Chosen Olympiad Geometry Configurations

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These are my personal notes on a few relatively obscure geometric configurations which I found were worth knowing.

Notations

The circle passing through the points X, Y, Z is denote by (XYZ) and the circle with diameter XY is denote (XY). If T is a point on a circle ω, then the antipode of T is the point on ω diametrically opposite to T. Let ABC be a fixed triangle with orthocenter H, circumcenter O, incenter I, incircle (I), centroid G, $A-$excenter I_A, $M_A = AI \cap (ABC)$, M_{BC} be the antipode of M_A in (ABC).

§1 Foot from $A-$intouch point

Let D, E, F be the intouch points of $\triangle ABC$. Let P be the perpendicular foot on EF from D and $BP \cap AC = Y, CP \cap AB = Z$.

- $BCYZ$ is a bicentric quadrilateral.
- $BC, M_{BC}I, AP$ are concurrent.
- Discover harmonic bundles.
- IMO Shortlist 2002/G8
- Brazil MO 2013/6
- RMM 2012/6

§2 The Feuerbach Point

Let F_e be the Feuerbach point, D be the $A-$intouch point, and M be the midpoint of BC.

- $\triangle AIO \sim \triangle F_e DM$
- $F_e \in (DT)$, where T is the midpoint of AI.

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§3 Special Poncelet's Porisms

Poncelet's porism states that given any point $X \in (ABC)$, we can pick points $Y, Z \in (ABC)$ such that (I) is also the incircle of $\triangle XYZ$.

- Take $X = A'$ where A' is the $A-$antipode.
- Take $X = T_A$ where T_A is the $A-$Mixtilinear touch point.

§4 Six-point circle

Let P be a point with isogonal conjugate P^*. For a point X, the pedal circle of X is the circumcirle of its pedal triangle.

- The pedal circles of P and P^* are the same.
- The center of the common pedal circle is the midpoint of PP^*.

§5 Incircle-circumcircle collinearity

Let A' be the $A-$antipode. Let D, E, F be the $A, B, C-$intouch points, respectively, P be the foot from D to EF.

- $A', I, P, (AEF) \cap (ABC)$ are collinear.

§6 Schwatt line

Let M be the midpoint of $A-$altitude and N be the midpoint of BC. The line MN is called the $A-$Schwatt line.

- The symmedian point K lies on MN.
- MN is the locus of the centers of rectangles inscribed in the triangle.

§7 A line perpendicular to OI

Let the circle centered at B passing through C intersect AB at P and similar define Q.

- $PQ \perp OI$

§8 Isogonal conjugate of the isotomic conjugate of H

Let the isogonal conjugate of the isotomic conjugate of H be X.

- X is the homothety center of the intouch triangle and excentral triangle $I_AI_BI_C$.
- X lies on the Euler line.

§9 Tangential quadrilaterals

Let $ABCD$ be a tangential quadrilateral with incenter I and incircle (I). Let E, F, G, H be the intouch points.

- Define Ex-tangential quadrilaterals by extrapolating. Find which properties carry on.
- AC, BD, EG and FH are concurrent.
- If $ABCD$ is cyclic with circumcenter O then OI is perpendicular to the third diagonal.

§10 Three orthic incircles (Own and Tumon2001)

Let J_A, J_B, J_C be the incenters of $\triangle BHC, \triangle AHC, \triangle AHB$ respectively. Let H' be the orthocenter of $\triangle J_A J_B J_C$. Let T_A be the $H-$intouch point of triangle BHC. Let $K_A = AH' \cap (ABC)$.

- Let the perpendicular to AI_A at I_A meet BC at M. Let N be the point such that $AI_A MN$ is a rectangle. Prove that N lies on the line joining the incenters of $\triangle ABH$ and $\triangle ACH$.
- $I_A J_A \perp J_B J_C$
- K_A, T_A, M_{BC} are collinear.
- The line of collinearity of K_A, T_A, M_{BC} is parallel to DJ_A.
- The radical center of the three incircles of $\triangle AHC, \triangle BHC$ and $\triangle AHB$ is the nine point center of $\triangle J_A J_B J_C$.
- Let I', O' be the incenter and circumcenter of $\triangle J_A J_B J_C$ respectively, then the lines $I'H, OO'$ and $G_e I$ are parallel, where G_e is the Gergonne point of $\triangle ABC$.
- The distance between lines OO' and $I'H$ is same as the distance between lines OO' and $G_e I$.

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