{F}_p^\times$, because of existence of primitive roots.

**3.3. Lemma. —** *Any smooth representation $\pi$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ contains a weight, i.e., a subrepresentation of $\pi|_{\mathrm{GL}_n(\mathbb{Z}_p)}$ is a weight.*

*Proof.* We use ideas analogous to the proof of Proposition 3.1. Pick a nonzero $x \in \pi^{K(1)}$. The $\mathrm{GL}_n(\mathbb{Z}_p)$-orbit of $x$ spans a finite dimensional $\mathrm{GL}_n(\mathbb{Z}_p)$-subrepresentation of $\pi|_{\mathrm{GL}_n(\mathbb{Z}_p)}$, and therefore contains an irreducible subrepresentation. □

From now on, we focus on $n = 2$. Define

$$G_p = \mathrm{GL}_2(\mathbb{F}_p), \quad B_p = \left\{\begin{bmatrix} a & b \\ 0 & d\end{bmatrix} : a, d \in \mathbb{F}_p^\times, b \in \mathbb{F}_p\right\}, \quad T_p = \left\{\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix} : a, d \in \mathbb{F}_p^\times\right\},$$
$$U_p = \left\{\begin{bmatrix} 1 & b \\ 0 & 1\end{bmatrix} : b \in \mathbb{F}_p\right\}.$$

**3.4. Theorem (Weights of $\mathrm{GL}_2$). —** *The weights of $\mathrm{GL}_2(\mathbb{F}_p)$ are*

$$F(a,b) := \mathrm{Sym}^{a-b} k^2 \otimes \det{}^b,$$

*where $0 \leqslant a-b \leqslant p-1$ and $0 \leqslant b < p-1$, where $k^2$ is the natural injection $\mathrm{GL}_2(\mathbb{F}_p) \hookrightarrow \mathrm{GL}_2(k)$ and $\det^b$ is the representation $\mathrm{GL}_2(\mathbb{F}_p) \to k^\times$, $A \mapsto (\det A)^b$.*

*Proof.* We divide the proof into steps.

*General observations.* The symmetric algebra $\mathrm{Sym}^{a-b} k^2$ can be identified with $k[X,Y]_{a-b}$, the $a-b$ graded component of the standard graded $k$-algebra $k[X,Y]$. The action of $\mathrm{GL}_2(\mathbb{F}_p)$ on $F(a,b) \cong k[X,Y]_{a-b}$ is given by

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta\end{bmatrix} \cdot f(X,Y) = f(\alpha X + \gamma Y, \beta X + \delta Y)(\alpha\delta - \beta\gamma)^b.$$

Observe that $F(a+p-1, b+p-1) \cong F(a,b)$ as $G_p$-representations. We may twist and assume $b=0$ without any loss of generality to show irreducibility of $F(a,b)$. Here we are using that tensor product of irreducible representations of finite groups is irreducible.

*$F(a,b)$ is irreducible.* It suffices to show that $V := k[X,Y]_d \cong \mathrm{Sym}^d k^2, 0 \leqslant d \leqslant p-1$, is irreducible.

**Claim 1.** $V^{U_p} = kX^d$.

Let $f \in V^{U_p}$. Then $f(X, uX+Y) = f(X,Y)$ for each $u \in \mathbb{F}_p$. Consider $g(Y) := f(X,Y) - f(X,0) \in k(X)[Y]$. Note that $g(-uX) = 0$ and $\deg g < p$. Thus, $g = 0$ and $f$ is a scalar multiple of $X^d$.

**Claim 2.** *The* $\mathrm{GL}_2(\mathbb{F}_p)$*-orbit of* $X^d$ *spans* $V$.

Indeed, it is easy to see that the polynomials $(X+uY)^d$, $u \in \mathbb{F}_p$ generates $V$. This follows from nonvanishing of a Vandermonde determinant. Put $d = a-b$. Note that $d < p$. Consider the $\overline{U}_p$ orbit of $X^{a-b}$:

$$(X+uY)^d = \sum_{i=0}^{d} u^{d-i} \binom{d}{i} X^i Y^{d-i},$$

for each $u \in \mathbb{F}_p$. The Vandermonde determinant

$$\left| u^{d-i} \right|_{\substack{u=0,1,\dots,d \\ i=0,1,\dots,d}} = \prod_{d \geqslant i > j \geqslant 0} (u^i - u^j) \neq 0.$$

Therefore, by theory of linear equations, it follows that $(X+uY)^d, u = 0,1,\dots,d$ span the same space as $X^iY^{d-i}, i = 0,1,\dots,d$. Note that we don't necessarily need all the $p-1$ equations.

Finally, if $W$ is a nonzero subrepresentation of $V$ then by Lemma 2.12, $W^{U_p} \neq 0$. So $W^{U_p} = kX^d$. But by Claim 2, the orbit of this space spans $V$. Hence, $W = V$ and $V$ is irreducible.

*The $F(a,b)$ are distinct.* Since $T_p$ is a subgroup of the normalizer of $U_p$, it acts on $F(a,b)^{U_p} = kX^{a-b}$. This action is given by

$$\mathrm{diag}(\alpha,\delta) X^{a-b} = \alpha^a \delta^b X^{a-b}.$$

If $F(a,b) \cong F(a',b')$ we must have $a-b = a'-b'$ because of dimension reasons. We must also have $a \equiv a' \pmod{p-1}$ and $b \equiv b' \pmod{p-1}$. Hence, $a = a'$ and $b = b'$.

*These are all the weights.* There are two ways to do this–

- Let $V$ be any irreducible $G_p$-representation. Then $V^{U_p}$ is a nonzero $T_p$-representation. As $T_p$ is abelian of order coprime to $p$, by Maschke's theorem, this representation decomposes as a direct sum of characters. If $\chi$ is one such character, then $\chi \hookrightarrow V^{U_p}$ as $T_p$-representations. This lifts to $\chi \hookrightarrow V$ as $B_p$-representations. By Frobenius reciprocity, we get a nonzero map $\mathrm{Ind}_{B_p}^{G_p} \chi \to V$, which has to be surjective since $V$ is irreducible. Therefore, it suffices to show that irreducible quotients of $\mathrm{Ind}_{B_p}^{G_p} \chi$ are of the form $F(a,b)$ for each $T_p$-character $\chi$. The proof is now complete by Lemmas 3.5 and 3.6 below.

- *Using modular representation theory (See Sections 4 for the background)*. The number of $p$-modular representations of the finite group $G_p$ is equal to the number of $p$-regular conjugacy classes in $G_p$. Using Jordan canonical form, it follows that the latter number is exactly $p(p-1)$, which shows that the representations $F(a,b)$ with $0 \leqslant a-b \leqslant p-1$ and $0 \leqslant b < p-1$ form a full system of representatives for the $p$-modular representations of $G_p$. $\square$

**3.5. Lemma. —** *We have $F(a,b)_{\overline{U}_p} \cong \chi_{a,b}$ as $T_p$-representations, where $\chi_{a,b}$ denotes the character $\mathrm{diag}(d_1,d_2) \mapsto d_1^a d_2^b$. Further, the natural $T_p$-linear map*

$$F(a,b)^{U_p} \hookrightarrow F(a,b) \twoheadrightarrow F(a,b)_{\overline{U}_p}$$

*is an isomorphism.*

*Proof.* We may assume $b=0$ without any loss of generality and identify $F(a,0) = \mathrm{Sym}^a k^2$ with $k[X,Y]_a$ as before. Consider the following equations

$$\begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} X^a - X^a = \sum_{i=1}^{a} \binom{a}{i} u^i X^{a-i} Y^i$$

$$\begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} X^{a-i} Y^i - X^{a-i} Y^i = \sum_{k=1}^{a} \binom{a-i}{k} u^k X^{a-i-k} Y^{i+k}.$$

where $1 \leqslant i \leqslant a$. This shows that $\mathrm{Ker}(F(a,0) \twoheadrightarrow F(a,0)_{\overline{U}_p})$ is spanned by $X^{a-1}Y, X^{a-2}Y^2, \ldots, Y^a$. Hence, $F(a,b)_{\overline{U}_p} \cong kX^a$. We have seen in Claim 1 in the proof of Theorem 3.4 that $F(a,b)^{U_p} = kX^a$. This proves the lemma. $\square$

**3.6. Lemma. —** *Let $a,b \in \mathbb{Z}$ with $0 \leqslant a-b < p-1$. Any irreducible quotient of $\mathrm{Ind}_{B_p}^{G_p} \chi_{a,b}$ is $F(a,b)$ and also $F(a+p-1,b)$ if $a=b$. Any irreducible subrepresentation of the same is either $F(b+p-1,a)$ and also $F(b,a)$ if $a=b$.*

*Proof.* Frobenius reciprocity tells us that $\mathrm{Ind}_{B_p}^{G_p} \twoheadrightarrow F(a',b')$ if and only iff $\chi_{a,b} \hookrightarrow F(a',b')^{U_p} \cong \chi_{a',b'}$ (see Lemma 3.5). Similarly, $F(a',b') \hookrightarrow \mathrm{Ind}_{B_p}^{G_p} \chi_{a,b}$ if and only if $\chi_{b',a'} \cong F(a',b')_{U_p} \twoheadrightarrow \chi_{a,b}$. Hence, there do exist quotients and subrepresentations as in the statement of the lemma. Now,

$$\dim F(a,b) + \dim F(a+p-1,a) = (a-b+1) + (b+p-a) = p+1$$

and $\dim \mathrm{Ind}_{B_p}^{G_p} \chi_{a,b} = \dim \chi_{a,b} \otimes_{\mathbb{F}_p[B_p]} \mathbb{F}_p[G_p] = [G_p : B_p] = p+1$. $\square$

**3.7. Principal series representations.** Let $\chi_1, \chi_2 \colon \mathbb{Q}_p^\times \to k^\times$ be two smooth characters. We then have a smooth character $\chi_1 \otimes \chi_2 \colon T \to k^\times$. This inflates to a smooth character

$\chi_1 \otimes \chi_2 \colon \overline{B} \to k^\times$, which in turn induces a smooth $G$-representation $\chi := \mathrm{Ind}_{\overline{B}}^G \chi_1 \otimes \chi_2$. This is an infinite dimensional representation and it's called a **principal series representation**.

**3.8. Proposition (weights of principal series representations). —** *Let $\chi = \chi_1 \otimes \chi_2 \colon T \to k^\times$ be a smooth character as above. We can consider $\chi$ as a smooth $\overline{B}$-representation. Then*

$$\dim_k \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}}^G \chi|_K) \leqslant 1$$

*for all weights $V$. If $\chi_1|_{\mathbb{Z}_p^\times} \neq \chi_2|_{\mathbb{Z}_p^\times}$, then there is precisely one $V$ such that equality holds, and $\dim_k V > 1$. If $\chi_1|_{\mathbb{Z}_p^\times} = \chi_2|_{\mathbb{Z}_p^\times}$, then two choices of $V$ such that equality holds, and either* $\dim_k V = 1$ *or* $\dim_k V = p$.

*Proof.* Let $V$ be a weight. The restriction map $\mathrm{Ind}_{\overline{B}}^G \chi \to \mathrm{Ind}_{\overline{B}\cap K}^K \chi$ is an isomorphism of $K$-representations. This can be checked manually by using the Iwasawa decomposition (or see Mackey decomposition 3.11). By Proposition 3.1, we know that $\chi_i|_{\mathbb{Z}_p^\times}$ factor through $\mathbb{F}_p^\times$. Let's write $\chi_i|_{\mathbb{Z}_p^\times} \colon \mathbb{Z}_p^\times \twoheadrightarrow \mathbb{F}_p^\times \xrightarrow{\eta_i} k^\times$. Thus,

$$\begin{aligned}
\mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}}^G \chi|_K) &\cong \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}\cap K}^K (\chi_1|_{\mathbb{Z}_p^\times} \otimes \chi_2|_{\mathbb{Z}_p^\times}) \\
&\cong \mathrm{Hom}_{\overline{B}\cap K}(V|_{\overline{B}\cap K}, \chi_1|_{\mathbb{Z}_p^\times} \otimes \chi_2|_{\mathbb{Z}_p^\times}) && \text{(Adjunction)} \\
&\cong \mathrm{Hom}_{\overline{B}_p}(V|_{\overline{B}_p}, \eta_1 \otimes \eta_2) \\
&\cong \mathrm{Hom}_{T_p}(V_{\overline{U}_p}, \eta_1 \otimes \eta_2) && (\overline{U}_p \text{ acts trivially})
\end{aligned}$$

Let $V = F(a,b)$. By a brute-force computation, one can check that

$$F(a,b)_{\overline{U}_p} = F(a,b)/(kX^iY^{a-b-i} \colon 0 \leqslant i < a-b).$$

As a $T_p$-representation, this is isomorphic to the tensor product of two smooth characters, precisely, the $a$th-power character and the $b$th-power character. Hence, $\mathrm{Hom}_{T_p}(V_{\overline{U}_p}, \eta_1 \otimes \eta_2)$ is one-dimensional when $V_{\overline{U}_p} \cong \eta_1 \otimes \eta_2$ and zero otherwise. Therefore, $V$ is fully determined by $\eta_i$. If $\eta_1 = \eta_2$ then it's clear that $V \cong F(a,b)$ for $a \neq b$ and hence $\dim_k V > 1$. On the other hand, if $\eta_1 = \eta_2$ then $V \in \{F(a,a), F(p-1,0)\}$. The former has dimension 1 and the latter is $p$-dimensional. □

**3.9.** *Example. Let $\chi_1, \chi_2 \colon \mathbb{Q}_p^\times \to k^\times$ be two smooth characters. Consider the smooth representation*

$$\chi \colon \overline{B} \to E^\times, \qquad \begin{bmatrix} \alpha & 0 \\ \gamma & \delta \end{bmatrix} \mapsto \chi_1(\alpha)\chi_2(\delta).$$

*Then there is a vector space isomorphism:*

$$\begin{aligned}
\{f \in \mathrm{Ind}_{\overline{B}}^G \chi \colon \mathrm{Supp}\, f \subseteq \overline{B}U\} &\longrightarrow \mathscr{C}_c^\infty(\mathbb{Q}_p, k) \\
f &\longmapsto \left(f \mapsto f\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right),
\end{aligned}$$

*where $\mathscr{C}_c^{\infty}(\mathbb{Q}_p, k)$ is the space of all locally constant, compactly supported functions $\mathbb{Q}_p \to k$. In particular, principal series representations are infinite-dimensional.*

*Proof.* The LHS is indeed a vector space because $\operatorname{Supp}(f+g) \subset \operatorname{Supp} f \cup \operatorname{Supp} g$. First we show that map

$$f \longmapsto \left(x \mapsto f\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\right)$$

is well-defined– $f$ being a smooth vector implies that $f\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} w\right) = f\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\right)$ holds for all $w$ in an open subgroup, say $V$. The arithmetic of $U$ is essentially same as $(\mathbb{Q}_p, +)$:

$$\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & x-y\\ 1 & 0\end{bmatrix}.$$

So we can choose $y$ so close to $x$ that $||x-y||$ is very small and the right hand side is contained in $V$. Hence, the image of $f$ is indeed locally constant. Further, $f$ being a smooth vector also implies that $f^{-1}(0)$ is open. Therefore, $\overline{B}\backslash \operatorname{Supp} f$ is closed in $\overline{B}\backslash G$. By Iwasawa decomposition, $\overline{B}\backslash G \cong (\overline{B} \cap K)\backslash K$, which is compact because $\overline{B} \cap K$ is closed in the compact group $K$. In particular, $\overline{B}\backslash \operatorname{Supp} f$ is compact. As $\overline{B}\backslash \overline{B}U \cong U$, it follows that $\operatorname{Supp} f \cap U$ is compact as well. This proves that the image of $f$ indeed has compact support.

The given map is clearly linear. If some $g \mapsto 0$ then $\operatorname{Supp} g$ cannnot be contained in $\overline{B}U$. So it's injective. Now, we need to check surjectivity. Let $\phi$ be any locally constant, compactly supported, $k$-valued map from $\mathbb{Q}_p$. Define $f\colon G \to k$ as follows– let $f\colon U \to k$ be given by

$$f\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} = \phi(x).$$

Note that $\overline{B} \cap U = \{1\}$. Hence we can extend it to a map $f\colon \overline{B}U \to k$

$$f\left(T\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\right) = T\phi(x), \qquad \text{for all } T \in \overline{B}.$$

Now just extend by zero to get a function $f\colon G \to k$. It's clear that $\operatorname{Supp} f \subseteq \overline{B}U$ and that $\operatorname{Supp} f$ is closed. Choose $n$ so large so that

- the compact open subgroup $K(n)$ is contained in $G \setminus \operatorname{Supp} f$ (possible because $G \setminus \operatorname{Supp} f$ is open).
- $\chi|_{\overline{B}\cap K(n)}$ is trivial (possible since $\chi$ is smooth).
- $\phi(xa) = \phi(x)$ for each $x \in \mathbb{Q}_p$ and each $a \in 1 + p^n\mathbb{Z}_p$. Indeed, locally constant + compactly supported implies that $\phi$ takes only finitely many values.

It is now easy to observe that $f$ is fixed by $K(n)$, which shows that $f$ is smooth. □

We end this section by stating two very fundamental but extremely useful results–

**3.10. Schur's lemma. —** *Let $T$ be a group and $V, W$ be irreducible $T$-representations.*

- *If $V$ and $W$ are not isomorphic, then there are no nontrivial $T$-linear maps between them.*
- *If $V \cong W$ as $T$-representations, and they are finite-dimensional over an algebraically closed field, then the only nontrivial $G$-linear maps is a scalar multiple of the identity.*

*Proof.* See [**Ser77**, p. 13]. □

For any $H$-representation $\rho$ over some field, where for $g \in G$ we put ${}^gH := gHg^{-1}$ and ${}^g\rho$ is a ${}^gH$-representation defined as ${}^g\rho(ghg^{-1}) = \rho(h)$ for any $h \in H$.

**3.11. Mackey decomposition. —** *Let $G$ be a locally profinite group. Let $H$ and $K$ be closed subgroups in $G$. Let $\rho$ be a smooth representation of $H$ over some commutative ring $R$ with unit.*

*(1) If either $H$ or $K$ is open in $G$, then we have the Mackey decomposition*

$$\mathrm{Res}^G_K \, \text{c-Ind}^G_H \rho \cong \bigoplus_{g \in K \backslash G / H} \text{c-Ind}^K_{K \cap {}^gH} \mathrm{Res}^{{}^gH}_{K \cap {}^gH} {}^g\rho.$$

*(2) If $K$ is open in $G$, then we have the Mackey decomposition*

$$\mathrm{Res}^G_K \mathrm{Ind}^G_H \rho \cong \left( \prod_{g \in K \backslash G / H} \mathrm{Ind}^K_{K \cap {}^gH} \mathrm{Res}^{{}^gH}_{K \cap {}^gH} {}^g\rho \right)^{\infty},$$

*where for a $K$-representation $\tau$, we denote by $\tau^{\infty}$ the $K$-smooth part of $\tau$.*

*(3) If we omit the assumption that $K$ is open, there exists an example such that $H$ is open in $G$ and the isomorphism in (2) does not hold.*

*Proof.* See [**Yam**]. □

## 4. Associative algebras and conjugacy classes

The purpose of this section is to provide necessary background for the proof of Theorem 3.4 using modular representation theory. The reader is free to skip it.

**4.1. Proposition. —** *Every reduced algebra is semisimple.*

**4.2. Wedderburn's theorem. —** *Every simple $k$-algebra is isomorphic to $\mathrm{Mat}_{n\times n}(k)$ for some positive integer $n$.*

**4.3. Proposition. —** *Let $A$ be a finite-dimensional (not necessarily commutative) algebra over an algebraically closed field $k$ of characteristic $p > 0$. Let $S = \mathrm{Span}_k\{ab - ba \colon a, b \in A\}$ and $T = \{r \in A \colon r^q \in S$ for some power $q$ of $p\}$. Then $T$ is a subspace of $A$, and the number of isomorphism classes of simple $A$-modules is $\dim_k(A/T)$.*

*Proof.* As $A$ is commutative modulo $S$, it follows that $(a+b)^p \equiv a^p + b^p \pmod S$. Therefore, $T$ is a vector subspace of $A$. Also, it's easy to check that $S \subset T$. For now, assume that $A$ is simple. By Wedderburn's theorem, we may identify $A$ with some matrix algebra $M_n(k)$. By routine linear algebra, $S$ can be identified with the subspace of traceless matrices. This can be checked at matrix units. We have $\dim A/S = 1$ by first isomorphism theorem on Trace: $M_n(k) \to k$. Therefore, either $T = A$ or $\dim A/T = 1$. As the matrix unit $e_{1,1}$ is not in $T$, it follows that $T \neq A$ and $\dim A/T = 1$. At this point, we have checked the desired result for simple algebras $A$.

For the general case, of course $\operatorname{rad} A \subset T$, where $\operatorname{rad} A$ denotes the maximal left ideal of nilpotent elements. It turns that it is a two-sided ideal. Furthermore, it is well-known that reduced algebras are semisimple. Note that $\operatorname{rad} A$ acts trivally on every simple $A$-module. Hence, we may replace $A$ by $A/\operatorname{rad} A$. So we may assume $A$ is semisimple. Write $A = A_1 \oplus \cdots \oplus A_r$ as a direct sum of simple algebras. It is not hard to see that $T = T_1 \oplus \cdots \oplus T_r$. Therefore, $\dim A/T = r$. Thus, the required result holds since simple modules over $A$ are just $A_i$. □

**4.4. Proposition. —** *Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $A = k[G]$. Then the number of $p$-regular conjugacy classes in G is same as* $\dim A/T$.

*Proof.* Every $x \in G$ can be written as $x = st$ where $s$ is $p$-regular and the order of $t$ is a power of $p$. If the order of $x$ is $np^e$ for $p \nmid n$ then one can take $s = x^{ap^e}$ and $t = x^{bn}$ for $ap^e + bn = 1$ (Bezout). Observe that $st - s \in T$. This means that every element of $G$ is $p$-regular modulo $T$. Therefore, there are at least $\dim A/T$ $p$-regular conjugacy classes in $G$. Let $r_1, r_2, \ldots, r_n$ be a set of representatives of $p$-regular conjugacy classes. Suppose $\sum a_i r_i \in T$ for some coefficients $a_r \in k$. Fix a $q$, power of $p$, such that $r_i^q = r_i$ for each $i$. Choose $q$ so large that $(\sum_i a_i r_i)^q \in S$. Then

$$0 \equiv (\sum a_i r_i)^q \equiv \sum a_i^q r \pmod S.$$

Note that $S$ is spanned by expressions of the form $ab - ba$, and both $ab$ and $ba$ are in the same conjugacy class. Hence, expressions in $S$ have the property that the sum of the coefficients of all elements in each conjugacy class is zero. Therefore, $a_i^q = 0 \implies a_i = 0$. The proof is complete. □

**4.5. Corollary (Brauer's theorem). —** *The number of isomorphism classes of simple $k[G]$-modules is equal to the number of $p$-regular conjugacy classes in $G$.*

## 5. Hecke Algebras for $\mathrm{GL}_2$

We fix a smooth $G$-representation $\pi$. We have seen that $\pi|_K$ contains a weight, say $V$. The multiplicity of this weight is given by $\dim_k \mathrm{Hom}_K(V, \pi|_K)$. By Frobenius reciprocity,

$$\mathrm{Hom}_K(V, \pi|_K) \cong \mathrm{Hom}_G(\text{c-Ind}_K^G V, \pi).$$

**5.1. Definition.** The **Hecke algebra** of the weight $V$ is $\mathscr{H}_G V := \mathrm{End}_G \, \text{c-Ind}_K^G V$.

It is clear that $\mathrm{Hom}_G(\text{c-Ind}_K^G V, \pi)$ is a right $\mathscr{H}_G V$-module via pre-composition.

**5.2. Proposition. —** *There is an isomorphism of algebras*

$$\mathscr{H}_G V \cong \{\varphi\colon G \to \mathrm{End}_k V : \varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2 \text{ for all } k_1, k_2 \in K,\ g \in G,\ K \backslash \operatorname{Supp} \varphi / K \text{ is finite}\} \tag{5.2.1}$$

*The binary operation on this algebra is the convolution*

$$(\varphi_1 \star \varphi_2)(g) = \sum_{\gamma \in K\backslash G} \varphi_1(g\gamma^{-1})\varphi_2(\gamma).$$

*Here, $K\backslash G$ denotes a set of right-coset representatives and $K \backslash \operatorname{Supp}\varphi / K$ denotes double cosets.*

**5.3.** *Remark.* We address well-definedness of the convolution sum in Proposition 5.2 and alternative definitions of the same. Recall the main differences in definitions:

(a) Breuil's definition [**Br07**]: Support is compact modulo center. (the author considers a general local field $F/\mathbb{Q}_p$)
(b) Herzig's [**Herz**]: Image of support in $K\backslash G/K$ is finite. (only $F = \mathbb{Q}_p$)
(c) Herzig's [**Herz+**]: Support is compact. ($F = \mathbb{Q}_p$)

I will show that all three are equivalent when $F = \mathbb{Q}_p$.

- (a) $\Longrightarrow$ (b). Let $f$ be a function whose support is compact modulo the center. Let's write $\operatorname{Supp} f = \bigsqcup_a KaK$, union of disjoint double $K$-cosets. Certainly, $KaK$ is open compact. Indeed, it is a continuous image of the compact set $K \times K$ and it is open because $K$ and $aK$ are open. Further, two disjoint cosets remain disjoint when considered modulo $\mathbb{Q}_p^\times$. Quotient maps of topological groups are open. Therefore, the images of $KaK$ under $\mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{GL}_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times$ are disjoint compact opens. If this image is compact, then the union $\operatorname{Supp} f = \bigsqcup_a KaK$ must be finite.
- (b) $\Longrightarrow$ (c). $\operatorname{Supp} f$ is a finite disjoint union of double cosets, say $\bigsqcup_{i=1}^n Ka_iK$. Then $\operatorname{Supp} f$ is certainly compact because any double coset $KaK$, being a continuous image of $K \times K$ , is compact too.
- (c) $\Longrightarrow$ (a). This is trivial because continuous image of compact is compact.

Now one has to check whether

$$\sum_{\gamma H\in G/H}\varphi_1(\gamma)\circ\varphi_2(\gamma^{-1}g)$$

is a finite sum. Note that the sum is independent of representatives. Now $\operatorname{Supp}\varphi_1$ is compact so is only contained in finitely many left cosets $\gamma H$ (recall $H$ is open and the cosets form an open cover of $G$, whence of $\operatorname{Supp}(\varphi_1)$). So the first term $\varphi_1(\gamma)$ in the sum vanishes outside finitely many left cosets $\gamma H$. So composition $\varphi_1(\gamma)\circ\varphi_2(\gamma^{-1}g)$ is also finite outside finitely many cosets $\gamma H$ (don't need to argue with the second term). Thus the sum is finite.

**5.4.** *Proof of Proposition 5.2.* First we sketch the main idea.

$$\mathscr{H}_G V \overset{\text{Adjunction}}{\cong} \operatorname{Hom}_K(V,\text{c-Ind}_K^G V|_K)\subset \operatorname{Mor}_{\mathsf{Set}}(V,\operatorname{Mor}_{\mathsf{Set}}(G,V))\cong \operatorname{Mor}_{\mathsf{Set}}(G,\operatorname{Mor}_{\mathsf{Set}}(V,V)).$$

The right hand side of (5.2.1) is a subset of $\operatorname{Mor}_{\mathsf{Set}}(G,\operatorname{Mor}_{\mathsf{Set}}(V,V))$. Take some

$$(v\mapsto f_v)\in \operatorname{Hom}_K(V,\text{c-Ind}_K^G V|_K).$$

The image in $\operatorname{Mor}_{\mathsf{Set}}(G,\operatorname{Mor}_{\mathsf{Set}}(V,V))$ is a function $\varphi$ given by $\varphi(g)(v)=f_v(g)$. From $K$-linearity and definition of induction, we have

$$\varphi(k_1gk_2)(v)=f_v(k_1gk_2)=k_1f_{k_2v}(g)=k_1(\varphi(g)(k_2v)).$$

Therefore, $\operatorname{Supp}\varphi=G\setminus\{g\in G\colon g\in\operatorname{Ker}f_v \text{ for all } v\in V\}=\bigcup_{v\in B_1(V,0)}\operatorname{Supp}f_v$, which is compact as the unit ball is compact. We now check the product. Let $\varphi_i\in\operatorname{Mor}(G,\operatorname{End}V)$ be corresponding to $\psi_i'\in\operatorname{End}_G\text{c-Ind}_K^G V$ and $\psi_i\in\operatorname{Hom}_K(V,\text{c-Ind}_K^G V)$ for $i\in\{1,2\}$. By definition, $\psi_i(x)=\psi_i'[1,x]$. Note that $\psi_i'([1,x])(\gamma)=\psi_i(x)(\gamma)=\varphi_i(\gamma)(x)$. Therefore,

$$\psi_i'([1,x])=\sum_{\gamma\in K\backslash G}[\gamma^{-1},\varphi_i(\gamma)(x)]=\sum_{\gamma\in K\backslash G}\gamma^{-1}[1,\varphi_i(\gamma)(x)].$$

Composing,

$$\begin{aligned}\psi_1'(\psi_2'([1,x]))=\psi_1'\left(\sum_{\gamma\in K\backslash G}\gamma^{-1}[1,\varphi_2(\gamma)(x)]\right)&=\sum_{\gamma_2\in K\backslash G}\gamma_2^{-1}\sum_{\gamma_1\in K\backslash G}[\gamma_1^{-1},(\varphi_1(\gamma_1)\circ\varphi_2(\gamma_2))(x)]\\&=\sum_{\gamma_1,\gamma_2\in K\backslash G}[\gamma_2^{-1}\gamma_1^{-1},(\varphi_1(\gamma_1)\circ\varphi_2(\gamma_2))(x)]\\&=\sum_{\gamma\in K\backslash G}\left[\gamma^{-1},\sum_{\gamma_2\in K\backslash G}\varphi_1(\gamma\gamma_2^{-1})\circ\varphi_2(\gamma_2)(x)\right],\end{aligned}$$

via the change of variable $\gamma=\gamma_1\gamma_2$. □

Given a **Hecke operator** $\varphi\in\mathscr{H}_G V$ and $f\in\operatorname{Hom}_K(V,\pi|_K)$, the right action of the Hecke algebra is given by

$$(f\cdot\varphi)(v)=\sum_{g\in K\backslash G}g^{-1}f(\varphi(g)v).$$

To see this, Let $\psi'$ and $\psi$ correspond to $\varphi$ as in the above proof. Suppose $f' \in \mathrm{Hom}_G(\text{c-Ind}_K^G V, \pi)$ corresponds to $f$. Using the results in the previous proof,

$$f'(\psi(x)) = f'\left(\sum_{\gamma \in K\backslash G} \gamma^{-1} \cdot [1, \varphi(\gamma)(x)]\right) = \sum_{\gamma \in K\backslash G} \gamma^{-1} \cdot f(\varphi(\gamma)(x)).$$

We get a better understanding of $K\backslash G/K$ from the following lemma

**5.5. Cartan decomposition. —** $G = \bigsqcup_{r\leqslant s} K \operatorname{diag}(p^r, p^s) K$.

*Proof.* By the theory of smith normal forms, we know that $G = \bigcup_{r\leqslant s} K \operatorname{diag}(p^r, p^s) K$ (c.f. Proposition 1.1). Given distinct pairs $(r,s), (r',s'), r \leqslant s, r' \leqslant s'$ we need to show that $K \operatorname{diag}(p^r, p^s) K \neq K \operatorname{diag}(p^{r'}, p^{s'}) K$. Suppose not. Looking at $p$-adic valuation of determinants, we have $r + s = r' + s'$. Further, looking at the norm of both sides, we get $r = r'$. Proved. □

**5.6. Theorem (Hecke algebras for $\mathrm{GL}_2$). —**

*(1) For any pair of integers $r \leqslant s$, there is a unique Hecke operator $T_{r,s} \in \mathcal{H}_G V$ such that $\operatorname{Supp} T_{r,s} = K \operatorname{diag}(p^r, p^s) K$ and $T_{r,s} \operatorname{diag}(p^r, p^s) \in \operatorname{End}_k V$ is a linear projection.*
*(2) $\{T_{r,s}\}_{r\leqslant s}$ forms a basis for $\mathcal{H}_G V$.*
*(3) We have an isomorphism of $k$-algebras $\mathcal{H}_G V \cong k[T_1, T_2, T_2^{-1}]$ where $T_1 = T_{0,1}$ and $T_2 = T_{1,1}$. In particular, $\mathcal{H}_G V$ is commutative.*

*Proof.*

(1) Because of how Hecke operators are defined, $\varphi \in \mathcal{H}_G V$ is determined by the choice of its values at matrices of the form $\operatorname{diag}(p^r, p^s)$. Pick some nonzero matrix $A$ such that $k_1 \operatorname{diag}(p^r, p^s) = \operatorname{diag}(p^r, p^s) k_2$ implies $k_1 A = A k_2$. Define $\varphi \colon G \to \operatorname{End}_k V$ as

$$\varphi(g) = \begin{cases} k_1 A k_2, & \text{if} g \in k_1 \operatorname{diag}(p^r, p^s) k_2 \\ 0, & \text{otherwise} \end{cases}.$$

It is easily seen that this is a well-defined Hecke operator supported at $K \operatorname{diag}(p^r, p^s) K$, provided such an $A$ exists. We also need to show that such $A$ is unique upto scaling.

- $r = s$. We show that $A$ is a nonzero multiple of identity. We must have $k \circ A = A \circ k$ for all $k \in K$. Then $V \xrightarrow{A} V$ is a nonzero $K$-module map, which by Schur's lemma must be a nonzero multiple of identity.
- $r < s$. Note that existence of a $k_2$ such that $k_1 \operatorname{diag}(p^r, p^s) = \operatorname{diag}(p^r, p^s) k_2$ is equivalent to

$$k_1 \in K \cap \operatorname{diag}(p^r, p^s) K \operatorname{diag}(p^r, p^s)^{-1} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K \colon a, d \in \mathbb{Z}_p^\times, b \in \mathbb{Z}_p, c \in p^{s-r}\mathbb{Z}_p \right\}.$$

An analogous statement holds for $k_2$. So we may write

$$k_1 = \begin{bmatrix} a & b \\ p^{s-r}c & d \end{bmatrix}, \qquad k_2 = \begin{bmatrix} a & p^{s-r}b \\ c & d \end{bmatrix}$$

for $a, b, c, d$ as above. By Proposition 3.1, the condition $k_1 A = Ak_2$ is equivalent to

$$\begin{bmatrix} \overline{a} & \overline{b} \\ 0 & \overline{d} \end{bmatrix} A = A \begin{bmatrix} \overline{a} & 0 \\ \overline{c} & \overline{d} \end{bmatrix},$$

where the bar means "reduction modulo $p$". We may rewrite this as

$$\begin{bmatrix} 1 & \overline{b} \\ 0 & 1 \end{bmatrix} A = A, \quad A = A \begin{bmatrix} 1 & 0 \\ \overline{c} & 1 \end{bmatrix}, \quad \mathrm{diag}(\overline{a}, \overline{d})A = A\,\mathrm{diag}(\overline{a}, \overline{d}).$$

This implies that there is a commutative square

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d] & V \\ V_{\overline{U}_p} \arrow[r, "\overline{A}"'] & V^{U_p}. \arrow[u] \end{tikzcd}$$

The third condition says that $\overline{A}$ is $T_p$-linear. By Lemma 3.5, it follows that the space of such $A$ is at most 1-dimensional and we may take $T_{r,s}$ to be corresponding to any such nonzero $A$. It's also clear that such $A$'s are linear projections.

(2) Any $f \in \mathscr{H}_G V$ can be written as

$$f = \sum_{r \leqslant s} f|_{K\,\mathrm{diag}(p^r, p^s)K}.$$

(3) Consider the following claims.

(a) $T_{i,i}T_{r,s} = T_{r+i,s+i} = T_{r,s}T_{i,i}$. In particular, $T_{0,0}$ is the identity, $T_2$ is invertible and $T_2^r = T_{r,r}$.

(b) $T_{r,s}T_1 = T_{r,s+1} + \sum_{i>0} a_i T_{r+i,s+1-i}$ for some $a_i \in k$.

We first outline the proof. Consider the following claims.

**Claim 1.** $T_{i,i}T_{r,s} = T_{r+i,s+i} = T_{r,s}T_{i,i}$.

In particular, $T_{0,0}$ is the identity, $T_2$ is central and invertible, and $T_2^r = T_{r,r}$.

**Claim 2.** $T_{r,s}T_1 = T_{r,s+1} + \sum_{i>0} a_i T_{r+i,s+1-i}$ *for some* $a_i \in k$.

Combining these two results, we have, for $r \leqslant s$,

$$T_1^{s-r}T_2^r = T_2^r T_1^{s-r} = T_{r,s} + \sum_{1 \leqslant i \leqslant (s-r)/2} a'_i T_{r+1,s-i}$$

for some $a'_i \in k$. This is shown by a simple induction on $s-r$. Therefore, $\{T_1^{s-r}T_2^r\}_{r \leqslant s}$ forms a basis for $\mathscr{H}_G V$. This proves the desired result.

*Proof of Claim 1.*

$$\begin{aligned}
T_{r,s}T_{i,i}(g) &= \sum_{\gamma\in K\backslash G} T_{r,s}(g\gamma^{-1})T_{i,i}(\gamma) \\
&= T_{r,s}\left(g\,\mathrm{diag}(p^i,p^i)^{-1}\right) \\
&= \begin{cases} 1, & \text{if } r=s \text{ and } g\in K\begin{bmatrix} p^{r+i} & 0 \\ 0 & p^{s+i}\end{bmatrix}K \\ k_1Pk_2, & \text{if } r<s \text{ and } g=k_1\begin{bmatrix} p^{r+i} & 0 \\ 0 & p^{s+i}\end{bmatrix}k_2 \\ 0, & \text{otherwise}\end{cases} \\
&= T_{r+i,s+i}(g).
\end{aligned}$$

Similarly, one can check that $T_{r+i,s+i} = T_{r,s}T_{i,i}$.

*Proof of Claim 2.* Using Claim 1, we may multiply out by $T_2^{-r} = T_{-r,-r}$ to assume $r=0$ without any loss in generality. The convolution formula says

$$(T_{0,s}\star T_1)(g) = \sum_{\gamma\in K\backslash G} T_{0,s}(g\gamma^{-1})T_1(\gamma).$$

Hence,

$$\mathrm{Supp}\,T_{0,s}T_1 \subset K\,\mathrm{diag}(1,p^s)K\,\mathrm{diag}(1,p)K.$$

By looking at the determinant and using Cartan decomposition,

$$\mathrm{Supp}\,T_{0,s}T_1 \subset \bigsqcup_{0\leqslant i\leqslant (s+1)/2} K\,\mathrm{diag}(p^i,p^{s+1-i})K.$$

So we can find $a_i\in E$ such that

$$T_{0,s}T_1 = \sum_{0\leqslant i\leqslant (s+1)/1} a_iT_{i,s+1-i}.$$

We now need to show that $a_0=1$. Indeed,

$$\begin{aligned}
T_{0,s}T_1\,\mathrm{diag}(1,p^{s+1}) &= \sum_{\gamma\in K\backslash G} T_{0,s}(\mathrm{diag}(1,p^{s+1})\gamma^{-1})T_1(\gamma) \\
&= \sum_{\gamma\in K\backslash K\,\mathrm{diag}(1,p)K} T_{0,s}(\mathrm{diag}(1,p^{s+1})\gamma^{-1})P \\
&= \sum_{\gamma\in\left\{\left[\begin{smallmatrix} 0 & 1 \\ -p & 0\end{smallmatrix}\right]\right\}\cup\left\{\left[\begin{smallmatrix} 1 & u \\ 0 & p\end{smallmatrix}\right]:\,0\leqslant u\leqslant p-1\right\}} T_{0,s}(\mathrm{diag}(1,p^{s+1})\gamma^{-1})P \\
&= T_{0,s}\begin{bmatrix} 0 & p^{-1} \\ p^{s+1} & 0\end{bmatrix}T_1\begin{bmatrix} 0 & 1 \\ -p & 0\end{bmatrix} + \sum_{u=0}^{p-1} T_{0,s}\begin{bmatrix} 1 & -up^{-1} \\ 0 & p^s\end{bmatrix}T_1\begin{bmatrix} 1 & u \\ 0 & p\end{bmatrix} \\
&= T_{0,s}\,\mathrm{diag}(1,p^s)T_1\,\mathrm{diag}(1,p) = T_{0,s+1}\,\mathrm{diag}(1,p^{s+1}).
\end{aligned}$$

□

**5.7. Definition.** Let $V$ be a weight. A $K$-linear map $f\colon V\to\pi$ is an **eigenvector** of the Hecke operator $\varphi\in\mathscr{H}_GV$ with **eigenvalue** $\lambda\in k$ if $f\cdot\varphi=\lambda f$. If $f$ is an eigenvector for each

$\varphi$ then $f$ is called a **Hecke eigenvector** and the eigenvalues corresponding to $T_1$ and $T_2$ are called its **eigenvalues**.

**5.8. Proposition (Hecke eigenvectors).** — *Suppose $\chi\colon T \to k^\times$ is a smooth character and $f\colon V \hookrightarrow \mathrm{Ind}_B^G \chi$ is a $K$-linear embedding of the weight $V$. Then $f$ is Hecke eigenvector of $\mathscr{H}_G V$, precisely,*

$$f \cdot T_1 = \chi(\mathrm{diag}(1,p)^{-1}) f$$
$$f \cdot T_2 = \chi(\mathrm{diag}(p,p)^{-1}) f.$$

*Proof.* From Proposition 3.8, it follows that $f$ is a Hecke eigenvector. To find the eigenvalues, one can perform a direct, but long, calculation. For the second equality,

$$(f \cdot T_2)(x) = \sum_{g \in K\backslash G} g^{-1} f(T_2(g)x) = \mathrm{diag}\, p, p^{-1} f(T_2(\mathrm{diag}(p,p))x) = \mathrm{diag}(p,p)^{-1} f(x).$$

The eigenvector corresponding to $T_1$ is the ratio between $(f \cdot T_1)(x)(1)$ and $f(x)(1)$. By the Hecke-action formula,

$$\mathrm{S} = (f \cdot T_1)(x)(1) = \sum_{\gamma \in K\backslash G} f(T_1(\gamma)(x))(\gamma^{-1}).$$

By definition, $T_1$ is supported at $K\,\mathrm{diag}(1,p)K$. We have

$$K\,\mathrm{diag}(1,p)K = K\begin{bmatrix} 0 & 1 \\ -p & 0 \end{bmatrix} \sqcup \bigsqcup_{0 \leqslant u \leqslant p-1} K \begin{bmatrix} 1 & u \\ 0 & p \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathrm{S} &= \sum_{u=0}^{p-1} f\left(T_1 \begin{bmatrix} 1 & u \\ 0 & p \end{bmatrix} x\right) \begin{bmatrix} 1 & -up^{-1} \\ 0 & p^{-1} \end{bmatrix} + f\left(T_1 \begin{bmatrix} 0 & 1 \\ -p & 0 \end{bmatrix} x\right) \begin{bmatrix} 0 & -p^{-1} \\ 1 & 0 \end{bmatrix} \\ &= \sum_{u=0}^{p-1} f(x)\left(\begin{bmatrix} 1 & -up^{-1} \\ 0 & p^{-1} \end{bmatrix}\right) + f\left(P \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x\right)\left(\begin{bmatrix} 0 & -p^{-1} \\ 1 & 0 \end{bmatrix}\right), \end{aligned}$$

where $P = T_1\,\mathrm{diag}(1,p)$ and we are using the factorisation

$$\begin{bmatrix} 1 & u \\ 0 & p \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ -p & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that

$$\begin{bmatrix} 1 & -up^{-1} \\ 0 & p^{-1} \end{bmatrix} = \begin{bmatrix} up^{-1} & 0 \\ -p^{-1} & u^{-1} \end{bmatrix}\begin{bmatrix} 1 & pu^{-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore,

$$f(x)\left(\begin{bmatrix} 1 & -up^{-1} \\ 0 & p^{-1} \end{bmatrix}\right) = \begin{cases} \chi(\mathrm{diag}(1,p)^{-1}) f(x)(1), & \text{if } u = 0 \\ \chi(\mathrm{diag}(up^{-1}, u^{-1})) f(x)\left(\begin{bmatrix} 1 & pu^{-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right), & \text{otherwise.} \end{cases}$$

It is easy to verify using the definitions that $f(x)\left(\begin{bmatrix}1 & pu^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right) = 0$ if $\dim V > 1$. Therefore, the result is verified when $\dim V > 1$. When $\dim V = 1$, we know that $V \cong \det^{\otimes b}$. Therefore,

$$\begin{aligned} S &= \chi(\mathrm{diag}(1,p)^{-1})f(x)(1) + \sum_{u=1}^{p-1}\chi(\mathrm{diag}(up^{-1},u^{-1}))f(x)\left(\begin{bmatrix}1 & pu^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right) \\ &\quad + f(x)\left(\begin{bmatrix}0 & -p^{-1}\\ 1 & 0\end{bmatrix}\right) \\ &= \chi(\mathrm{diag}(1,p)^{-1})f(x)(1). \end{aligned}$$

□

## 6. Mod-$p$ Satake transform

Denote by $\mathsf{Vec}_k$ the category of all $k$-vector spaces.

**6.1. Lemma. —** *There is a natural isomorphism of functors* $\mathsf{Mod}^{\mathsf{sm}}{}_T \to \mathsf{Vec}_k$

$$\mathrm{Hom}_G(\text{c-Ind}_K^G V, \mathrm{Ind}_{\overline{B}}^G(-)) \cong \mathrm{Hom}_T(\text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}, -).$$

*We denote this map by* $f \mapsto f_T$.

*Proof.* Observe that

$$\begin{aligned} \mathrm{Hom}_G(\text{c-Ind}_K^G V, \mathrm{Ind}_{\overline{B}}^G(-)) &\cong \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}}^G(-)|_K) && \text{(Adjunction)} \\ &\cong \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}\cap K}^K(-|_{T\cap K})) && \text{(Iwasawa decomposition)} \\ &\cong \mathrm{Hom}_{\overline{B}\cap K}(V, -|_{T_p}) && \text{(Adjunction)} \\ &\cong \mathrm{Hom}_{T\cap K}(V_{\overline{U}_p}, -|_{T\cap K}) \\ &\cong \mathrm{Hom}_M(\text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}, -). && \text{(Adjunction)} \end{aligned}$$

□

Any Hecke operator $\varphi \in \mathscr{H}_G V$ gives a natural endomorphism, i.e., a natural transformation to itself, of the functor $\mathrm{Hom}_G(\text{c-Ind}_K^G V, \mathrm{Ind}_{\overline{B}}^G(-))$. By the above lemma, this gives a natural endomorphism of $\mathrm{Hom}_T(\text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}, -)$. All endormorphisms of the same are given by the Hecke algebra $\mathscr{H}_T V_{\overline{U}_p} := \mathrm{End}_T\, \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}$ by the Yoneda lemma. Thus, we obtain a natural map

$$\mathcal{S}_G \colon \mathscr{H}_G V \longrightarrow \mathscr{H}_T V_{\overline{U}_p}.$$

This is called the **mod-$p$ Satake transform**. We remark that

$$\mathscr{H}_T V_{\overline{U}_p} \cong \left\{ \begin{array}{c} \varphi\colon T \to \mathrm{End}_k V_{\overline{U}_p} : \varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2 \\ \text{for all } k_1, k_2 \in K\cap T,\ g \in T,\ (K\cap T)\backslash \mathrm{Supp}\,\varphi/(K\cap T) \text{ is finite} \end{array} \right\}$$

follows by imitating the proof of Proposition 5.2. It's clear that $(f\circ\varphi)_T = f_T\circ\mathcal{S}_G(\varphi)$ for each $f\in\mathrm{Hom}_G(\text{c-Ind}_K^G V, \mathrm{Ind}_{\overline{B}}^G W)$, $W\in\mathrm{Ob}(\mathrm{Mod}^{\mathrm{sm}}{}_T)$.

**6.2. Proposition. —** *$\mathcal{S}_G$ is a homomorphism of algebras.*

*Proof.* To check that it respects multiplication, note that

$$f_T\circ\mathcal{S}_G(\varphi_1\circ\varphi_2) = (f\circ\varphi_1\circ\varphi_2)_T = (f\circ\varphi_1)_T\circ\mathcal{S}_G(\varphi_2) = f_T\circ\mathcal{S}_G(\varphi_1)\circ\mathcal{S}_G(\varphi_2).$$

Similarly, additivity and identity can be checked. □

**6.3. Proposition. —** *We have the following explicit formula*

$$\mathcal{S}_G(\varphi)(t) = \sum_{\overline{u}\in(\overline{U}\cap K)\backslash\overline{U}} \mathrm{pr}_{\overline{U}}\circ\varphi(\overline{u}t).$$

*Here, $\mathrm{pr}_{\overline{U}}\colon V\twoheadrightarrow V_{\overline{U}_p}$ and we are viewing $\varphi$ and $\mathcal{S}_G(\varphi)$ as functions $G\to\mathrm{End}_k V$ and $T\to\mathrm{End}_T V_{\overline{U}_p}$, respectively. The expression on the right hand side is a sum of linear maps $V\twoheadrightarrow V_{\overline{U}_p}$ and implicit in the statement is the assertion that the sum factors through $V\twoheadrightarrow V_{\overline{U}_p}$ to a map $V_{\overline{U}_p}\to V_{\overline{U}_p}$.*

*Proof.* The proof of this is just a long computation. See [**Herz+**, Proposition 27]. □

**6.4.** *Remark.* We verify well-definedness of the explicit formula for $\mathcal{S}_G$ given in Proposition 6.3.

(1) *Finiteness of the sum.* Suppose $(\overline{U}\cap K)a$ and $(\overline{U}\cap K)b$ be two distinct cosets where $\varphi$ is nonzero. Since they are distinct, we must have $ab^{-1}\notin\overline{U}\cap K$. This means that $KaK$ and $KbK$ are distinct double cosets in $\mathrm{Supp}\,\varphi$. Since $\mathrm{Supp}\,\varphi$ consists of finitely many distinct double cosets of $K$, the sum is finite.

(2) *The sum is independent of coset representatives.* Let $A\in\overline{U}\cap K$. We want to show that

$$\mathrm{pr}_{\overline{U}}\circ A\circ\varphi\circ\overline{u}t = \mathrm{pr}_{\overline{U}}\circ\varphi\circ\overline{u}t.$$

Let $x\in V$. Put $\varphi\circ\overline{u}tx = v$. Then we wish to show that $\mathrm{pr}_{\overline{U}}(Av) = \mathrm{pr}_{\overline{U}}(v)$, which is obviously true because $Av - v = 0$ in $V_{\overline{U}}$.

(3) *Factors through coinvariants.* Let $A\in\overline{U}\cap K$. By universal property of coinvariants, it is enough to check that $\mathcal{S}_G(\varphi)(Ax-x)=0$ for a general $x\in V$. But this is trivial by observations in (2).

One can obtain an analog of Theorem 5.6 quite easily because $T$ is abelian. Define $\tau_{r,s}\in\mathscr{H}_T V_{\overline{U}_p}$ by

$$\mathrm{Supp}\,\tau_{r,s} = (T\cap K)\,\mathrm{diag}(p^r,p^s)(T\cap K),\qquad \tau_{r,s}\,\mathrm{diag}(p^r,p^s)=1.$$

Set $\tau_1 := \tau_{0,1}$ and $\tau_2 := \tau_{1,1}$. Then $\tau_{r,s} = \tau_1^{s-r}\tau_2^r$ and $\mathscr{H}_T V_{\overline{U}_p}\cong k[\tau_1,\tau_2,\tau_1^{-1},\tau_2^{-1}]$. It is just a matter to computation to show that $\mathcal{S}_G(T_i)=\tau_i$ for $i\in\{1,2\}$.

**6.5. Theorem. —** *The Satake transform $\mathcal{S}_G\colon\mathscr{H}_G V\to\mathscr{H}_T V_{\overline{U}_p}$ is injective.*

*Proof.* We employ the universal property. We know that $(f \circ \varphi)_T = f_T \circ \mathcal{S}_G(\varphi)$ for any $f \in \mathrm{Hom}_G(\text{c-Ind}_K^G V, \mathrm{Ind}_{\overline{B}}^G W)$ where $W$ is a $\overline{B}$-representation. Suppose $\mathcal{S}_G(\varphi) = 0$. Then $f \circ \varphi = 0$ for each $\overline{B}$-representation $W$ and $f \in \mathrm{Hom}_G(\text{c-Ind}_K^G V, \mathrm{Ind}_{\overline{B}}^G W)$. We know by Lemma 6.1 that

$$\mathrm{Hom}_G(\text{c-Ind}_K^G V, \mathrm{Ind}_{\overline{B}}^G W) \cong \mathrm{Hom}_{T\cap K}(V_{\overline{U}_p}, W|_{T\cap K}).$$

By symmetry, it suffices to find a nonzero $f$. Lemma 3.5 tells us that $\dim_k V_{\overline{U}_p} = \dim_k \chi_{a,b} = 1$. Hence, it suffices to find $W$ so that $\mathrm{Hom}_{T\cap K}(\chi_{a,b}, W|_{T\cap K}) \neq 0$. Just choose $W = \chi_{a,b}$ viewed as a $\overline{B}$-representation through $\overline{B} \twoheadrightarrow T$. The proof is complete. See [**Herz+**, Theorem 29] for an alternative approach. □

**6.6. Theorem. —** *The image of $\mathcal{S}_G$ is $k[\tau_1, \tau_2, \tau_2^{-1}]$.*

*Proof.* By Theorems 6.5 and 5.6, it is sufficient to verify that $\mathcal{S}_G(T_i) = \tau_i$ for $i = 1,2$. This follows almost immediately by the explicit formula. □

Thus, the map $\mathcal{S}_G$ corresponds to the natural injection of $k$-algebras

$$k[T_1, T_2, T_2^{-1}] \hookrightarrow k[\tau_1, \tau_2, \tau_1^{-1}, \tau_2^{-1}], \qquad T_1 \mapsto \tau_1,\ T_2 \mapsto \tau_2.$$

In other words, $\mathcal{S}_G$ is the localization map which inverts $T_1$.

## 7. Comparison between compact and parabolic induction

We describe comparison isomorphisms between compact and parabolic induction. Fix a character $\chi\colon T \to k^\times$, a weight $V$, and a $K$-module embedding into the parabolic induction $f\colon V \hookrightarrow \mathrm{Ind}_{\overline{B}}^G \chi$. Frobenius reciprocity gives us a nonzero $G$-linear map $\tilde{f}\colon \text{c-Ind}_K^G V \to \mathrm{Ind}_{\overline{B}}^G \chi$. Since $f$ is a Hecke eigenvector, $\tilde{f}$ is too. This is easily seen by identifying $\text{c-Ind}_K^G V \cong k[G] \otimes_{k[K]} V$ and using explicit formulas. Define the character $\chi'\colon \mathcal{H}_G V \to k^\times$ by

$$\chi'(T_1) = \chi\,\mathrm{diag}(1,p)^{-1}, \qquad \chi'(T_2) = \chi\,\mathrm{diag}(p,p)^{-1}.$$

By universal property of tensor products, $\tilde{f}$ factors as

$$\hat{f}\colon \text{c-Ind}_K^G V \otimes_{\mathcal{H}_G V} \chi' \longrightarrow \mathrm{Ind}_{\overline{B}}^G \chi.$$

Our aim is to prove the following theorem.

**7.1. Theorem. —** *This map is an isomorphism if* $\dim V > 1$.

Consider the natural $G$-linear map

$$F\colon \text{c-Ind}_K^G V \to \mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}$$

which comes from the natural $T\cap K$-linear map $V_{\overline{U}_p} \to \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}$, $x \mapsto [1,x]$, via the following natural isomorphisms

$$\mathrm{Hom}_{T\cap K}(V_{\overline{U}_p}, \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}) \cong \mathrm{Hom}_{\overline{B}\cap K}(V, \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}) \cong$$
$$\mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}\cap K}^K \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}) \cong \mathrm{Hom}_G(\text{c-Ind}_K^G V, \mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}).$$

**7.2. Lemma. —** *$F$ is $\mathscr{H}_G V$-linear with respect to $\mathcal{S}_G$, i.e., $F\circ\varphi = \mathrm{Ind}_{\overline{B}}^G \mathcal{S}_G(\varphi)\circ F$ for each $\varphi \in \mathscr{H}_G V$. (Remember that $\mathrm{Ind}_{\overline{B}}^G$ is a functor)*

*Proof.* It suffices to check that $F\circ T_i = \mathrm{Ind}_{\overline{B}}^G \tau_i \circ F$ for $i \in \{1,2\}$. Let $F' \in \mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p})$ correspond to $F$ via the series of isomorphisms discussed before. We know from Propositon 5.8 that $F'$ is Hecke-linear. We are now done because the Frobenius reciprocity isomorphism

$$\mathrm{Hom}_K(V, \mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}) \cong \mathrm{Hom}_G(\text{c-Ind}_K^G V, \mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p})$$

is clearly Hecke-linear. □

Since $\mathcal{S}_G(T_1) = \tau_1$ is invertible, we get an induced $G$-linear and $(\mathscr{H}_G V)[T_1^{-1}]$-linear map

$$\overline{F}\colon \text{c-Ind}_K^G V \otimes_{\mathscr{H}_G V} (\mathscr{H}_G V)[T_1^{-1}] \longrightarrow \mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}.$$

**7.3. Lemma. —** *$\overline{F}$ is injective, and is an isomorphism if* $\dim V > 1$.

We first see how the above result proves Theorem 7.1.

**7.4.** *Proof of Theorem 7.1.* From Lemma 7.3, we have the isomorphism

$$\overline{F} \otimes_{(\mathscr{H}_G V)[T_1^{-1}]} \chi' \colon \text{c-Ind}_K^G V \otimes_{\mathscr{H}_G V} \chi' \longrightarrow (\mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}) \otimes_{(\mathscr{H}_G V)[T_1^{-1}]} \chi'.$$

By Corollary 2.8,

$$(\mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}) \otimes_{(\mathscr{H}_G V)[T_1^{-1}]} \chi' \cong \mathrm{Ind}_{\overline{B}}^G (\text{c-Ind}_{T\cap K}^T V_{\overline{U}_p} \otimes_{(\mathscr{H}_G V)[T_1^{-1}]} \chi').$$

So, it suffices to show that

$$\text{c-Ind}_{T\cap K}^T V_{\overline{U}_p} \otimes_{(\mathscr{H}_G V)[T_1^{-1}]} \chi' \cong \chi,$$

which is obvious because of the identifications

$$\mathscr{H}_G V \cong k[T_1, T_2, T_2^{-1}], \qquad \chi' \cong k[T_1, T_2, T_2^{-1}]/(T_1 - (\chi\,\mathrm{diag}(1,p))^{-1}, T_2 - (\chi\,\mathrm{diag}(p,p))^{-1}).$$ □

We will use the following result in the proof of Lemma 7.3.

**7.5. Bruhat decomposition. —** *$G = B \sqcup BwB$, where $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.*

*Proof.* This just row reduction from rudimentary linear algebra. The rightmost $B$ is the echelon form of a matrix, the factor of $w$ accounts for rows having to be reordered, and the leftmost $B$ is the coefficient matrix of the Gauss-Jordan elimination algorithm. □

**7.6.** *Proof of Lemma 7.3.* We show injectivity by using the theory of mod-$p$ Satake transform. It suffices to show that
$$F\colon \text{c-Ind}_K^G V \to \mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}$$
is injective. Indeed, the Satake map $\mathcal{S}_G$ is a localization and $\mathscr{H}_G V$ is a domain. Suppose $F$ is not injective and there is a weight $V' \hookrightarrow (\operatorname{Ker} F)|_K$. By Frobenius reciprocity, this induces a nonzero $G$-linear map
$$\theta\colon \text{c-Ind}_K^G V' \longrightarrow \operatorname{Ker} F.$$
Because of funtoriality of $\mathrm{Ind}_{\overline{B}}^G$ and $\mathcal{S}_G$ and Lemma 7.2, we have a commutative square
$$\begin{tikzcd} \text{c-Ind}_K^G V \arrow[r, "F"] & \mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p} \\ \text{c-Ind}_K^G V \arrow[u, "\theta"] \arrow[r, "F'"] & \mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V'_{\overline{U}_p}. \arrow[u, "\mathrm{Ind}_{\overline{B}}^G \mathcal{S}_G(\theta)"'] \end{tikzcd}$$
Hence, $\mathrm{Ind}_{\overline{B}}^G \mathcal{S}_G(\theta)\circ F' = 0$. With notations of Lemma 6.1, $F'_T$ is identity. Thus,
$$\mathcal{S}_G(\theta) = \mathrm{Ind}_{\overline{B}}^G \mathcal{S}_G(\theta)_T = 0.$$
We are now done by Theorem 6.5.

We follow [**Herz+**, p. 18-19] to prove surjectivity. Fix $x \in V$. Then $f := F([1,x])$ is a $\overline{B}$-equivariant function $G \to \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}$ mapping $t\overline{u}k$ to $[t, \overline{kx}]$, $t\in T, \overline{u}\in\overline{U}, k\in K$. To compute $\operatorname{Supp} f$, we want to determine when $\overline{kx}\neq 0$ in $V_{\overline{U}_p}$. By Bruhat decomposition, after multiplying by $w := \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}$, $G_p = wB_p \sqcup \overline{B}_p B_p$. For $\overline{kx}\neq 0$, we want $k \in \overline{B}_p B_p = \overline{B}_p U_p$ because $wx \in \operatorname{Ker}(V \twoheadrightarrow V_{\overline{U}_p})$. Note that $\overline{I}(U\cap K) = (\overline{B}\cap K)(U\cap \overline{I})(U\cap K) = (\overline{B}\cap K)(U\cap K)$. Therefore, $\operatorname{Supp} f \subseteq \overline{B}(U\cap K)$. However,
$$f\begin{bmatrix} 1 & a \\ 0 & 1\end{bmatrix} = [1,\overline{x}].$$
So $\operatorname{Supp} f = \overline{B}(U\cap K)$. Consider
$$\begin{aligned} \{f \in \mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p} : \operatorname{Supp} f \subset \overline{B}U\} &\cong \mathscr{C}^\infty_{\text{cpt}}(\mathbb{Q}_p, \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}) \\ g &\longmapsto \left(g^\dagger\colon a \mapsto g\begin{bmatrix} 1 & a \\ 0 & 1\end{bmatrix}\right). \end{aligned}$$
This is a $B$-linear isomorphism (c.f. Example 3.9). The $B$-actions are
$$\begin{bmatrix} 1 & u \\ 0 & 1\end{bmatrix} g^\dagger(a) = g^\dagger(a+u), \qquad \begin{bmatrix} x & 0 \\ 0 & y\end{bmatrix} g^\dagger(a) = g^\dagger(ay/x).$$
Also, $f^\dagger(t)$ equals $[1,\overline{x}]$ if $t\in\mathbb{Z}_p$ and 0 otherwise. We show that acting by $(\mathscr{H}_G V)[T_1^{-1}]$ and $G$ gives us all functions. Observe that $(\mathscr{H}_G V)[T_1^{-1}]$-span gives all functions supported and constant on $\mathbb{Z}_p$. By scaling, i.e., $T$-action, we get any function supported and constant on

any $p^n\mathbb{Z}_p$. Furthermore, $U$-action behaves as translation, so we get all functions supported and constant on a set of the form $a+p^n\mathbb{Z}_p$, for any $a\in\mathbb{Q}_p$ and $n\in\mathbb{Z}$. It is easy to see that these functions span $\mathscr{C}^\infty_{\mathrm{cpt}}(\mathbb{Q}_p, \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p})$. So, we see that $f$ generates all functions in $\mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}$ supported on $\overline{B}U$. Acting by $G$, we get any function supported on $\overline{B}Ug^{-1}$ for each $g\in G$. Since $\overline{B}\setminus G$ is compact by Iwasawa decomposition, it follows that these functions span all of $\mathrm{Ind}_{\overline{B}}^G \text{c-Ind}_{T\cap K}^T V_{\overline{U}_p}$. □

**7.7. Corollary. —** *If* $\dim V > 1$, *the weight* $f(V)$ *generates* $\mathrm{Ind}_{\overline{B}}^G \chi$ *as a G-representation.*

*Proof.* The of $V$ in $\text{c-Ind}_K^G V$ generates it as a $G$-representation. This is easily seen by identifying $\text{c-Ind}_K^G V$ with $k[G]\otimes_{k[K]} V$. The desired result is clear from the following commutative diagram.

$$\begin{tikzcd}
V \arrow[d, hook] \arrow[drr, hook] & & \\
\text{c-Ind}_K^G V \arrow[rr, hook] \arrow[d, two heads] & & \mathrm{Ind}_{\overline{B}}^G \chi \\
\text{c-Ind}_G^V K \otimes_{\mathscr{H}_G V} \chi'. \arrow[urr, "\sim"'] & &
\end{tikzcd}$$

□

**7.8. Corollary (irreducible principal series representations). —** *Let* $\chi_1, \chi_2\colon \mathbb{Q}_p^\times \to k^\times$ *be two smooth characters, and* $\chi_1|_{\mathbb{Z}_p^\times} \neq \chi_2|_{\mathbb{Z}_p^\times}$. *Then* $\mathrm{Ind}_{\overline{B}}^G(\chi_1\otimes\chi_2)$ *is an irreducible G-representation.*

*Proof.* We are in the setup of Proposition 3.8. So there is a unique weight $V$ and it satisfies $\dim V > 1$. By Corollary 7.7, $V$ generates $\mathrm{Ind}_{\overline{B}}^G(\chi_1\otimes\chi_2)$. Any nonzero subrepresentation must contain the unique weight $V$. Hence $\mathrm{Ind}_{\overline{B}}^G(\chi_1\otimes\chi_2)$ is irreducible. □

## 8. Digression: Bruhat-Tits tree

A good reference for this section is [Ser80, Chapter 2]. Let $F/\mathbb{Q}_p$ be a finite extension of local fields with uniformizer $\pi$. Fix a two-dimensional $F$-vector space $W$.

**8.1. Definition.** A **lattice** in $W$ is a $\mathscr{O}_F$-submodule of $W$ which spans $W$ as an $F$-vector space.

We say two lattices $L$ and $L'$ are equivalent or homothetic if $L = cL'$ for some $c\in F^\times$. Denote by $\mathscr{X}$ the set of all equivalence classes of lattices. We endow $\mathscr{X}$ with a graph structure as follows– the vertices are the equivalence classes $[L]$ and two distinct vertices $[L]$ and $[L']$

are adjacent by a unique edge if there exists representatives $L$ and $L'$ such that $\pi L \subset L' \subset L$. Why are not defining it as a directed graph? because $\pi L \subset L' \subset L$ implies $L' \subset L \subset \pi^{-1}L'$. Then $\mathscr{X}$ is called the **Bruhat-Tits tree** corresponding to $\mathrm{GL}_2$. The name is justified by the following proposition–

**8.2. Proposition. —** *$\mathscr{X}$ is a tree.*

*Proof.* Let $[L]$ and $[L']$ be two vertices of $\mathscr{X}$. Since $L \cap L'$ is also a lattice by standard linear algebra, we may replace $L'$ by a different representative so that $L' \subset L$. Then $L/L'$ is a finite $\mathscr{O}_F$-module. Hence, by Jordan-Hölder filtration, there is a sequence of lattices $L' = L_0 \subset \cdots \subset L_n = L$ such that $L_n/L_{n-1} \cong \mathscr{O}_F/\pi$. This shows that $\mathscr{X}$ is connected. For the second part, consider a path $L_0 \subset \cdots \subset L_{n-1} \subset L_n \subset \cdots \subset L_r$ in $\mathscr{X}$. In particular, $L_n/L_{n-1} \cong \mathscr{O}_F/\pi$ and $[L_i] \neq [L_j]$ for each $i \neq j$. We show that $L_r/L_0 \cong \mathscr{O}_F/\pi^r$ by induction on $r$. There is nothing to do for $r = 1$. Assume as induction hypothesis that $L_k/L_\ell \cong \mathscr{O}_F/\pi^{k-\ell}$ for each $1 \leqslant k-\ell \leqslant r-1$. We have the following commutative diagram of $\mathscr{O}_F$-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & L_1/L_0 \arrow[r] & L_r/L_0 \arrow[r] & L_r/L_1 \arrow[r] & 0 \\
0 \arrow[r] & L_1/L_0 \arrow[r] \arrow[u, equal] & L_{r-1}/L_0 \arrow[r] \arrow[u, hook] & L_{r-1}/L_1 \arrow[r] \arrow[u, hook] & 0 \\
0 \arrow[r] & \mathscr{O}_F/\pi \arrow[r] \arrow[u, equal] & \mathscr{O}_F/\pi^{r-1} \arrow[r] \arrow[u, equal] & \mathscr{O}_F/\pi^{r-2} \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd}$$

If the top row splits then $L_{r-1}/L_0 \hookrightarrow L_r/L_0 \twoheadrightarrow L_r/L_1$ must be an isomorphism. But this is certainly not possible because the composition $L_1/L_0 \hookrightarrow L_{r-1}/L_0 \hookrightarrow L_r/L_0 \twoheadrightarrow L_r/L_1$ must be 0. Hence, the top row is non-split and $L_r/L_0 \cong \mathscr{O}_F/\pi^r$. Here, we are implicitly using PID structure theorem. $\square$

**8.3. Proposition. —** *$\mathscr{X}$ is a regular graph of degree $|\mathscr{O}_F/\pi|+1$, i.e., every vertex has degree $|\mathscr{O}_F/\pi|+1$.*

*Proof.* Fix a vertex $[L']$. Then $[L]$ is adjacent to $[L']$ if and only if $\pi L \subset L' \subset L$. That is, there is a one-one correspondence between $L$ and one-dimensional $\mathscr{O}_F/\pi$-subspaces of $\pi^{-1}L'/L' \cong L'/\omega L' \cong (\mathscr{O}_F/\pi)^{\oplus 2}$. The proposition follows by the observation that $(\mathscr{O}_F/\pi)^{\oplus 2}$ has exactly $|\mathscr{O}_F/\pi|+1$ lines. $\square$

In fact, the structure of $\mathscr{X}$ is completely determined by the properties: (a) it is a tree; (b) every vertex has $|\mathscr{O}_F/\pi|+1$ neighbors.

**8.4. Action of $\mathrm{GL}_2(F)$.** The general linear group $\mathrm{Aut}_F W$ acts on lattices. If $L_1 \sim L_2$ are equivalent lattices then it is easy to show that $AL_1 \sim AL_2$ for each $A \in \mathrm{Aut}_F W$. Also, if

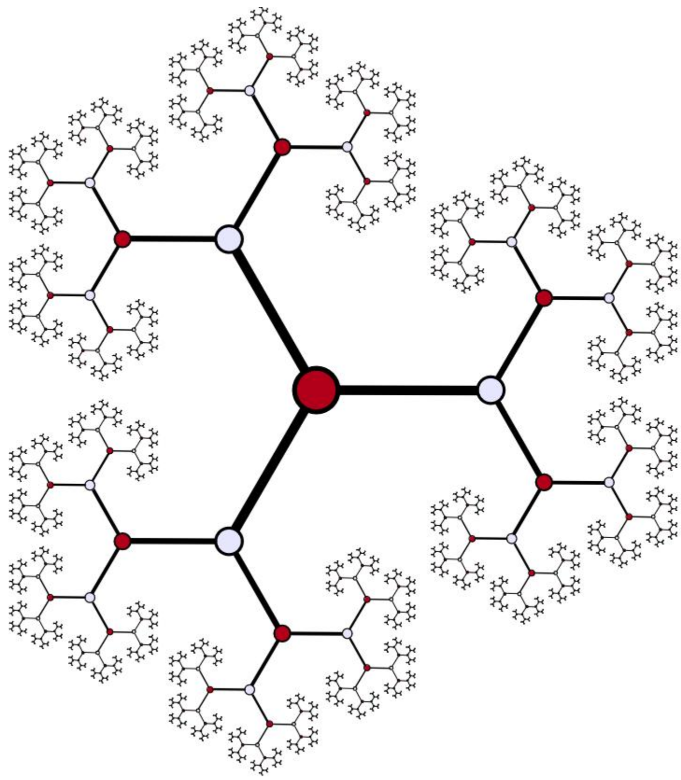

FIGURE 8.1. A Bruhat-Tits tree when $\mathscr{O}_F/\pi \cong \mathbb{F}_2$ [**Cas**, p. 7].

$[L_1] \sim [L_2]$ are adjacent then $[AL_1]$ and $[AL_2]$ are also adjacent. Therefore, after fixing a basis $e_1, e_2 \in V$, it follows that $\mathrm{GL}_2(F)$ acts on $\mathscr{X}$.

**8.5. Proposition. —** *The action of $G$ on $\mathscr{X}$ is transitive on (ordered) edges.*

*Proof.* Let us assume $F = \mathbb{Q}_p$. The same proof works in the general case. It is clear that $G$ acts transitively on nodes. Therefore, it suffices to show the following claim–

**Claim 1.** *Let $L$ be a lattice. Denote $G_L := \mathrm{Stab}_G L$. Then $G_L$ acts transitively on the edges emanating from the node $[L]$.*

Again, because the $G$-action is transitive on nodes, we may assume $L = \mathbb{Z}_p^{\oplus 2}$ without any loss of generality. Then $G_L = K = \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$. This is same as showing that $G_L$ acts transitively on the set of sub-lattices $L' \subset \mathbb{Z}_p^{\oplus 2}$ satisfying $\mathbb{Z}_p^{\oplus 2}/L' \cong \mathbb{F}_p$. Such $L'$ are of the form $A(\mathbb{Z}_p^{\oplus 2})$ for some $A \in \mathrm{Mat}_2(\mathbb{Z}_p)$ such that $\det A = p$. If $B$ is another matrix with integral entries such that $\det B = p$ then it's clear that $AB^{-1} \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times = G_L$. Hence the claim. □

**8.6. Stabilizers of edges as Iwahori subgroups.** Choose a basis $e_1, e_2$. Take the adjacent lattices $L_1 = \mathscr{O}_F e_1 + \mathscr{O}_F e_2$ and $L_2 = \mathscr{O}_F e_1 + \pi \mathscr{O}_F e_2$. Note that $L_2 = \alpha L_1$ where $\alpha = \mathrm{diag}(1, \pi)$.

The stabilizer of the edge $\overline{[L_1][L_2]}$ is equal to

$$\mathrm{Stab}_G\,\overline{[L_1][L_2]} = \mathrm{Stab}_G[L_1] \cap \mathrm{Stab}_G[L_2] = KF^\times \cap \alpha K \alpha^{-1} F^\times,$$

which can be easily verified to be equal to $IF^\times$ where $I = K \cap \alpha K \alpha^{-1}$ is an Iwahori subgroup. In terms of matrices: the preimage of the upper triangular subgroup $\begin{bmatrix} \star & \star \\ 0 & \star \end{bmatrix}$ of $\mathrm{GL}_2(\mathscr{O}_F/\pi)$.

**8.7. Corollary. —** *There is a bijection between ordered edges of $\mathscr{X}$ and $G/IF^\times$.*

*Proof.* Orbit-Stabilizer theorem. □

## 9. The Steinberg representation

The Steinberg representation arises from the principal series representation corresponding the trivial character $\mathbb{1}_T \colon T \to k^\times$, i.e., $\mathrm{Ind}_{\overline{B}}^G \mathbb{1}_T$.

**9.1. Definition.** The **Steinberg representation**, denoted St, is the unique representation which fits into the exact sequence

$$0 \to \mathbb{1}_G \to \mathrm{Ind}_{\overline{B}}^G \mathbb{1}_T \to \mathrm{St} \to 0,$$

where the first map comes from Frobenius reciprocity. Explicitly, one can write

$$\begin{aligned} \mathrm{Ind}_{\overline{B}}^G \mathbb{1}_T &= \{f \colon G \to k \colon f(xg) = f(g) \text{ for each } x \in \overline{B}\}^\infty \\ &\cong \{f \colon \overline{B}\backslash G \to k \colon f \text{ is locally constant}\}, \end{aligned}$$

where we are implicitly using compactness of $\overline{B} \backslash G \cong (\overline{B} \cap K)\backslash K$ (Iwasawa decomposition). Then $\mathbb{1}_G$ can be identified with the subset of constant functions $\overline{B}\backslash G \to k$.

**9.2.** *Remark.* One can alternatively define St via the exact sequence

$$0 \longrightarrow \mathrm{St} \longrightarrow \frac{\text{c-}\mathrm{Ind}_K^G \mathbb{1}_K}{(T_1 - 1, T_2 - 1)} \xrightarrow{\text{sum all nodes in } \mathscr{X}} \mathbb{1}_G \longrightarrow 0$$

Recall the pro-$p$ Iwahori subgroup $I(1)$.

**9.3. Lemma. —** *$\overline{B} \backslash G$ has exactly two $I(1)$-orbits. Furthermore,*

- *A fundamental domain for the $I(1)$-action is $\left\{\overline{B}, \overline{B}w\right\}$, where $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.*
- *The coset $\overline{B}$ is stabilized by $I(1) \cap \overline{B}$ and $\overline{B}w$ is stabilized by $I(1) \cap B$.*
- *The action of $I(1) \cap \overline{B}$ (respectively $I(1) \cap B$) is transitive on the $I(1)$-orbit of $\overline{B}w$ (respectively $\overline{B}$).*

*Proof.* We know that $\overline{B} \setminus G \cong \overline{B} \cap K \setminus K$. Consider the following identities:

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} az & aw \\ cz+dx & cw+dy \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax & ay \\ cx+dz & cy+dw \end{bmatrix},$$

where $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in I(1)$. We will either take $z = 0$ or $y = 0$ depending on the situation to prove the third assertion of the statement on the fly. It is not hard to observe that the RHS of the above two equalities cover all matrices in $K$. Indeed, to express an arbitrary matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix} \in K$ in the above form, we may proceed as follows–

- $v_p(e) > v_p(f)$. Consider the first identity and set $y = 0$. Take $a = p^{v_p(f)}u_1$ and $z = p^{v_p(e)-v_p(f)}u_2$ while adjusting $u_1, u_2, w \in \mathbb{Z}_p^\times$ so that $w \equiv 1 \pmod p$. Choose $c = h/w$. Then we want to ensure $cz + dx = g \iff g - hw^{-1}p^{v_p(e)-v_p(f)}u_2 = dx$. This is certainly possible.
- $v_p(e) \leqslant v_p(f)$. Consider the second identity and set $z = 0$. Take $a = p^{v_p(e)}u_1$ and $y = p^{v_p(f)-v_p(e)}u_2$ where $u_1, u_2 \in \mathbb{Z}_p^\times$ are chosen such that $x \equiv 1 \pmod p$. Set $c = g/x$. Then one can easily adjust $d$ and $w$ to ensure $dw = h - gx^{-1}p^{v_p(f)-v_p(e)}u_2$. □

**9.4. Lemma. —** $\dim \mathrm{St}^{I(1)} = 1$.

*Proof.* We have the long exact sequence in cohomology,

$$0 \to k \to \mathrm{Ind}_{\overline{B}}^G(\mathbb{1}_T)^{I(1)} \to \mathrm{St}^{I(1)} \to H^1(I(1), k) \to H^1(I(1), \mathrm{Ind}_{\overline{B}}^G \mathbb{1}_T) \to \cdots.$$

We know that $H^1(I(1), k) \cong \mathrm{Hom}_{\mathsf{Grp}}^{\mathrm{cont}}(I(1), k)$ holds because $k$ is a trivial $I(1)$-module. We claim that $H^1(I(1), k) \to H^1(I(1), \mathrm{Ind}_{\overline{B}}^G(\mathbb{1}_T))$ is injective. Suppose not. Let $c \in \mathrm{Hom}_{\mathsf{Grp}}^{\mathrm{cont}}(I(1), k)$ be such that there exists some $f \in \mathrm{Ind}_{\overline{B}}^G \mathbb{1}_T$ such that $f(xg) - f(x) = c(g)$ for each $x \in \overline{B} \setminus G$ and $g \in I(1)$. Given any coset $\overline{B}x$ in the $I(1)$-orbit of $\overline{B}$, by Lemma 9.3, we can find $i \in I(1) \cap B$ such that $i$ fixes $\overline{B}w$ and $\overline{B}i = \overline{B}x$. Therefore, $c(i) = f(\overline{B}i) - f(\overline{B}) = 0$. Hence, $0 = c(i) = f(\overline{B}i) - f(\overline{B}) = f(\overline{B}x) - f(\overline{B})$. This shows that $f$ is constant on the $I(1)$-orbit of $\overline{B}$. Similarly, $f$ is constant on the $I(1)$-orbit of $\overline{B}w$. Thus, $c = 0$ and we have the exact sequence

$$0 \to k \to \mathrm{Ind}_{\overline{B}}^G \mathbb{1}_T^{I(1)} \to \mathrm{St}^{I(1)} \to 0.$$

We are now done by Lemma 9.3 and the fact that dim is additive over exact sequences. □

**9.5.** *Remark.* It is worth mentioning that $H^1(I(1), k) \cong \mathrm{Hom}_{\mathsf{Grp}}^{\mathrm{cont}}(I(1), k)$ is far from 0 due to maps of the form $I(1) \twoheadrightarrow U_p \xrightarrow{\sim} \mathbb{F}_p \hookrightarrow k$.

**9.6. Theorem. —** St *is irreducible.*

*Proof.* Since $\dim \mathrm{St}^{I(1)} = 1$, it follows that St contains a unique weight. By Proposition 3.8, a weight of $\mathrm{Ind}_B^G \mathbb{1}_T$ has dimension either 1 or $p$. Let $V$ be the weight of $\mathrm{Ind}_B^G \mathbb{1}_T$ with $\dim V = p$. By left-exactness of $\mathrm{Hom}_K(V, -)$,

$$0 \to \mathrm{Hom}_K(V, \mathbb{1}_G|_K) \to \mathrm{Hom}_K(V, \mathrm{Ind}_B^G(\mathbb{1}_T)|_K) \to \mathrm{Hom}_K(V, \mathrm{St}|_K).$$

Of course $\mathrm{Hom}_K(V, \mathbb{1}_G|_K) = 0$. By Proposition 3.8, $\dim \mathrm{Hom}_K(V, \mathrm{Ind}_B^G(\mathbb{1}_T)|_K) = 1$. Therefore $V$ is also a weight of St, consequently, this is the only weight. By Corollary 7.7, $V$ generates the quotient St as a $G$-module. Any nonzero subrepresentation must contain the unique weight $V$. Thus, St is irreducible. □

**9.7.** *Remark.* The "Steinberg weight" $F(p-1,0)$ is a weight of $\mathrm{Ind}_B^G \mathbb{1}_T$ and is disjoint from $\mathbb{1}_G$, hence a weight of St and therefore the unique weight of St.

## 10. Change of weight

Fix weights $V$ and $V'$. Define the "relative Hecke object", also called the **module of interwiners**, as

$$\mathscr{H}_G(V, V') = \mathrm{Hom}_G(\text{c-Ind}_K^G V, \text{c-Ind}_K^G V').$$

It is a $(\mathscr{H}_G V', \mathscr{H}_G V)$-bimodule with pre- and post-composition.

**10.1. Proposition. —**

*(1)*

$$\mathscr{H}_G(V, V') \cong \left\{ \begin{matrix} \varphi\colon G \to \mathrm{Hom}_k(V, V') \colon \varphi(k_1 g k_2) = k_1 \varphi(g) k_2 \\ \text{for all } k_1, k_2 \in K,\ g \in G,\ K \setminus \mathrm{Supp}\,\varphi / K \text{ is finite} \end{matrix} \right\}$$

*(2) $\mathscr{H}_G(V, V') \neq 0$ if and only if $V_{\overline{U}_p} \cong V'_{\overline{U}_p}$ as $T_p$-representations.*

*(3) If $V \ncong V'$ and $V_{\overline{U}_p} \cong V'_{\overline{U}_p}$, there is a Hecke operator $\varphi\colon G \to \mathrm{Hom}_k(V, V')$ supported on $K \operatorname{diag}(p^r, p^s) K$ if and only if $r < s$, and it is unique up to scaling.*

*Proof.* The proof is essentially same as that of Proposition 5.2. □

In (3) of Proposition 10.1, the only possibility is $V = F(b, b)$ and $V' = F(b+p-1, b)$, upto symmetry. There exist "relative" Hecke operators $\varphi^-\colon \text{c-Ind}_K^G V' \to \text{c-Ind}_K^G V$ and $\varphi^+\colon \text{c-Ind}_K^G V \to \text{c-Ind}_K^G V'$ such that $\mathrm{Supp}\,\varphi^\pm = K \operatorname{diag}(1, p) K$. For the next proposition, we make the identifications

$$\mathscr{H}_G V \cong k[T_1, T_2, T_2^{-1}] \cong \mathscr{H}_G V'$$

and call both algebras $\mathscr{H}$. We think of $\varphi^- \circ \varphi^+$ and $\varphi^+ \circ \varphi^-$ as algebra endormorphisms of $\mathscr{H} = k[T_1, T_2, T_2^{-1}]$.

**10.2. Proposition. —** *The maps $\varphi^-$ and $\varphi^+$ commute and $\varphi^+ \circ \varphi^- = \varphi^- \circ \varphi^+ = T_1^2 - T_2$ upto scaling.*

*Sketch.* One can develop a "relative" mod-$p$ Satake transform giving an inclusion of algebras $\mathscr{H}_G(V, V') \hookrightarrow \mathscr{H}_T(V_{\overline{U}_p}, V'_{\overline{U}_p})$ using which one shows that $\varphi^\pm$ commute. The second assertion is proved by a direct calculation. □

**10.3. Corollary. —** *If $\chi' : \mathscr{H} \to k$ is an algebra homomorphism such that $\chi'(T_1^2 - T_2) \neq 0$ then*

$$\text{c-Ind}_K^G V \otimes_{\mathscr{H}} \chi' \cong \text{c-Ind}_K^G V' \otimes_{\mathscr{H}} \chi'.$$

*Proof.* We have maps $\varphi^\pm \otimes_{\mathscr{H}} \chi'$. By Proposition 10.2, note that $(\varphi^+ \circ \varphi^-) \otimes_{\mathscr{H}} \chi' = \chi'(T_1^2 - T_2) \neq 0$. Hence, their compositions act invertibly by the nonzero scalar $\chi'(T_1^2 - T_2)$. □

**10.4. Proposition. —** *Let $\chi = \chi_1 \otimes \chi_2 \colon T \to k^\times$ be a smooth character and $\chi_1 \neq \chi_2$. Then* $\mathrm{Ind}_B^G \chi$ *is irreducible.*

*Proof.* The case $\chi_1|_{\mathbb{Z}_p^\times} \neq \chi_2|_{\mathbb{Z}_p^\times}$ is already done. So let us assume $\chi_1|_{\mathbb{Z}_p^\times} = \chi_2|_{\mathbb{Z}_p^\times}$ and $\chi_1(p) \neq \chi_2(p)$ We know that $\pi := \mathrm{Ind}_B^G \chi$ contains two weights, say $V$ and $V'$, of the form $F(b, b)$ and $F(p-1+b, b)$, respectively, with $0 \leqslant b < p-1$. Corollary 7.7 tells us that $V'$ generates $\pi$ as a $G$-module. As usual, let's suppose $\sigma \subseteq \pi$ is a nonzero $G$-subrepresentation. It suffices to show that $V' \subset \sigma$. Indeed, if this were not true then there is a $K$-linear inclusion $V \hookrightarrow \sigma|_K$. This is simply because $\sigma|_K$ must contain a weight. By Frobenius reciprocity, this gives a $G$-linear map $\text{c-Ind}_K^G V \to \sigma$, which factors as $\text{c-Ind}_K^G V \otimes_{\mathscr{H}} \chi' \to \sigma$ for some Hecke-character $\chi'$, as $\mathrm{Hom}_K(V, \pi|_K)$ is 1-dimensional. By Proposition 5.8, $\chi'$ is given by $\chi'(T_1) = \chi_2(p)^{-1}$ and $\chi'(T_2) = \chi_1(p)^{-1}\chi_2(p)^{-1}$. Note that $\chi'(T_1^2 - T_2) = \chi_2(p)^{-1}(\chi(p)^{-1} - \chi_1(p)^{-1}) \neq 0$. Therefore, Corollary 10.3 shows that we have a nonzero $G$-linear map $\text{c-Ind}_K^G V' \otimes_{\mathscr{H}} \chi' \to \sigma$, which gives rise to a $K$-linear inclusion $V' \hookrightarrow \sigma|_K$. Thus, $\sigma$ generates $\pi$. □

## 11. Classification

In this section, we classify all smooth irreducible (admissible) representations of $G$. We remark that this admissiblity assumption is not necessary. Let $\Gamma \subset G$ be a closed subgroup.

**11.1. Definition.** A smooth $\Gamma$-representation $\pi$ is called **admissible** if $\dim \pi^W$ is finite for all open subgroups $W$ of $\Gamma$.

**11.2. Definition.** Let $\pi$ be an irreducible admissible $G$-representation. We say $\pi$ is **supersingular** if for any weight $V$ the action of $T_1$ on $\mathrm{Hom}_K(V, \pi|_K)$ is nilpotent, i.e., all eigenvalues of $T_1$ are zero.

Let us state the main theorem of this chapter.

**11.3. Theorem (Barthel-Livné).** — *Every irreducible admissible $G$-representation falls into one of the following disjoint families:*

*(1) principal series:* $\mathrm{Ind}_B^G \chi_1 \otimes \chi_2, \chi_1 \neq \chi_2$,
*(2) smooth characters:* $\chi \circ \det$,
*(3) twists of Steinberg:* $\mathrm{St} \otimes (\chi \circ \det)$,
*(4) the supersingular representations.*

**11.4. Proposition.** — *Let $\pi$ be a smooth $\Gamma$-representation. Then $\pi$ is admissible if and only if* $\dim \pi^W$ *is finite for some open pro-$p$ subgroup $W$ of $\Gamma$.*

*Proof.* Let $W$ be an open pro-$p$ subgroup such that $\dim \pi^W < \infty$ and $W'$ be an arbitrary open subgroup. By replacing $W'$ by $W' \cap W$ we may assume $W' \subset W$. Since $W$ is pro-$p$, $W'$ must have finite index. Note that $\pi^{W'} = \mathrm{Hom}_{W'}(\mathbb{1}_{W'}, \pi|_{W'}) \cong \mathrm{Hom}_W(\text{c-Ind}_{W'}^W \mathbb{1}_{W'}, \pi|_W)$. We are now done by Lemma 11.5. □

**11.5. Lemma.** — *Let $\pi$ be a smooth $\Gamma$-representation such that* $\dim \pi^W < \infty$ *for some open pro-$p$ subgroup $W$ of $\Gamma$. Then* $\dim \mathrm{Hom}_W(M, \pi|_W)$ *is finite-dimensional for any finite-dimensional smooth $W$-representation $M$.*

*Proof.* We induct on $\dim M$. Since $W$ is pro-$p$, there is an exact sequence

$$0 \to \mathbb{1}_W \to M \to M/\mathbb{1}_W \to 0.$$

Applying $\mathrm{Hom}_W(-, \pi|_W)$ gives us the desired result by induction hypothesis. □

**11.6. Proposition.** — *Let $\pi$ be smooth representation of $G$.*

*(1) The representation $\pi$ is admissible if and only if* $\dim \mathrm{Hom}_K(V, \pi|_K)$ *is finite for any weight $V$.*
*(2) If $\pi$ is admissible, then $\pi$ possesses a central character.*

*Proof.*

(1) One direction is done by Lemma 11.5. For the converse, let us suppose that $\dim \mathrm{Hom}_K(V, \pi|_K) < \infty$ for all weights $V \hookrightarrow \pi|_K$. It is enough to show that $\pi^{K(1)}$ is finite-dimensional. Note that $\pi^{K(1)} \cong \mathrm{Hom}_K(\text{c-Ind}_{K(1)}^K \mathbb{1}_{K(1)}, \pi)$ by Frobenius reciprocity. We know that $\text{c-Ind}_{K(1)}^K \mathbb{1}_{K(1)}$ is finite-dimensional. The proof is complete by Lemma 11.5.
(2) There is a natural map $\mathbb{Q}_p^\times \cong Z(G) \to \mathrm{Aut}_G \pi$. By Schur's lemma, $\mathrm{Aut}_G \pi = k^\times$. So we obtain a character $\mathbb{Q}_p^\times \to k^\times$. □

**11.7. Corollary. —** *All principal series representations and all representations of the form* $\mathrm{St}\otimes(\chi\circ\det)$ *with* $\chi\colon\mathbb{Q}_p^\times\to k^\times$ *a smooth character, are admissible.*

*Proof.* We have already characterised weights of principal series representations. So Proposition 11.6 (i) implies that principal series are admissible. By the proof of Theorem 9.6, we get the second result. □

**11.8.** *Remark and definition.* Let $V$ be a finite dimensional representation of $\Gamma$. A filtration $0=V_0\subset V_1\subset\cdots\subset V_\ell=V$ of $V$ is called a Jordan-Holder series if all the $V_j$ are subrepresentations and each $V_j/V_{j-1}$ is simple. The **semisimplification** $V^{\mathrm{ss}}$ of $V$ is defined to be $\bigoplus_{j=1}^{\ell}V_j/V_{j-1}$. This is independent of the choice of Jordan-Holder filtation.

**11.9.** *Proof of Theorem 11.3.* Let $\pi$ be an irreducible admissible representation and let $V$ be a weight of $V$. The (finite-dimensional) multiplicity space $\mathrm{Hom}_K(V,\pi|_K)$ contains a common Hecke eigenvector $f\colon V\hookrightarrow\pi|_K$. Let this eigenvector be given by the algebra homomorphism $\chi'\colon\mathcal{H}_GV\to k$. If $\chi'(T_1)=0$ for all $V$ then $\pi$ is supersingular. So let us assume $\chi'(T_1)\neq0$. Then there is a *nonzero* $G$-linear surjection $\text{c-Ind}_K^GV\otimes_{\mathcal{H}_GV}\chi'\twoheadrightarrow\pi$. We consider several possiblities:

- If $\dim V>1$ then $\text{c-Ind}_K^GV\otimes_{\mathcal{H}_GV}\chi'\cong\mathrm{Ind}_{\overline{B}}^G\chi_1\otimes\chi_2$ for some choice of $\chi_i$. See Theorem 7.1. Therefore, $\pi$ is either an irreducible principal series or a twist of Steinberg in this case.
- If $\dim V=1$ and $\chi'(T_1^2-T_2)\neq0$ then we have $\text{c-Ind}_K^GV\otimes_{\mathcal{H}_GV}\chi'\cong\text{c-Ind}_K^GV'\otimes_{\mathcal{H}_GV}\chi'$ for some $p$-dimensional weight $V'$ from Corollary 10.3. One can now proceed as in the previous case.
- If $\dim V=1$ and $\chi'(T_1^2-T_2)=0$, then we may assume $V=\mathbb{1}_K$ simply by twisting by a character of the form $\eta\circ\det$. We may further arrange $\chi'(T_1)=\chi'(T_2)=1$ by ensuring $\eta(p)=\chi'(T_1)$. One can prove that there is an exact sequence
$$0\longrightarrow\mathrm{St}\longrightarrow\frac{\text{c-Ind}_K^G\mathbb{1}_K}{(T_1-1,T_2-1)}\xrightarrow[\text{in Bruhat-Tits tree}]{\text{sum all nodes}}\mathbb{1}_G\longrightarrow0.$$
The middle term is the same as $\text{c-Ind}_K^G\mathbb{1}_K\otimes_{\mathcal{H}_GV}\chi'$. It follows that $\pi\otimes(\eta\circ\det)$ is the trivial character. Twisting back, $\pi$ is a character of the desired form.

We now show that the four families discussed above are disjoint. We do an analysis on their weights and Hecke eigenvalues. Firstly, if $\pi$ is any subquotient of a principal series and $V$ is any weight of $\pi$ then $V_{\overline{U}_p}\cong\chi_1|_{\mathbb{Z}_p^\times}\otimes\chi_2|_{\mathbb{Z}_p^\times}$ as $T_p$-representations. This gives that the Hecke eigenvalues on $\mathrm{Hom}_K(V,\pi|_K)$ are given by $\chi'(T_1)=\chi_2(p)^{-1}$ and $\chi'(T_2)=\chi_1(p)^{-1}\chi_2(p)^{-1}$.

- The condition $\chi'(T_1)=0$ distinguishes the supersingular family.
- The irreducible principal series representations are distinguished by the condition $1<\dim V<p$ or $\chi'(T_1^2-T_2)\neq0$.
- Finally, the characters of $G$ are determined by $\dim V=1$ and $\chi'(T_1^2-T_2)=0$,
- while twists of the Steinberg are determined by $\dim V=p$ and $\chi'(T_1^2-T_2)=0$. □

**11.10. Definition.** An irreducible admissible $G$-representation is called **supercuspidal** if it is not a subquotient of a principal series representation.

**11.11. Corollary. —** *If $\pi$ is an irreducible admissible representation of $G$ then $\pi$ is supercuspidal if and only if $\pi$ is supersingular.*

*Proof.* Immediate from Theorem 11.3. □

We end this chapter by stating a theorem which characterises supersingular representations.

**11.12. Theorem (Breuil). —** *The irreducible supersingular representations of $G$ are exactly*

$$\frac{\text{c-Ind}_{K\mathbb{Q}_p^\times}^G \operatorname{Sym}^r \overline{\mathbb{F}}_p^2}{(T_1)} \otimes (\eta \circ \det),$$

*where $\eta\colon \mathbb{Q}_p^\times \to \overline{\mathbb{F}}_p^\times$ is a character.*

*Proof.* See [**Herz+**, Section 10] or [**Em08**]. □

# Chapter 3

# Mod $p$ local Galois representations

In this chapter, we classify all irreducible mod $p$ representations of the local Galois group $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over $k$.

## 1. Structure theory of local field extensions

For this section, fix a local field $(K, |\cdot|)$, i.e., a complete non-Archimedean field with a discrete valuation $|\cdot|$. Let $R$ be its valuation ring and $k$ the residue field. We assume $k$ to be perfect and of positive characteristic $p$. We also fix an algebraic closure $\overline{K}$ of $K$. For basic definitions of ramification theory, see [**Ser80**, Chapter 1].

**1.1. Definition.** A class of field extensions $\mathscr{C} = \{L/K\}$ is said to be **distinguished** if it satisfies the following two conditions:

- (transitive in towers) If $K/F$ and $L/K$ are in $\mathscr{C}$ then $L/F$ is in $\mathscr{C}$.
- (base change) Suppose $E, F, K$ are subfields of a common field, and $F \subset K, F \subset ()E$, and $K/F \in \mathscr{C}$. Then $EK/E \in \mathscr{C}$.
- (Redundant) Suppose $K, L_1, L_2$ are subfields of a common field, with $K \subset L_1 \cap L_2$ and that $L_1/K, L_2/K \in \mathscr{C}$. Then $L_1L_2/K \in \mathscr{C}$.

Well-known examples of distinguished classes include finite extensions, separable extensions, and purely inseparable extensions. Both unramified and tamely ramified extensions are distinguished classes of field extensions in the sense of [**Lang**]–

**1.2. Theorem. —**

- *The class of unramified extensions is a distinguished class.*
- *The class of tamely ramified extensions is a distinguished class.*

Then a formal consequence is that there exists a unique maximal unramified extension, denoted $K^{\mathrm{unr}}$, and a maximal tamely ramified extension, denoted $K^{\mathrm{tame}}$. The residue field of $K^{\mathrm{unr}}$ is $\overline{k}$. This is because given any finite extension $\ell/k$, there exists an unramified extension $L/K$ with residue field extension $\ell/k$. This comes from the theory of Witt vectors. The extension $K^{\mathrm{sep}}/K^{\mathrm{unr}}$ is Galois and totally ramified. Also, there is a natural isomorphism $\mathrm{Gal}(K^{\mathrm{unr}}, K) \to \mathrm{Gal}(\overline{k}/k)$. We have the usual short exact sequence of Galois groups

$$1 \to \mathrm{Gal}(K^{\mathrm{tame}}/K^{\mathrm{unr}}) \to \mathrm{Gal}(K^{\mathrm{tame}}/K) \to \mathrm{Gal}(K^{\mathrm{unr}}/K) \to 1.$$

An extension is called **wildly ramified** if it is not tamely ramified. Informally, $K^{\mathrm{sep}}/K^{\mathrm{tame}}$ is the wildly ramified "part" of $K^{\mathrm{sep}}/K$.

**1.3. Theorem (maximal tamely ramified extension). —** *For each positive integer $e$ not divisible by $p$, there exists a unique degree $e$ tamely ramified extension $L_e/K^{\mathrm{unr}}$, obtained by adjoining the eth root of any uniformer of $K^{\mathrm{unr}}$. Moreoever, $K^{\mathrm{tame}} = \bigcup_e L_e$ and $\mathrm{Gal}(K^{\mathrm{tame}}/K^{\mathrm{unr}}) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$.*

*Proof.* We may relabel $K^{\mathrm{unr}}$ by $K$. The residue field of $K$ is algebraically closed, so $K$ contains all roots of unity of order prime to $p$ by the theory of Teichmüller lifts. Also, all extensions of $K$ are totally ramified because no residue field extension is possible. By Theorem 1.4, we are almost done, except that we now need to show $K[\pi^{1/e}] = K[\pi'^{1/e}]$ holds for two uniformizers $\pi, \pi' \in K$. This is immediate because $\pi/\pi' \in \mathcal{O}_K^\times$ is an $e$th power. Indeed, this comes from a quick application of Hensel's lemma and the fact that $k$ is algebraically closed. Now, let $L_e = K[\pi^{1/e}]$ be the unique degree $e$ tamely ramified extension of $K$. There is an isomorphism $\mathrm{Gal}(L_e/K) \cong \mathbb{Z}/e\mathbb{Z}$ functorial in $L_e$ and $e$. Taking limits, we get $\mathrm{Gal}(K^{\mathrm{tame}}/K) = \lim \mathbb{Z}/e\mathbb{Z} = \prod_{\ell \neq p} \mathbb{Z}_\ell$. □

**1.4. Theorem (characterisation of totally tamely ramified extensions). —** *Let $L/K$ be totally tamely ramified with $[L\colon K] = e$. Then there exists a uniformizer $\pi$ of $K$ and a uniformizer $\Pi$ of $L$ such that $\Pi^e = \pi$. That is, $L = K[\pi^{1/e}]$. Conversely, if $p \nmid e$ then $K[\pi^{1/e}]/K$ is a totally tamely ramified extension of degree $e$.*

*Proof.* See [**Lang**, pp. 52-53]. □

**1.5. Theorem. —** *The wild ramification group $\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{tame}})$ is pro-$p$.*

*Proof.* A finite quotient of $\mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{tame}})$ corresponds to a finite Galois extension of $K^{\mathrm{tame}}$. Such an extension is purely wildly ramified and hence must be of $p$-power degree. □

By techniques similar to above, one can prove

**1.6. Theorem (maximal unramified extension). —** *Suppose $k \cong \mathbb{F}_{p^f}$. There exists a unique unramified extension $L_e/K$ of degree $e$ which is obtained by adjoining to $K$ all roots of $X^{p^{fe}} - X$. Moreover, $K^{\mathrm{unr}} = \bigcup_e L_e$ and $\mathrm{Gal}(K^{\mathrm{unr}}/K) \cong \widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$, which is topologically cyclically generated[1] by the Frobenius $x \mapsto x^{p^f}$ of $\overline{k}/k$.*

[1] i.e., $\widehat{\mathbb{Z}}$ has a dense cyclic subgroup generated by...

Assuming the residue field of $K$ to be finite, we summarise the main facts in a diagram:

$$\begin{array}{c} K^{\text{sep}} \\ \Big|\ \text{pro-}p \\ K^{\text{tame}} \\ \Big|\ \prod_{\ell\neq p}\mathbb{Z}_\ell \\ K^{\text{unr}} \\ \Big|\ \widehat{\mathbb{Z}} \\ K \end{array}$$

## 2. Serre's fundamental characters

We return to working with $\mathbb{Q}_p$. Let us setup some notations–

- $\sigma$ is the Frobenius of $\overline{\mathbb{F}}_p/\mathbb{F}_p$.
- The **absolute inertia group** $I_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\text{unr}})$. Of course, $G_{\mathbb{Q}_p}/I_{\mathbb{Q}_p} = \mathrm{Gal}(\mathbb{Q}_p^{\text{unr}}/\mathbb{Q}_p)$, which is topologically generated by $\sigma$.
- $I_t = \mathrm{Gal}(\mathbb{Q}_p^{\text{tame}}/\mathbb{Q}_p^{\text{unr}})$ and $I_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\text{tame}})$.

**2.1. Definition.** Let $X$ be set with an action of $G_{\mathbb{Q}_p}$. We call the action **unramified** if $I_{\mathbb{Q}_p}$ acts trivially.

As a special case we can define what it means for a Galois representation to be unramified. If $\eta\colon G_{\mathbb{Q}_p} \to k^\times$ is an unramified character then it is fully determined by the choice of $\eta(\sigma)$. For $\lambda \in k^\times$, we denote by $\mu_\lambda$ the unique unramified character given by $\sigma \mapsto \lambda^{-1}$. For each $n \geqslant 1$, fix a $\pi_n \in \overline{\mathbb{Q}}_p$ such that $\pi_n^{p^n-1} = -p$. Then $\mathbb{Q}_p^{\text{tame}} = \bigcup_{n\geqslant 1}\mathbb{Q}_p^{\text{unr}}(\pi_n)$. To see this, use Theorem 1.3 and Fermat's little theorem. Furthermore, if $g \in G_{\mathbb{Q}_p}$ then $g(\pi_n)/\pi_n$ is a $p^n-1$-th root of unity, so there exists a character, independent of the choice of $\pi_n$, $\omega_n\colon G_{\mathbb{Q}_{p^n}} \to \mathbb{F}_{p^n}^\times$ such that $g(\pi_n) = \omega_n(g)\pi_n$. Here, $\mathbb{Q}_{p^n}$ denotes the unique unramified extension extension of $\mathbb{Q}_p$ of degree $n$. It follows that $(\zeta_p-1)^{p-1}/p \equiv -1 \pmod{\zeta_p - 1}$ from the following calculation

$$\frac{1}{p}(\zeta_p-1)^{p-1} = \frac{(\zeta_p-1)^{p-1}}{(\zeta_p-1)(\zeta_p^2-1)\cdots(\zeta_p^{p-1}-1)} = \prod_{n=1}^{p-2}\frac{1}{1+\zeta_p+\cdots+\zeta_p^n} \equiv \frac{1}{(p-1)!} \overset{\text{Wilson's}}{\equiv} -1 \pmod{\zeta_p-1}.$$

Thus, $\omega_1$ is the familiar mod $p$ cyclotomic character which we denote simply by $\omega$.

Of course $\omega_n$ is trivial on the wild ramification "part" $I_p$. Therefore, $\omega_n$ is in fact a character of $I_t$. If $d \mid n$ it's clear that $\omega_n^{(p^n-1)/(p^d-1)} = \omega_d$. For a given $n$ and $h \in \mathbb{Z}$, we say $h$ is **primitive** if it is not divisible by $(p^n-1)/(p^d-1)$ for any $d < n$. Thus, we conclude that every mod $p$ character of $I_t$ is of the form $\omega_n^h$ for some well-defined $n$ and primitive $h$. Such a character said to be of **level** $n$.

**2.2. Classification.** Let $g \in G_{\mathbb{Q}_p}$. Observe that $\tau \mapsto \omega_n^h(g\tau g^{-1})$ is a character of $G_{\mathbb{Q}_{p^n}}$ which is of the form $\omega_n^{p^j h}$. If $h$ is primitive then the characters $\omega_n^h, \omega_n^{ph}, \dots, \omega_n^{p^{n-1}h}$ are pairwise distinct. Therefore, by Mackey's irreducibility criterion **??**, $\mathrm{Ind}_{G_{\mathbb{Q}_{p^n}}}^{G_{\mathbb{Q}_p}} \omega_n^h$ is irreducible. Note that these inductions are finite-dimensional, and since $G_{\mathbb{Q}_{p^n}}$ has finite index in $G_{\mathbb{Q}_p}$, compact and smooth inductions are equal.

**2.3. Lemma. —** *We can twist* $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2$ *by an unramified character so that its determinant is* $\omega$. *We denote the resulting representation by* $\mathrm{Ind}\,\omega_2$.

*Proof.* It is easy to see that an unramified character $\chi\colon G_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^\times$ is actually a character on $G_{\mathbb{Q}_p}/I_{\text{abs}} = \mathrm{Gal}(\mathbb{Q}_p^{\text{unr}}/\mathbb{Q}_p) \cong \widehat{\mathbb{Z}}$. So $\chi$, being continuous, is determined by its value at the Frobenius $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^{\text{unr}}/\mathbb{Q}_p)$ since $\mathbb{Z} \subset \widehat{\mathbb{Z}}$ is a dense cyclic subgroup. We have

$$\det(\chi \otimes \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2) = \chi^2 \det(\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2).$$

It suffices to show that the determinant of $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2$ on $I_{\text{abs}} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\text{unr}})$ is $\omega$. Indeed, then it's clear from the above formula that we can choose $\chi$ to be an unramified character so $\det(\chi \otimes \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2) = \omega$. So we are done by the following lemma. □

**2.4. Lemma. —** *Determinant of* $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2$ *on* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\text{unr}})$ *is* $\omega$.

*Proof.* Because of how $\omega_2$ is defined, $\omega_2$ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\text{tame}})$ is trivial. So $\omega_2|_{I_{\text{abs}}}$ is a character on $\mathrm{Gal}(\mathbb{Q}_p^{\text{tame}}/\mathbb{Q}_p^{\text{unr}}) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$, where $\tilde{\sigma}\tau\tilde{\sigma}^{-1} = \tau^p$ for $\tau \in \mathrm{Gal}(\mathbb{Q}_p^{\text{tame}}/\mathbb{Q}_p^{\text{unr}})$ and $\tilde{\sigma} \in \mathrm{Gal}(\mathbb{Q}_p^{\text{tame}}/\mathbb{Q}_p)$ is a lift of The Frobenius $\sigma$. Choose a lift $\overline{\sigma} \in G_{\mathbb{Q}_p}$ of $\sigma$. Then $G_{\mathbb{Q}_p} = G_{\mathbb{Q}_{p^2}} \sqcup G_{\mathbb{Q}_{p^2}}\overline{\sigma}$. By explicit description of induction, it can be seen that, for $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\text{unr}})$,

$$\det(\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2)(g) = \det(\omega_2(g))\det(\omega_2(\overline{\sigma}g\overline{\sigma}^{-1})) = \omega_2(g)\omega_2(g^p) = \omega_2^{p+1}(g) = \omega(g).$$

The lemma is proved. □

It is easy to see that one can prove Lemma 2.3 in a more general setting where $\mathbb{Q}_{p^2}$ is replaced by $\mathbb{Q}_{p^n}$.

**2.5. Theorem (irreducible mod $p$ representations of $G_{\mathbb{Q}_p}$). —** *If $W$ is an $n$-dimensional irreducible representation of $G_{\mathbb{Q}_p}$ over $k$ then there exists $\lambda \in \overline{\mathbb{F}}_p^\times$ such that*

$$W \cong \mathrm{Ind}\,\omega_n^h \otimes \mu_\lambda$$

*for primitive* $1 \leqslant h \leqslant p^n - 2$.

*Proof.* Since $I_p$ is pro-$p$ and $W$ is irreducible, $W = W^{I_p}$. Hence, $W|_{I_{\mathbb{Q}_p}}$ is a representation of $I_t$. Since $I_t$ is abelian of pro-order relatively prime to $p$, it follows that $W|_{I_t}$ is a direct sum of characters. Let $\chi$ be such a character. Then it of some level $m$ so that it extends to $G_{\mathbb{Q}_{p^m}}$. Consequently, $W|_{G_{\mathbb{Q}_{p^m}}}$ contains $\chi$ and so by Frobenius reciprocity $W$ contains $\mathrm{Ind}_{G_{\mathbb{Q}_{p^m}}}^{G_{\mathbb{Q}_p}} \chi$. Since $W$ is irreducible, it follows that $n = m$ and $\chi = \omega_n^h$ times some unramified character. □

## 3. The mod $p$ local Langlands correspondence

We follow [**Br10**].

**3.1. Theorem (Breuil). —** *There exists an explicit bijection*

$$\left\{\begin{array}{c} \textit{irreducible (admissible)} \\ \textit{supersingular representations of} \\ \mathrm{GL}_2(\mathbb{Q}_p) \textit{ over } \overline{\mathbb{F}}_p \textit{ upto isomorphism.} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} \textit{irreducible continuous} \\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p) \\ \textit{upto isomorphism.} \end{array}\right\}$$

*Sketch.* The bijection is the following:

$$\frac{\text{c-Ind}_{K\mathbb{Q}_p^\times}^{G} \mathrm{Sym}^r \overline{\mathbb{F}}_p^2}{(T_1)} \otimes (\eta \circ \det) \longleftrightarrow \mathrm{Ind}\, \omega_2^{r+1} \otimes \eta,$$

where on the RHS, $\eta$ acts on $\overline{\mathbb{F}}_p$ via local class field theory. □

To the representation $\pi$ which is obtained as the extension

$$0 \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2 \varepsilon^{-1} \to \pi \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_2 \otimes \chi_1 \varepsilon^{-1} \to 0$$

we associate the Galois representation $\overline{\rho}$ which is obtained as an extension $0 \to \chi_1 \to \overline{\rho} \to \chi_2 \to 0$. Here $\varepsilon$ is the reduction mod $p$ cyclotomic character, and $\chi_1$ and $\chi_2$ are characters $\mathbb{Q}_p^\times \to \mathbb{F}^\times$ which are not equal to each other nor to the product of the other by the $p$-adic cyclotomic character or its inverse.

For more general 2-dimensional reducible representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, the corresponding representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ are a bit more subtle to define and we refer the reader to [**Em10**] or [**Col**, §VII]. The above correspondence can be realized using the theory of $(\varphi, \Gamma)$-modules, which makes it much more natural.

# Chapter 4

# Fontaine's $(\varphi, \Gamma)$-modules

### 1. $p$-adic complex numbers

We first prove two basic lemmas.

**1.1. Lemma. —** *If $F$ is a complete valued field, then the completion of the algebraic closure $\overline{F}$ is algebraically closed.*

*Proof.* Denote the completion of the algebraic closure by $E$. Let $P \in E[X]$ be a polynomial of minimal degree $\geqslant 1$ which doesn't have a zero. By a suitable scalar change of variables, we may assume that $P \in \mathcal{O}_E[X]$ and $P$ is monic. Choose a sequence $P_n \in \overline{F}[X]$ such that $\lim_{n\to\infty} P_n = P$. $\square$

**1.2. Ax's theorem. —** *If $F$ is a valued field of characteristic $p$, then $F^{\text{sep}}$ is dense in $\overline{F}$.*

*Proof.* If $y \in \overline{F}$ then there exists $n \geqslant 1$ with $\alpha := y^{p^n} \in F^{\text{sep}}$ by field theory. Pick any $\sigma$ with a positive valuation. Let $y_i$ be a zero of the separable polynomial $X^{p^n} - \sigma^i X - \alpha$. It is now clear that $y_i \to y$. $\square$

We recall that a ring $R$ of characteristic $p > 0$ is called **perfect** if the ring endomorphism $x \mapsto x^p$ is bijective. Define $\mathbb{C}_p$ as the $p$-adic completion of $\overline{\mathbb{Q}}_p$. The above results tell us that $\mathbb{C}_p$ is algebraically closed.

$$\begin{tikzcd}
 & \overline{\mathbb{Q}}_p \arrow[r] & \mathbb{C}_p \\
\mathbb{Q} \arrow[r] & \mathbb{Q}_p \arrow[u] &
\end{tikzcd}$$

FIGURE 1.1. Horizontal and verticle arrows denote completions and algebraic closure, respectively.

## 2. Period rings in characteristic $p$

It turns out the $\mathscr{O}_{\mathbb{C}_p}$ is not a ***p*-ring**, i.e., $\mathscr{O}_{\mathbb{C}_p}/p$ is not perfect. Define

$$\mathscr{O}_{\mathbb{C}_p}^{\flat} := \lim_{x\mapsto x^p} \mathscr{O}_{\mathbb{C}_p}/p = \{(x^{(0)}, x^{(1)}, \dots) : x^{(i)} \in \mathscr{O}_{\mathbb{C}_p}/p,\ (x^{(i+1)})^p = x^{(i)}\},$$

that is, the limit of the inverse system

$$\cdots \longrightarrow \mathscr{O}_{\mathbb{C}_p}/p \xrightarrow{x\mapsto x^p} \mathscr{O}_{\mathbb{C}_p}/p \xrightarrow{x\mapsto x^p} \mathscr{O}_{\mathbb{C}_p}/p.$$

Also define

$$\widetilde{\mathbf{E}}^+ := \lim_{x\mapsto x^p} \mathscr{O}_{\mathbb{C}_p} = \{(x_0, x_1, \dots) : x_i \in \mathscr{O}_{\mathbb{C}_p},\ x_{i+1}^p = x_i\}$$

where the ring operations are defined by setting

$$(xy)_i = x_i y_i, \qquad (x+y)^i = \lim_{k\to\infty} (x_{i+k} + y_{i+k})^{p^k}.$$

It is easily verified that $\mathscr{O}_{\mathbb{C}_p}^{\flat} \cong \widetilde{\mathbf{E}}^+$ through the natural map $\widetilde{\mathbf{E}}^+ \to \mathscr{O}_{\mathbb{C}_p}^{\flat}$ given by

$$(x_0, x_1, \dots) \mapsto (x_0 \pmod p), x_1 \pmod p), \dots).$$

From now on, we work with $\widetilde{\mathbf{E}}^+$. Define a valuation $\mathrm{val}_{\mathbf{E}} \colon \widetilde{\mathbf{E}}^+ \setminus \{0\} \to \mathbb{R}_{\geqslant 0}$ as $\mathrm{val}_{\mathbf{E}}\, x := \mathrm{val}_p\, x_0$. We introduce the following notation.

- $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in \widetilde{\mathbf{E}}^+$ where $\zeta_{p^n}$ is a primitive $p^n$th root of unity.
- $X := \varepsilon - 1$.
- $\widetilde{\mathbf{E}} := \widetilde{\mathbf{E}}^+[1/X]$.
- $\mathbf{E} := \mathbb{F}_p((X))^{\mathrm{sep}} \subset \widetilde{\mathbf{E}}$ (we will see later that $\widetilde{\mathbf{E}}$ is an algebraically closed field).
- The Frobenius map $\varphi \colon x \mapsto x^p$ on $\widetilde{\mathbf{E}}^+$.
- $\widetilde{\mathbf{E}}^{\varphi^f=1} := \{0\} \cup (\varphi^f)^{-1}(1)$.
- Define $\theta_n \colon \widetilde{\mathbf{E}}^+ \to \mathscr{O}_{\mathbb{C}_p}/p$ as the composite $\widetilde{\mathbf{E}}^+ \xrightarrow{\sim} \mathscr{O}_{\mathbb{C}_p}^{\flat} \xrightarrow{\pi_n} \mathscr{O}_{\mathbb{C}_p}/p$ where $\pi_n$ denotes "projection onto the $n$th coordinate".

We record some basic properties in the following proposition.

**2.1. Proposition. —**

*(1)* $\mathrm{val}_{\mathbf{E}}\, X = p/(p-1)$.
*(2)* $\widetilde{\mathbf{E}}^+$ *is perfect ring of characteristic $p$.*
*(3)* $\widetilde{\mathbf{E}}$ *is a field containing* $:= \mathbb{F}_p((X))$.
*(4)* $\widetilde{\mathbf{E}}^{\varphi^f=1} = \mathbb{F}_{p^f}$ *where $\overline{\mathbb{F}}_p$ sits inside $\widetilde{\mathbf{E}}^+$ via* $\alpha \mapsto (\alpha^{1/p^n})_{n\geqslant 0}$.

*Proof.*

(1) By definition, $\mathrm{val}_{\mathbf{E}}\, X = \lim_{n\to\infty} (\zeta_{p^n} - 1)^{p^n}$. Valuation is Galois-invariant. Therefore, $\mathrm{val}_p(\zeta_{p^n} - 1) = \frac{1}{\phi(p^n)} \mathrm{val}_p\, \Psi_{p^n}(1) = 1/p^{n-1}(p-1)$, where $\Psi_{p^n}$ denotes the $p^n$th cyclotomic polynomial. The result follows.
(2) Obvious since $\varphi$ acts as shifting coordinates.
(4) Easily verified. $\square$

**2.2. Theorem. —** $\widetilde{\mathbf{E}}$ *is algebraically closed.*

*Proof.* It is enough to show that every monic polynomial $P(T) \in \widetilde{\mathbf{E}}^+[T]$ has a root in $\widetilde{\mathbf{E}}$. Put $d = \deg P$. Denote $P_n(T) = \theta_n(P) \in (\mathscr{O}_{\mathbb{C}_p}/p)[T]$ and choose monic lifts $\widetilde{P}_n(T) \in \mathscr{O}_{\mathbb{C}_p}[T]$ respectively. Since $\mathbb{C}_p$ is algebraically closed, the polynomial $\widetilde{P}_n$ has zeroes $\alpha_{1,n},\dots,\alpha_{d,n}$. If $k \geqslant 1$, define $S_{n,k} \subset \mathscr{O}_{\mathbb{C}_p}/p = \{\alpha_{i,n+k}^{p^k} \pmod p\}$. We claim that if $p^k \geqslant d$ then $S_{n,k}$ does not depend on the choice of $\widetilde{P}_n(T)$. Indeed, if $\alpha \in \mathscr{O}_{\mathbb{C}_p}$ is such that $\widetilde{P}_{n+k}(\alpha) \in p\mathscr{O}_{\mathbb{C}_p}$ then $\prod_{i=1}^{(} \alpha - \alpha_{i,n+k}) \in p\mathscr{O}_{\mathbb{C}_p}$ so that there is some $i$ such that $\mathrm{val}_p(\alpha - \alpha_{i,n+k} \geqslant 1/d$ and a root $\alpha = \alpha'_{i,n+k}$ of another lift $\widetilde{P}'_{n+k}(T)$ of $P_{n+k}(T)$ so that $S'_{n,k} \subset S_{n,k}$ and we have equality by symmetry. Likewise, we have $S_{n,k+1} \subset S_{n,k}$ and since $S_{n+1,k}^p = S_{n,k+1}$ this tells us that the sets $\{S_{n,k}\}_{n\geqslant 0}$ form a compatible system of nonempty sets of cardinal at most $d$ so that their inverse limit is nonempty. Since $P_n(\alpha_{i,n+k}^{p^k}) = P_{n+k}(\alpha_{i,n+k})^{p^k}$ in $\mathscr{O}_{\mathbb{C}_p}/p$, an element of that inverse limit is a root of $P(T)$ which completes the proof. □

**2.3. Theorem. —** $\mathbf{E}$ *is dense in* $\widetilde{\mathbf{E}}$.

*Proof.* See [**Ber10**]. □

## 3. Galois action on $\widetilde{\mathbf{E}}$

The absolute Galois group action $G_{\mathbb{Q}_p}$ on $\overline{\mathbb{Q}}_p$ extends to a continuous action on $\mathbb{C}_p$. This in turn extends to a continuous $G_{\mathbb{Q}_p}$-action on $\widetilde{\mathbf{E}}$. We know that $G_{\mathbb{Q}_p}$ acts on $\mathbb{Q}_p(\zeta_{p^\infty}) := \mathbb{Q}_p(\zeta_p,\zeta_{p^2},\dots)$ via the cyclotomic character $\chi_{\mathrm{cycl}}\colon G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$. Therefore, for $g \in G_{\mathbb{Q}_p}$, we have

$$g \cdot X = (1+X)^{\chi_{\mathrm{cycl}}(g)} - 1.$$

Also define $\mathscr{H}_{\mathbb{Q}_p} := \ker \chi_{\mathrm{cycl}} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_{p^\infty}))$. Then the action of $\mathscr{H}_{\mathbb{Q}_p}$ on $\mathbf{E} = \mathbb{F}_p((X))$ is trivial, and furthermore, if $h \in \mathscr{H}_{\mathbb{Q}_p}$ and $K$ is a separable extension of $\mathbb{F}_p((X))$ then $h(K)$ is another separable extension of $\mathbb{F}_p((X))$. Therefore, we have a map $\mathscr{H}_{\mathbb{Q}_p} \to G_{\mathbb{F}_p((X))} := \mathrm{Gal}(\mathbf{E}/\mathbb{F}_p((X)))$.

**3.1. Proposition. —** *The map* $\mathscr{H}_{\mathbb{Q}_p} \to G_{\mathbb{F}_p((X))}$ *is an isomorphism.*

*Proof.* If $h \in \mathscr{H}_{\mathbb{Q}_p}$ acts trivially on $\mathbf{E}$ then by Theorem 2.3, $h$ acts trivially on $\widetilde{\mathbf{E}}$, and therefore on $\mathscr{O}_{\mathbb{C}_p}$, so $h = \mathrm{id}$. On the other hand, if $\alpha$ is an automorphism of $\mathbf{E}$ then it extends by continuity to an automorphism of $\widetilde{\mathbf{E}}$. This automorphism must be trivial on $\mathbb{F}_p((X))^{\mathrm{perf}}$. We know that finite extensions of $\mathbb{F}_p((X))^{\mathrm{perf}}$ are of the form $\widetilde{\mathbf{E}}_K$ where $K/\mathbb{Q}_p$ is a finite extension. However, $\widetilde{\mathbf{E}}_K$ only depends on $K_\infty$ so that $\alpha$ is the image of some element $h \in \mathscr{H}_{\mathbb{Q}_p}$. □

## 4. $(\varphi,\Gamma)$-modules

If $W$ is an $\mathbb{F}_p$-representation of $G_{\mathbb{Q}_p}$ then the $\mathbb{F}_p((X))$-vector space of diagonal $\mathscr{H}_{\mathbb{Q}_p}$-action invariants

$$D(W) := (\mathbf{E} \otimes_{\mathbb{F}_p} W)^{\mathscr{H}_{\mathbb{Q}_p}}$$

inherits the Frobenius $\varphi$ of $\mathbf{E}$ and an action of $\Gamma := G_{\mathbb{Q}_p}/\mathscr{H}_{\mathbb{Q}_p}$.

**4.1. Definition.** A $(\varphi,\Gamma)$**-module** over $\mathbb{F}_p((X))$ is a finite dimensional $\mathbb{F}_p((X))$-vector space endowed with a semilinear Frobenius $\varphi$ such that the (formal) matrix of $\varphi$ with respect to an arbitrary basis is invertible, and there is a continuous semilinear action of $\Gamma$ which commutes with $\varphi$.

If $E/\mathbb{F}_p$ is an extension then we endow it with the trivial $\varphi$ and the trivial action of $\Gamma$ so that we may talk about $(\varphi,\Gamma)$-modules over $E((X)) := E \otimes_{\mathbb{F}_p} \mathbb{F}_p((X))$.

**4.2. Fontaine's equivalence of categories. —** *The functor $W \mapsto D(W)$ gives an equivalence of categories between the category of $k$-representations of $G_{\mathbb{Q}_p}$ and the category of $(\varphi,\Gamma)$-modules over $k((X))$.*

*Proof.* Without loss of any generality, assume $k = \mathbb{F}_p$. By Hilbert's theorem 90, $H^1(\mathscr{H}_{\mathbb{Q}_p}, \mathrm{GL}_d(\mathbf{E}))$ is the trivial group, where we only consider cocycles which are trivial on an open subgroup of $\mathscr{H}_{\mathbb{Q}_p}$. So, if $W$ is an $\mathbb{F}_p$-representation of $\mathscr{H}_{\mathbb{Q}_p}$ then $\mathbf{E} \otimes_{\mathbb{F}_p} W \cong \mathbf{E}^{\dim W}$ as a representation of $\mathscr{H}_{\mathbb{Q}_p}$. It follows that $\dim_{\mathbb{F}_p((X))} D(W) = \dim W$. In particular, $D(W)$ is a $(\varphi,\Gamma)$-module and one can recover $W$ using the formula $W = (\mathbf{E} \otimes_{\mathbb{F}_p((X))} D(W))^{\varphi=1}$. On the other hand, if $D$ is a $(\varphi,\Gamma)$-module of dimension $d$ over $\mathbb{F}_P((X))$, then we set

$$W(D) := (\mathbf{E} \otimes_{\mathbb{F}_p((X))} D(W))^{\varphi=1}.$$

Choose a basis $\{d_i\}_{1\leqslant i\leqslant d}$ of $D$ and let $\mathrm{Mat}(\varphi)^{-1} = (q_{ij})_{1\leqslant i,j\leqslant d}$ with respect to that basis. It is easily checked that $\sum_{i=1}^d \lambda_i \otimes d_i \in (\mathbf{E} \otimes_{\mathbb{F}_p((X))} D)^{\varphi=1}$ if and only if $\lambda_k^p = \sum_{i=1}^d q_{ki}\lambda_i$ for all $1 \leqslant k \leqslant d$. Observe that $\mathbf{E}[X_1,\ldots,X_d]/(X_k^p - \sum_{i=1}^d q_{ki}X_i)_{1\leqslant k\leqslant d}$ is an étale $\mathbf{E}$-algebra of dimension $p^d$ (this is easily checked using the Jacobian criterion from algebraic geometry). Since $\mathbf{E}$ is separably closed, it is isomorphic to $\mathbf{E}^{p^d}$ from general theory of étale algebras over a field. This identification gives us $p^d$ elements in $W$ so that $W$ is an $\mathbb{F}_p$-vector space of dimension $d$. It is then easy to verify that the functors $W \mapsto D(W)$ and $D \mapsto W(D)$ are inverses of each other. $\square$

## 5. Colmez' functor

We study how one can construct representations of $B$, the upper triangular Borel subgroup, from the data of a $(\varphi,\Gamma)$-module using Colmez' functor.

**5.1. The operator $\psi$.** Since $1, X, \dots, X^{p-1}$ is a basis for $k((X))$ as a $k((X^p))$-vector space, and by nonvanishing of the Vandermonde determinant, so is $1, 1+X, \dots, (1+X)^{p-1}$. If $\alpha \in k((X))$, we can therefore write $\alpha(X) = \sum_{j=0}^{p-1}(1+X)^j \alpha_j(X^p)$ in a unique manner. We define $\psi(\alpha) = \alpha_0(X)$. A $(\varphi,\Gamma)$-module $D$ has a basis whose elements belong to $\varphi(D)$. Indeed, this is equivalent to saying $\mathrm{Mat}(\varphi)$ is invertible for a choice of basis. If $y \in D$ then we can write $y = \sum_{j=0}^{p-1}(1+X)^j\varphi(y_j)$ and we set $\psi(y) = y_0$.

**5.2. Lemma. —** *$\psi$ commutes with $\Gamma$ and satisfies*

$$\psi(\alpha(X)\varphi(y)) = \psi(\alpha(X))y \text{ and } \psi(\alpha(X^p)y) = \alpha(X)\psi(y).$$

*Proof.* Straightforward computation. □

**5.3. Lemma. —** *Every $(\varphi,\Gamma)$-module $D$ admits a $k[[X]]$-lattice stable under $\psi$.*

*Proof.* Let $\mathscr{L}$ be a lattice of $D$. Denote by $\varphi^*(\mathscr{L})$ the $k[[X]]$-module generated by $\varphi(\mathscr{L})$. There exists $h \geqslant 0$ with $X^{h(p-1)}\mathscr{L} \subset \varphi^*(\mathscr{L})$. This gives that $X^{-h}M \subset \varphi^*(X^{-h}\mathscr{L})$ and hence $X^{-h}\mathscr{L}$ is the desired stable lattice. □

If $D$ is a $(\varphi,\Gamma)$-module, let $N$ denote a fixed $k[[X]]$-lattice stable under $\psi$. The inverse limit $\lim_\psi D$ denote the set of sequences $y = (y_n)_{n\geqslant 0}$ such that $\psi(y_{n+1}) = y_n$ for all $n \geqslant 0$. Denote by

$$(\lim_\psi D)^{\mathrm{bdd}} = \{y \in \lim_\psi D \colon y \text{ is bounded in } X\text{-adic topology}\},$$

i.e., there exists $j$, depending on $y$, such that $y_n \in X^{-j}N$ for each $n \geqslant 0$. The set $(\lim_\psi D)^{\mathrm{bdd}}$ is $\psi$-stable and $\psi$ is bijective on it. Of course, $(\lim_\psi D)^{\mathrm{bdd}}$ is $\Gamma$-stable because $\Gamma$ preserves $\psi$-stability of lattices. We define the action of $\alpha \in k[[X]]$ on $y \in (\lim_\psi D)^{\mathrm{bdd}}$ by $(\alpha y)_n = \varphi^n(\alpha)y_n$.

Denote by $D^\sharp$ the set of $y_0$ for all $y \in (\lim_\psi D)^{\mathrm{bdd}}$. This is a $k[[X]]$-module stable under $\psi$ and $\Gamma$. Further, $\psi$ is surjection on $D^\sharp$. Observe that $D^\sharp \subset X^{-1}N$ because $\psi(X^{-j}N) \subset X^{-\lceil j/p \rceil}N$. The natural map $\lim_\psi D^\sharp \to (\lim_\psi D)^{\mathrm{bdd}}$ is an isomorphism, and $D^\sharp$ is the largest bounded $k[[X]]$-module of $D$ which is stable under $\psi$ and $\Gamma$ and on which $\psi$ is surjective.

**5.4.** *Example.* If $D = D(\omega^s\mu_\lambda) = k[[X]]\cdot e$ with $\varphi(e) = \lambda e$ and $\gamma(e) = \omega^s(\gamma)e$, then $D^\sharp = X^{-1}k[[X]]\cdot e$.

**5.5. Representations of $B$.** Each element of $B$ can be written as a product of matrices $\mathrm{diag}(x,x)$ for $x \in \mathbb{Q}_p^\times$, $\mathrm{diag}(1,p^j)$, $j \in \mathbb{Z}$, $\mathrm{diag}(1,u)$ for $u \in \mathbb{Z}_p^\times$, and $\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$ with $z \in \mathbb{Z}_p$. Let $W$ be a representation of $G_{\mathbb{Q}_p}$ and $D(W)$ be the associated $(\varphi,\Gamma)$-module. For any smooth $\mathbb{Q}_p^\times$-character $\chi$, we endow $\lim_\psi D^\sharp(W)$ with a $B$-action as follows:

- $\mathrm{diag}(x,x)\cdot y)_i = \chi^{-1}(x)y_i$
- $\mathrm{diag}(1,p^j)\cdot y)_i = y_{i-j}$
- $\mathrm{diag}(1,a)\cdot y)_i = \gamma_{a^{-1}}(y_i)$, where $\gamma_{a^{-1}} \in \Gamma$ is such that $\chi_{\mathrm{cycl}}(\gamma_{a^{-1}}) = a^{-1} \in \mathbb{Z}_p^\times$.

- $\left(\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \cdot y\right)_i = (1+X)^{p^i z} y_i.$

We then define $\Omega(W) = (\lim_{\psi} D^{\sharp}(W))^{\star}$ to obtain a smooth representation, where $(-)^{\star}$ denotes the continuous dual.

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