HODGE-TATE DECOMPOSITION FOR ABELIAN VARIETIES WITH GOOD REDUCTION

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Abstract. Let K be a finite extension of $\mathbb{Q}_p$ and A/K an abelian variety with good reduction. We give a brief account of Hodge-Tate decomposition for the p-adic étale cohomology of A following Fontaine [Fon82].

1. Introduction

Fix a finite extension $K/\mathbb{Q}_p$, a completed algebraic closure $\overline{K} \hookrightarrow \mathbb{C}_p$, and Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$. It is well-known that $\mathbb{C}_p$ is algebraically closed. Since G_K acts on $\overline{K}$ by isometries, it follows that the action of G extends to $\mathbb{C}_p$. Given a finite-dimensional representation V of G_K over $\mathbb{Q}_p$, one can consider the diagonal action of G_K on $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. Of course, it is $\mathbb{Q}_p$-linear but not $\mathbb{C}_p$-linear. Such a $\mathbb{C}_p$-vector space is called a *semi-linear* Galois representation.

Theorem 1.1 (Hodge-Tate decomposition)**.** *Let A be an abelian variety over K with good reduction. Then there is a canonical isomorphism of semi-linear Galois representations*

$$\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq (\mathrm{H}^1(A, \mathscr{O}_A) \oplus (\mathrm{H}^0(A, \Omega^1_{A/K}) \otimes_{\mathbb{Z}_p} \chi_{\mathrm{cyc}}^\vee)) \otimes_K \mathbb{C}_p,$$

where $\chi_{\mathrm{cyc}}^\vee$ is the dual of the p-adic cyclotomic character.

Remark 1.2. For an abelian variety, the cohomology ring $\mathrm{H}^\bullet_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ is canonically isomorphic to the exterior algebra over $\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$. Therefore, it suffices to understand the first cohomology group.

2. Preliminaries

2.1. Tate modules.

Definition 2.1.1. Given an abelian group A and a prime p, the **p-adic Tate module** of A is defined as the inverse limit

$$T_p A = \lim_{n} A[p^n],$$

with transition morphisms given by the multiplication-by-p map $A[p^{n+1}] \to A[p^n]$. It is naturally a torsion-free $\mathbb{Z}_p$-module.

Example 2.1.2. The p-adic cyclotomic character $\chi_{\mathrm{cyc}} \colon G_K \to \mathbb{Z}_p^\times$ is defined as $T_p \mathbb{G}_{m,K}(\overline{K}) = T_p \overline{K}^\times$, with the obvious Galois action, together with the choice of a (suitable) element. From now onwards, we denote the p-adic cyclotomic character as $\mathbb{Z}_p(1)$, also known as a **Tate twist**. Naturally, set $\mathbb{Z}_p(-1) = \mathbb{Z}_p(1)^\vee := \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(1), \mathbb{Z}_p)$, $\mathbb{Z}_p(n) := \mathbb{Z}_p(1)^{\otimes n}$ and $\mathbb{C}_p(n) := \mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n)$ for all integers n.

Theorem 2.1.3 ([Tate, Theorems 1-2])**.** *(i)* $\dim_K \mathrm{H}^i(G_K, \mathbb{C}_p) = \begin{cases} 1, & i = 0,1 \\ 0, & i \geqslant 2 \end{cases}$.

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(ii) If n is nonzero, then $\mathrm{H}^i(G_K, \mathbb{C}_p(n)) = 0$ *for all* $i \geqslant 0$.

Corresponding to any abelian variety A, there is a dual abelian variety $A^\vee := \operatorname{Pic}^0 A$. There is a biduality isomorphism $A \xrightarrow{\sim} A^{\vee\vee}$.

Theorem 2.1.4. *Let A be an abelian variety over K. Then* $\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Z}_p) \simeq T_p A^\vee$.

Sketch. Start with the Kummer sequence:

$$0 \to \mu_{p^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 0.$$

Passing to cohomology and using $\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{G}_m) \simeq \operatorname{Pic} A$, one derives $\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mu_{p^n}) \simeq (\operatorname{Pic}^0 A_{\overline{K}})[p^n]$. Taking inverse limits, we get the desired result. □

2.2. **Weil pairing.** It turns out that n-torsion group schemes of A and $A^\vee$ are Cartier duals of each other. There is a Weil (perfect) pairing

$$A[p^n](\overline{K}) \times A^\vee[p^n](\overline{K}) \to \mu_{p^n}(\overline{K})$$

compatible with the Galois action. Passing to Tate modules, one obtains a Galois-equivariant perfect pairing

$$T_p A \times T_p A^\vee \to \mathbb{Z}_p(1).$$

Consequently,

$$\mathrm{H}^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Z}_p) \simeq T_p A^\vee \simeq \operatorname{Hom}_{\mathbb{Z}_p}(T_p A, \mathbb{Z}_p(1)) \simeq (T_p A)^\vee \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1).$$

2.3. ***p*-adic periods.** Consider the $\mathcal{O}_{\overline{K}}$-module of Kähler differentials $\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. Since $\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}\left[\frac{1}{p}\right] = \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \Omega_{\overline{K}/K} = 0$, it follows that $\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ is p-torsion. The following theorem of Fontaine gives a complete understanding of the rational Tate module of $\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$.

Theorem 2.3.1 ([Fon82, §1])**.** *There is a natural isomorphism of G_K-modules*

$$T_p \mathbb{G}_{m,K}(\overline{K}) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \xrightarrow{\sim} T_p \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

induced by $\operatorname{d}\log\colon \mathcal{O}_{\overline{K}}^\times \to \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$, $f \mapsto \frac{\mathrm{d}f}{f}$.

Note that the domain of the above isomorphism is (non-canonically) isomorphic to $\mathbb{C}_p(1)$.

3. **Proof of Hodge-Tate decomposition**

Let $\mathcal{A} \to \operatorname{Spec} \mathcal{O}_K$ be an abelian scheme with generic fiber $A \to \operatorname{Spec} K$. Any point $P \in A(\overline{K})$ extends to $\overline{P} \in \mathcal{A}(\mathcal{O}_{\overline{K}})$ by valuative criterion of properness. Evaluating differentials at $\overline{P}$ induces a morphism $\overline{P}^* \Omega_{\mathcal{A}/\mathcal{O}_K} \to \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. This induces a map

$$A(\overline{K}) \times \mathrm{H}^0(\mathcal{A}, \Omega_{\mathcal{A}/\mathcal{O}_K}) \to \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}.$$

This is actually a Galois-equivariant bilinear pairing due to translation invariance of global differential forms. Passing to Tate modules, we get a canonical map

$$F\colon \mathrm{H}^0(\mathcal{A}, \Omega_{\mathcal{A}/\mathcal{O}_K}) \to \operatorname{Hom}_{\mathbb{Z}_p[G_K]}(T_p A, T_p \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}).$$

It is a routine check that the above morphism is independent of the choice of model $\mathcal{A}$.

Proposition 3.1. *F is injective.*

The above result is proven by passing to formal completions, which we postpone to Section 4. Using Theorem 2.1.4, F can be rewritten as

$$\mathrm{H}^0(\mathcal{A},\Omega_{\mathcal{A}/\mathcal{O}_K}) \hookrightarrow \mathrm{H}^1_{\text{ét}}(A_{\overline{K}},\mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p[G_k]} T_p\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}.$$

Inverting p, using Theorem 2.3.1, and twisting by $\mathbb{C}_p(-1)$, we get an injection

$$\beta_A\colon \mathrm{H}^0(A,\Omega_{A/K}) \otimes_K \mathbb{C}_p(-1) \hookrightarrow \mathrm{H}^1_{\text{ét}}(A_{\overline{K}},\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p.$$

Similarly,

$$\beta_{A^\vee}\colon \mathrm{H}^0(A^\vee,\Omega_{A^\vee/K}) \otimes_K \mathbb{C}_p(-1) \hookrightarrow \mathrm{H}^1_{\text{ét}}(A^\vee_{\overline{K}},\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p.$$

Dualizing, using $\mathrm{H}^1(A,\mathcal{O}_A)^\vee \simeq \mathrm{H}^0(A^\vee,\Omega_{A^\vee/K})$ ([Mum, §13]), and the Weil pairing, we obtain a surjection of Galois modules

$$\alpha_A\colon \mathrm{H}^1_{\text{ét}}(A_{\overline{K}},\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \twoheadrightarrow \mathrm{H}^1(A,\mathcal{O}_A) \otimes_K \mathbb{C}_p.$$

Consider $(\alpha_A\circ\beta_A)\otimes\mathbb{C}_p(1)\colon \mathrm{H}^0(A,\Omega_{A/K})\otimes_K\mathbb{C}_p \to \mathrm{H}^1(A,\mathcal{O}_A)\otimes_K\mathbb{C}_p(1)$. Observe that any $e\otimes 1 \in \mathrm{H}^0(A,\Omega_{A/K})\otimes_K \mathbb{C}_p$ is G_K-invariant. Hence, the image of $e\otimes 1$ lies in $(\mathrm{H}^1(A,\mathcal{O}_A)\otimes_K\mathbb{C}_p(1))^{G_K}$, which is 0 by Theorem 2.1.3. Since elements of the form $e\otimes 1$ generate the domain, it follows that $\alpha_A\circ\beta_A = 0$. Because of dimension reasons, α_A and β_A give an *exact* sequence

$$0 \to \mathrm{H}^0(A,\Omega_{A/K}) \otimes_K \mathbb{C}_p(-1) \to \mathrm{H}^1_{\text{ét}}(A_{\overline{K}},\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \to \mathrm{H}^1(A,\mathcal{O}_A) \otimes_K \mathbb{C}_p \to 0.$$

It is now sufficient to show that $\mathrm{Ext}^1(\mathbb{C}_p,\mathbb{C}_p(-1)) = 0$ in the category of finite-dimensional semi-linear Galois representations over $\mathbb{C}_p$. However, observe that $\mathrm{Hom}(\mathbb{C}_p,-) = (-)^{G_K}$ in the same category. Thus, Theorem 2.1.3 (b) finishes the proof of Theorem 1.1. □

4. **Proof of Proposition 3.1**

We follow [Mon]. Define the 'perfection' $\tilde{A}$ as the inverse limit of the following system

$$\cdots \xrightarrow{\cdot p} A(\overline{K}) \xrightarrow{\cdot p} \cdots \xrightarrow{\cdot p} A(\overline{K}) \xrightarrow{\cdot p} A(\overline{K}) \xrightarrow{\cdot p} A(\overline{K}).$$

Denote by $\tilde{\Omega}$ the perfection of $\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. Similar to the construction of F, we can form a map

$$\tilde{F}\colon \mathrm{H}^0(\mathcal{A},\Omega_{\mathcal{A}/\mathcal{O}_K}) \to \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\tilde{A},\tilde{\Omega}).$$

We claim that there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^0(\mathcal{A},\Omega_{\mathcal{A}/\mathcal{O}_K}) & \xrightarrow{F_{\mathbb{Q}_p}} & \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(T_pA, T_p\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\
& \searrow^{\tilde{F}_{\mathbb{Q}_p}} & \uparrow \\
& & \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\tilde{A},\tilde{\Omega}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p
\end{array}$$

Since $\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ is p-torsion, the natural inclusion $T_p\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \hookrightarrow \tilde{\Omega}$ induces an isomorphism $\tilde{\Omega}[1/p] \simeq T_p\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{C}_p(1)$ by Theorem 2.3.1. We have a canonical exact sequence

$$0 \to T_pA \to \tilde{A} \to A(\overline{K}) \to 0.$$

Apply $\mathrm{Hom}_{\mathbb{Z}_p[G_K]}(-,\tilde{\Omega}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Observe that $\mathrm{Hom}_{\mathbb{Z}_p[G_K]}(A(\overline{K}),\tilde{\Omega}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(A(\overline{K}),\mathbb{C}_p(1)) = 0$. For if $\phi\colon A(\overline{K}) \to \mathbb{C}_p(1)$ is a Galois equivariant morphism, we have $\phi(A(L)) \subseteq \mathrm{H}^0(G_L,\mathbb{C}_p(1)) = 0$ for every finite extension L/K. Let θ be a non-vanishing global differential on $\mathcal{A}$. We need to show that $F(\theta) \neq 0$. If $\theta \in \mathfrak{m}_K\Omega_{\mathcal{A}/\mathcal{O}_K}$, we can divide θ by some power of the uniformizer of K so that θ is non-vanishing modulo $\mathfrak{m}_K$. This is ok to do because $T_p\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ is torsion-free. It suffices to show that $\tilde{F}(\theta)$ is not torsion. Let $\widehat{\mathcal{A}}$ be the formal completion of $\mathcal{A}$ along the unit section. By smoothness, $\widehat{\mathcal{A}}$ is isomorphic to $\mathrm{Spf}\,\mathcal{O}_K[\![x_1,\ldots,x_d]\!]$

where $d = \dim A$. Observe that $\widehat{\mathcal{A}}(\mathcal{O}_{\overline{K}}) \hookrightarrow \mathcal{A}(\mathcal{O}_{\overline{K}})$ and that $\widehat{\mathcal{A}}(\mathcal{O}_{\overline{K}})$ can be identified with $\mathfrak{m}_{\overline{K}} \times \cdots \times \mathfrak{m}_{\overline{K}}$. Expand θ in formal coordinates:

$$\theta = \sum_i F_i(x_1,\ldots,x_d)\mathrm{d}x_i, \qquad F_i \in \mathcal{O}_K[\![x_1,\ldots,x_d]\!].$$

Since we have assumed that the the reduction modulo $\mathfrak{m}_K$ of θ is nonzero, it follows that $F_i(0,\ldots,0)$ must be a unit for some i, say for $i = 1$. Then for any point $P = (x,0,\ldots,0) \in \mathfrak{m}_{\overline{K}} \times \cdots \times \mathfrak{m}_{\overline{K}}$ and any lift $\tilde{P} \in \tilde{A}$, we have $p_0(\tilde{F}(\theta)(\tilde{P})) = F_1(x,0,\ldots,0)\ \mathrm{d}x$ where $p_0 \colon \tilde{\Omega} \to \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ is the canonical projection. Since $F_1(x,0\ldots,0) \in \mathcal{O}_{\overline{K}}^{\times}$ and x can be varied over p-power roots of p, we conclude. □

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