MODULI SCHEMES OF ELLIPTIC CURVES

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Abstract. We give an account of the construction of certain moduli stacks of generalized elliptic curves which are Deligne-Mumford, proper smooth over $\operatorname{Spec}\mathbb{Z}$. Rigidifying level N structures are then imposed to get smooth projective moduli schemes over $\operatorname{Spec}\mathbb{Z}[1/N]$. The main reference is Deligne-Rapoport [DR].

Contents

1. Generalized elliptic curves

Definition 1.1. A **curve** over a scheme S is a morphism $C \to S$ that is separated, flat, and finitely presented with all fibers non-empty of pure dimension 1. A **DR semistable genus-1 curve** over S is a proper curve $f: C \to S$ such that the geometric fibers are connected and semistable with trivial dualizing sheaf.

Definition 1.2. Let $n \geqslant 1$. The **standard n-gon** (or a Néron polygon with n-sides, or a Néron n-gon) over $\operatorname{Spec}\mathbb{Z}$ is the proper $\mathbb{Z}$-curve obtained from $\mathbb{P}^1_{\mathbb{Z}} \times \mathbb{Z}/n\mathbb{Z}$ by gluing the ∞-section of $\mathbb{P}^1_{\mathbb{Z}} \times \{i\}$ to the 0-section of $\mathbb{P}^1_{\mathbb{Z}} \times \{i+1\}$ for each $i \in \mathbb{Z}/n\mathbb{Z}$. The standard n-gon over an arbitrary base S is obtained by extension of scalars.

We note that the standard 1-gon is isomorphic to the nodal cubic. The smooth locus C_n^{sm} of the standard n-gon (over $\mathbb{Z}$), say C_n, has a natural commutative group scheme structure– it is isomorphic to $\mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z}$. The multiplication map $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ extends to a morphism $\mathbb{G}_m \times \mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$. Thus, the addition map $+: C_n^{\mathrm{sm}} \times C_n^{\mathrm{sm}} \to C_n^{\mathrm{sm}}$ extends to $+: C_n^{\mathrm{sm}} \times C_n \to C_n$. The dualizing sheaf of a standard n-gon over an algebraically closed field is the trivial bundle. Indeed, consider the differentials $\mathrm{d}x/x$ on each copy of $\mathbb{P}^1_k$. Note that $\operatorname{div}\ \mathrm{d}x/x = [0] + [\infty]$, $\operatorname{Res}_0\ \mathrm{d}x/x = 1$, and $\operatorname{Res}_\infty\ \mathrm{d}x/x = -1$. Therefore, these glue to give a nowhere-vanishing differential η on C_n. See [DM, §1] for the precise fact used.

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Lemma 1.3. *Let C be a DR semistable genus-1 curve over an algebraically closed field k. Then C is either smooth or a Néron polygon.*

Proof. Let $\pi\colon \widetilde{C} \to C$ be the normalization. Write $\widetilde{C} = \sqcup_{i=1}^d C_i$ for smooth proper curves C_i. We use [Har77, Exercise IV.1.8]. That is, we have the exact sequence

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\widetilde{C}} \to \bigoplus_{j=1}^n k_j \to 0$$

where k_j denotes the skyscraper sheaf for k at the jth node. Taking cohomology,

$$0 \to k \to \bigoplus_{i=1}^d k \to \bigoplus_{j=1}^n k_j \to \mathrm{H}^1(C, \mathcal{O}_C) \to \bigoplus_{i=1}^d \mathrm{H}^1(C_i, \mathcal{O}_{C_i}) \to 0.$$

Taking dimensions, $d = n + \sum_{i=1}^d g(C_i)$. We also know that b is either $a-1$ or a. If $b = a-1$ then $a = 1$ and C is smooth. Otherwise if $b = a$ then $g(C_i) = 0$ for each i. Therefore $C_i \cong \mathbb{P}^1_k$. Let n_i be the number of points on C_i above nodes. Since $\omega_C \cong \mathcal{O}_C$, each $\omega_{C_i}(b_i) \cong \mathcal{O}_{\mathbb{P}^1}(b_i - 2)$ must have nonzero global sections, which implies that $b_i \geqslant 2$. This forces $b_i = 2$ for each i and the proof is complete. □

Definition 1.4. A **generalized elliptic curve** over S is a triple $(E, +, e)$ where E is a DR semistable genus-1 curve over S, $+ : E^{\mathrm{sm}} \times_S E \to E$ is an S-morphism, and $e \in E^{\mathrm{sm}}(S)$ is a section such that

- $+$ restricts to a commutative group scheme structure on E^{sm} with identity section e,
- $+$ is an action of E^{sm} on E such that on singular geometric fibers the translation action by each rational point in the smooth locus induces a rotation on the graph of irreducible components[1] (this forces the component groups of geometric fibers E^{sm}_s to be cyclic).

A morphism in the category of generalized elliptic curves over S is an S-morphism which restricts to a S-group morphism on the smooth loci.

Example 1.5. Let R be a DVR with uniformer π. Then

$$\mathrm{Proj}_R \frac{R[X,Y,Z]}{(Y^2 Z - X(X-Z)(X - \pi Z))} \to \operatorname{Spec} R$$

is a generalized elliptic curve whose special fiber is a 1-gon.

For a curve $f\colon C \to S$, the relative smooth locus $C^{\mathrm{sm},f}$ is exactly the locus where $\Omega^1_{C/S}$ is singly generated. Thus, the first Fitting ideal sheaf (see [Stacks, Tag 0C3C]) of $\Omega^1_{C/S}$ defines a closed subscheme structure on $C^{\mathrm{sing}} = C \setminus C^{\mathrm{sm},f}$.

Lemma 1.6 ([DR, II, 1.5])**.** *Let $f\colon C \to S$ be a proper flat map of finite presentation. The set of $s \in S$ such that C_s is a DR semistable genus-1 curve is open.*

Proof. It is well-known that fibral dimension is locally constant for proper flat maps; see, for example, [Stacks, Tag 0D4J]. Connected fibers are detected by fibral vector space dimension of $f_* \mathcal{O}_C$. Reducedness is detected by the support of $f_* \mathcal{N}_C$ where $\mathcal{N}_C$ denotes the sheaf of nilpotents of C. Openness of genus-1 locus comes from cohomology and base-change theorems. Therefore, we may assume that f is a proper flat family of connected reduced genus-1 curves. Consider the restrict of f to the singular locus $f'\colon C^{\mathrm{sing}} \to S$. This is a finite morphism. Consider the branch divisor of f'. Since f' is proper, we can find an open of S over which $C^{\mathrm{sing}} \to S$ is unramified. I claim that f' is DR semistable over this locus. Since all constructions in sight are compatible with base-change, it is harmless to assume

[1] For a topological space, the graph of irreducible components consists of the set of irreducible components as vertices where two vertices are adjacent if the corresponding irreducible components intersect.

$S = \operatorname{Spec} k$ for some algebraically closed field k. We want to show the following: let C be a projective curve over k and C^{sing} be its singular locus closed subscheme defined by the first Fitting ideal. Then the complement of the nonreduced points of C^{sing} is a nodal curve, which is same as saying that the first Fitting ideal of $\Omega_{C/k,p}$ in $\mathscr{O}_{C,p}$ is equal to the maximal ideal if and only if p is a node. Choose a presentation $k[\![x_0, x_1, \ldots, x_n]\!]/(f_1, \ldots, f_d)$, $d \geqslant n$, of $\hat{\mathscr{O}}_{C,p}$ such that n is minimal. Observe that no f_i can contain nonzero constant or linear terms. For if $f_i = X(c + g_1) + g_2$ where $c \in k$ then we can invert $c + g_1$ to contradict the assumption that n is minimal. It is well-known and easy to show that $\hat{\Omega}_{C,p}$ is isomorphic to the cokernel of the Jacobian matrix of $f_1, f_2, \ldots, f_d$, which is a matrix valued in $\mathfrak{m}_p$. The Fitting ideal is generated by $n \times n$ minors of this matrix. Therefore, for the Fitting ideal to equal $\mathfrak{m}_p$ we must at least have $n = 1$. By routine methods (for e.g., $k(x_0)[\![x_1]\!]$ is a PID, etc) we can take d to be 1. Therefore, we seek to understand when (x_0, x_1) equals $(\partial f_1/\partial x_0, \partial f_1/\partial x_1)$. It's clear that this happens when p is a node, that is, $f_1 = x_0 x_1$. Conversely, when $(x_0, x_1) = (\partial f_1/\partial x_0, \partial f_1/\partial x_1)$, x_0 and x_1 can be written as k-linear combinations of $\partial f_1/\partial x_0$ and $\partial f_1/\partial x_1$ due to degree reasons. Hence, we can make a linear change of variables so that $\partial f_1/\partial x_0 = x_1$ and $\partial f_1/\partial x_1 = x_0$. Then $f_1 = x_0 x_1$ and it follows that p is a node. □

Definition 1.7. For a curve $f\colon C \to S$, the locus of non-smoothness of f is the scheme-theoretic image $S^{\infty,f}$, or simply S^{∞} when f is clear from the context, of C^{sing}.

It is clear that S^{∞} is a closed subscheme and a fiber $C_{\bar{s}}$ for a geometric point $\bar{s} \in S$ is singular if and only if $\bar{s} \in S^{\infty}$. The following technical result is incredibly useful in understanding generalized elliptic curves.

Lemma 1.8 ([DR, II.1.15])**.** *Let $f\colon E \to S$ be a generalized elliptic curve. Then there is a locally finite family $(S_n^{\infty})_{n \geqslant 1}$ of closed subschemes of S such that $S^{\infty} = \bigcup S_n^{\infty}$ such that $E \times_S S_n^{\infty}$ is fppf-locally on S_n^{∞} isomorphic to the standard n-gon over S_n^{∞}.*

Lemma 1.9. *Let $f\colon E \to S$ be a generalized elliptic curve. The formation of S_n^{∞} is compatible with base change on S. In particular, the formation of S^{∞} is compatible with base change.*

Proof. Let $T \to S$ be a morphism and $f_T\colon E_T \to T$ be the base changed f and $E_T \to E$ be g. There is a canonical closed embedding $T^{\infty} \to S^{\infty} \times_S T$. Since $g^{-1} \operatorname{Fitt}_1 \Omega_{E/S} \cdot \mathscr{O}_{E_T} = \operatorname{Fitt}_1 \Omega_{E_T/T}$ [Stacks, Tag 0C3D], it follows that $E_T^{\text{sing}} = E^{\text{sing}} \times_S T$. After taking scheme-theoretic images, it follows that there is a closed embedding $T^{\infty} \hookrightarrow S^{\infty} \times_S T$. We wish to show that this is an isomorphism. One can do this fppf-locally on S^{∞}. By Lemma 1.8, this shows that formation of S^{∞} is compatible with base change on S. We rewrite our isomorphism as $\bigsqcup T_n^{\infty} \xrightarrow{\sim} \bigsqcup S_n^{\infty} \times_S T$. If $p \in T_i^{\infty} \cap (S_j^{\infty} \times_S T)$ for $i \neq j$ then the geometric fiber over p must simultaneously be an i-gon and a j-gon. It follows that $i = j$ and that $T_n^{\infty} = S_n^{\infty} \times_S T$. □

Let $(E, +, e)$ be a generalized elliptic curve over S. Then the morphism

$$n\colon E^{\text{sm}} \to E^{\text{sm}}, \quad x \mapsto nx$$

is fiberwise flat, hence flat (c.f. Theorem B.5). Therefore, the scheme-theoretic kernel $E[n]$ is a closed-subscheme and is S-flat. If E is the standard n-gon over S then $E^{\text{sm}} \cong \mathbb{G}_m \times \mathbb{Z}/m\mathbb{Z}$ and hence $E[n] \cong \mu_n \times \mathbb{Z}/\gcd(n,m)\mathbb{Z}$.

Lemma 1.10 ([DR, II.1.19])**.** *Let $f: X \to S$ be a quasi-finite, flat, and separated morphism, with S noetherian. If the rank of the fibers of f is constant, then f is finite.*

Proof. We first show that f is proper. According to the valuative criterion of properness, we can assume that $S = \operatorname{Spec} R$ for a DVR R. By Zariski's main theorem, X is an open subset of a finite S-scheme $\overline{X}$ that we can even take to be flat over S. Comparing the ranks of the special and generic fibers of X and $\overline{X}$, we find that $X = \overline{X}$. □

Corollary 1.11 ([DR, II.1.20], [KM, Theorem 2.3.1]). *Let $p : E \to S$ be a generalized elliptic curve and n an integer. We assume that, for every geometric point $\overline{s} \in S$, $E_{\overline{s}}$ is smooth, or an m-gon with $n \mid m$. Then, $E[n]$ is locally free of rank n^2 over S*

Lemma 1.12 ([DR, II.1.17]). *Let X be a generalized elliptic curve over S with a unit section e. Let $u : Y \to X$ be a finite étale covering of X with a section $e \in Y(S)$ above e. Suppose that the geometric fibers of Y/S are connected. Then, there exists a unique structure of a generalized elliptic curve on Y with unit e, such that the diagram*

$$\begin{array}{ccc} Y^{\mathrm{reg}} \times_S Y & \xrightarrow{+} & Y \\ \downarrow & & \downarrow u \\ X^{\mathrm{reg}} \times_S Y & \xrightarrow{+} & X \end{array}$$

is commutative.

Let R be a complete DVR with an algebraically closed residue field. Set $S = \operatorname{Spec} R$, η its generic point, and s its closed point. For any finite extension $\kappa(\eta')$ of $\kappa(\eta)$, we denote by (S', η', s') the normalization of S in η'. Let E_η be an elliptic curve over η . We say that E_η has **stable reduction** if the minimal model of E_η over S is DR semistable genus-1.

Theorem 1.13 ([DR, IV.1.6]). *(i) There exists a finite extension $\kappa(\eta')$ of $\kappa(\eta)$ such that the curve $E_{\eta'} = E_\eta \times_\eta \eta'$ has stable reduction over η'.*

(ii) If E_η has stable reduction, the minimal model $\mathscr{E}$ of E_η over S has a unique structure of a generalized elliptic curve extending that of E_η.

(iii) If, furthermore, the n-torsion points of E_η are defined over $\kappa(\eta)$, then either the minimal model $\mathscr{E}$ of E over S is smooth, or $\mathscr{E}_s$ is an m-gon, with $n \mid m$, and there exists a generalized elliptic curve E over S, with a smooth special fiber or an n-gon, and an isomorphism α of its generic fiber with E_η.

(iv) The pair (E, α) is unique up to a unique isomorphism.

If $C \to S$ a stable curve of genus one, with irreducible geometric fibers, and for $e \in C^{\mathrm{sm}}(S)$, there exists on C a unique structure of generalized elliptic curve with unit section u:

Theorem 1.14 ([DR, II.2.7]). *(i) Let $p\colon E \to S$ be a genus-1 curve with geometrically integral fibers and $e \in E^{\mathrm{sm}}(S)$. There exists a unique group law $+\colon E^{\mathrm{sm}} \times E \to E$ such that, on S and after any base change, for $x \in E^{\mathrm{sm}}(S)$ and $y \in E(S)$, one has*

$$m(x+y) \sim m(x) \otimes m(y) \otimes m(e)^\vee,$$

locally over S, where $m(s)$ is the ideal sheaf cutting out $s(S)$ for a section $s\colon S \to E$.

(ii) This law makes E^{sm} a commutative group scheme with unit e acting on E; for this action, E^{sm} acts trivially on $\operatorname{Pic}^0_{E/S}$.

(iii) The inversion $x \mapsto -x$ of E^{sm} extends to E, for $t \in E(S)$, locally on S, we have

$$m(-t) \simeq \mathscr{H}om(m(t), \mathscr{O}_E(-2e))$$

(iv) If E/S is stable, $+$ is the unique group structure on the generalized elliptic curve E with unit e.

2. Deformation theory of generalized elliptic curves

See Appendix A for a review of basic deformation theory. Fix a complete local ring Λ with residue field k. Let E_0 be a generalized elliptic curve over k. We suppose that E_0 is either smooth or the standard n-gon. The deformation functor $\widehat{\operatorname{Def}}_{E_0}$ of $(E_0, +)$ is defined as follows: for $A \in \operatorname{Ob} \hat{\mathrm{C}}_\Lambda$, $\widehat{\operatorname{Def}}_{E_0}(A)$ is the set of isomorphism classes of fibered diagrams

$$\begin{array}{ccc} (E_0,+) & \longrightarrow & (E,+) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} A \end{array}$$

where $E \to \operatorname{Spec} A$ is a generalized elliptic curve and $(E_0,+)$ its special fiber. The infinitesimal deformation functor Def_{E_0} is the restriction of $\widehat{\operatorname{Def}}_{E_0}$ to C_Λ. By taking $F = \operatorname{Def}_{E_0}$ in Appendix A, we can talk about local, infinitesimal, formal, versal, universal deformations, etc.

Theorem 2.1. *Suppose E_0 is smooth or n is relatively prime to $\operatorname{char} k$. Then*

(i) $(E_0,+)$ *has no nontrivial infinitesimal automorphisms, i.e., if* $(E,+) \to \operatorname{Spec} A$ *is an infinitesimal deformation of* $(E_0,+)$ *then any A-automorphism of* $(E,+)$ *is identity.*
(ii) Def_{E_0} *is is represented by a formal generalized elliptic curve over* $\operatorname{Spf}\Lambda[\![t]\!]$.
(iii) Def_{E_0} *is effectively prorepresentable (see Definition A.2).*
(iv) if E_0 is singular, we can choose t so that $t=0$ is the image of E_0^{sing}.

Proof. We first consider the case when E_0 is irreducible, i.e., either an elliptic curve or a nodal cubic. Thanks to Theorem 1.14, we can identify Def_{E_0} with the infinitesimal deformation functor $D = \operatorname{Def}_{(E_0,e)}$ of the pointed curve (E_0,e), where e is the unit section of E_0.

(i) One can use Theorem A.10 but we give a simpler proof. We can write the residue map $A \to k$ as a composite of small thickenings (see Definition A.7). Let $0 \to (t) \to \tilde{A} \to A \to 0$ be a small thickening in C_Λ and $\tilde{E} \to \operatorname{Spec}\tilde{A}$ and $E \to \operatorname{Spec} A$ be deformations compatible with $\tilde{A} \twoheadrightarrow A$. Let θ be an $\tilde{A}$-automorphism of $\tilde{E}$ which restricts to identity over $\operatorname{Spec} A$. Then we have an $\mathscr{O}_{\operatorname{Spec}\tilde{A}}$-algebra automorphism $\theta^*\colon \mathscr{O}_{\tilde{E}} \to \mathscr{O}_{\tilde{E}}$ which is identity modulo t. Therefore, we can write $\theta^*(x) = x + t\,\mathrm{d}x$ where $\mathrm{d}\colon \mathscr{O}_{\tilde{E}} \to \mathscr{O}_{E_0}$ is an $\tilde{A}$-derivation. However,

$$\operatorname{Der}_{\tilde{A}}(\mathscr{O}_{\tilde{E}},\mathscr{O}_{E_0}) = \operatorname{Hom}_{\mathscr{O}_{\tilde{E}}\text{-mod}}(\Omega^1_{\tilde{E}/\tilde{A}},\mathscr{O}_{E_0}) = \operatorname{Hom}_{\mathscr{O}_{E_0}\text{-mod}}(\Omega^1_{E_0/k},\mathscr{O}_{E_0}) = \operatorname{Der}_k(\mathscr{O}_{E_0},\mathscr{O}_{E_0}).$$

Since θ fixes the unit section, the admissible derivations $\mathrm{d}\colon \mathscr{O}_{E_0} \to \mathscr{O}_{E_0}$ must map the ideal sheaf cutting out the unit section into itself. By identifying $\operatorname{Der}_k(\mathscr{O}_{E_0},\mathscr{O}_{E_0}) \cong \mathrm{H}^0(\mathscr{O}_{E_0},\Omega^\vee_{E_0/k})$, this is same as specifying a vector field on E_0 vanishing at the unit section. If E_0 is smooth then all vector fields are constant since $\Omega_{E_0/k}$ is trivial, and hence $\mathrm{d}=0$. If E_0 is the 1-gon with p the node, consider its normalization $\mathbb{P}^1_k \to E_0$. We have an exact sequence $0 \to \mathscr{O}_{E_0,p} \to \mathscr{O}_{\mathbb{P}^1_k,0} \oplus \mathscr{O}_{\mathbb{P}^1_k,\infty} \to k \to 0$ which can be verified at completions, and via this we can view local uniformizers x and $1/x$ on $\mathbb{P}^1_k$ as elements of $\mathscr{O}_{E_0,p}$. Let $D \in \operatorname{Der}_k(\mathscr{O}_{E_0},\mathscr{O}_{E_0})$ vanishing at e. Then we have $0 = \frac{1}{x}D(x) + xD(1/x)$. This implies that $D(x) \in (x)$ and $D(1/x) \in (1/x)$. Thus, D defines a derivation on $\mathbb{P}^1$ which vanishes at 0, ∞, and the preimage of the unit section. Since $\deg\Omega^\vee_{\mathbb{P}^1_k} = 2$, it follows that $D=0$.

(ii) We show that D is prorepresentable using Schlessinger's criterion (Theorem A.8) and Theorem A.10. (H0) is trivial. (H3) is immediate because $\operatorname{Ext}^1_{\mathscr{O}_{E_0}}(\Omega_{E_0/k},\mathscr{O}_{E_0}(-e))$ is a finite dimensional vector space. The group $\operatorname{Ext}^2_{\mathscr{O}_{E_0}}(\Omega_{E_0/k},\mathscr{O}_{E_0}(-e)) \cong \operatorname{Ext}^2_{\mathscr{O}_{E_0}}(\Omega_{E_0/k}(e),\mathscr{O}_{E_0})$ is zero due to Serre duality and because the dualizing sheaf is trivial. This means that there is no obstruction to lifting infinitesimal deformations. We now verify (H4). Consider a fibered product diagram of Artin rings

$$\begin{array}{ccc} \tilde{A}_1 & \xrightarrow{\tilde{r}} & \tilde{A}_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ A_1 & \xrightarrow[r]{} & A_2 \end{array}$$

where π_i are small thickenings. We want to verify that $D(\tilde{A}_1) \to D(A_1) \times_{D(A_2)} D(\tilde{A}_2)$ is a bijection. We may assume that $D(A_1) \times_{D(A_2)} D(\tilde{A}_2)$ is nonempty because otherwise there is nothing to do. Let $(E_{A_1}, E_{\tilde{A}_2}) \in D(A_1) \times_{D(A_2)} D(\tilde{A}_2)$ and say (the class of) E_{A_1} restricts to (the class of) $E_{A_2} \in D(A_2)$. Because there is no obstruction to lifting, it follows that we have a composite isomorphism

$$D(\pi_2)^{-1}[E_{A_2}] \xrightarrow{\sim} \{\text{lifts of } E_{A_2} \text{ to } \tilde{A}_2\} \xleftarrow[-\otimes_{\tilde{A}_1}\tilde{A}_2]{\sim} \{\text{lifts of } E_{A_1} \text{ to } \tilde{A}_1\} \xrightarrow{\sim} D(\pi_1)^{-1}[E_{A_1}].$$

Hence, there is some class $[E_{\tilde{A}_1}] \in D(\pi_1)^{-1}[E_{A_1}]$ mapping to $E_{\tilde{A}_2}$. Therefore, the inverse of the above composite can be viewed as $D(\tilde{r})$ This shows that (H4) is satisfied. We conclude that D is prorepresentable. Due to no obstruction to liftings, D is formally smooth, which by Lemma A.9 means that D is prorepresented by a power series algebra over Λ. The number of indeterminants in this power series algebra is then precisely $\dim_k D(k[\varepsilon]/(\varepsilon^2)) = \dim_k \mathrm{Ext}^1(\Omega_{E_0/k}(e), \mathcal{O}_{E_0}) = \dim_k \mathrm{H}^0(E_0, \Omega_{E_0/k}(e))$ (Serre duality). If E_0 is smooth then this is just 1. Else if E_0 is the standard 1-gon, consider the normalization map $\pi\colon \tilde{E}_0 \to E_0$. We know that $\mathbb{P}^1 \cong \tilde{E}_0$. Then the map $\Omega_{E_0}(e) \to \pi_*\Omega_{\tilde{E}_0}(e)$ is a surjective map with a length 1 kernel supported at the node of E_0. By the long exact sequence in cohomology, it follows that $\mathrm{H}^0(E, \pi_*\Omega_{E_0}(e)) \cong \mathrm{H}^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(e)) = 0$.

(iii) We apply Grothendieck's existence theorem (Theorem A.6). If a formal generalized elliptic curve $\mathfrak{X} \to \mathrm{Spf}\,\Lambda[\![t]\!]$ prorepresents Def_{E_0} then may we take $\mathscr{L} = \mathcal{O}_{\mathfrak{X}}(\mathfrak{X}^{\mathrm{sm}}[n])$ which is relatively ample modulo each t^n (c.f. Theorem B.5). Therefore, we get a curve over $\mathrm{Spec}\,\Lambda[\![t]\!]$ with a designated 'unit' section, except that we need to show that it's DR semistable of genus 1 and 'algebraize' the formal group law too. It's DR semistable genus-1 by Lemma 1.6 because it is so over $\mathrm{Spec}\,\Lambda[\![t]\!]/(t)$. The group law is taken care of by Theorem 1.14.

(iv) Of course, E_0 is a 1-gon. The deformation theory of nodes (see [DR, I.5]) tells us that exists an element $u \in \Lambda[\![t]\!]$ such that $u = 0$ is the image of the non-smooth locus of E. By Lemma 1.8, on $\mathrm{Spec}\,\Lambda[\![t]\!]/(u)$, E is isomorphic to the inverse image of the standard 1-gon. The universal property of $\Lambda[\![t]\!]$ then implies that $\mathrm{Spec}\,\Lambda[\![t]\!]/(u)$ is unramified over $\mathrm{Spec}\,\Lambda$, hence $\Lambda[\![t]\!]$ is isomorphic to $\Lambda[\![u]\!]$. This completes the proof for the irreducible case of E_0.

The general case: Let E_0 be the standard n-gon over k and H_0 the cyclic subgroup $\{1\} \times \mathbb{Z}/n\mathbb{Z}$. Denote by $\mathrm{Def}_{(E_0,H_0)}$ the infinitesimal deformation functor of $(E_0, +, H_0)$: it takes $A \in \mathsf{C}_\Lambda$ to the set of all isomorphism classes of triplets $(E, +, H)$, where $(E, +)$ is a generalized elliptic curve over A and $H \subseteq E^{\mathrm{sm}}$ a subgroup isomorphic to $(\mathbb{Z}/n\mathbb{Z})_A$, together with an isomorphism of the special fiber with $(E_0, +, H_0)$. The canonical morphism $\mathrm{Def}_{(E_0,H_0)} \to \mathrm{Def}_{E_0}$ is an isomorphism. Indeed, $E[n]$ is a finite *étale* commutative group scheme over A since n is invertible on A. In particular, it must be the disjoint union of finitely many "thickened points"– this follows from imitating the standard proof of the structure theorem for finite étale algebras over a field. Therefore, given a subgroup of $E_0[n]$ we can extend it to a closed subgroup of $E[n]$ simply by replacing each point $p \in E_0[n]$ by the connected component of $E[n]$ it is contained in. It is easily seen that this extension is unique. Let $(E, +, H)$ be as above over $\mathrm{Spec}\,A$. The group H acts freely on E (since H_0 acts freely on E_0), so E/H (which exists as E is projective) is flat over A. The operation $+$ passes to the quotient, and $(E/H, +)$ is a deformation of the 1-gon $(E_0/H_0, +)$. Denoting D_1 as the infinitesimal deformation functor of $(E_0/H_0, +)$, we thus obtain a morphism $\mathrm{Def}_{(E_0,H_0)} \to D_1$. We claim that this is a natural isomorphism. Given an infinitesimal deformation $(E, +) \to \mathrm{Spec}\,A$ of the standard 1-gon $(E_0/H_0, +)$, consider the morphism $E_0 \to E_0/H_0 \hookrightarrow (E, +)$, by topological invariance of the étale site (Theorem B.1), it follows that there is a *unique* étale morphism $E' \to E$ extending $E_0 \to E$.

$$\begin{array}{ccc} E_0 & \longrightarrow & E' \\ \downarrow{\scriptstyle \text{ét.}} & & \downarrow{\scriptstyle \text{ét.}} \\ E_0/H_0 & \longrightarrow & E \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} A \end{array}$$

By Lemma 1.12, there is a unique structure of a generalized elliptic curve E' which is compatible with the above diagram. Then $\operatorname{Ker}(E' \to E)$ is a deformation of H_0 and it follows that $\operatorname{Def}_{(E_0,H_0)} \simeq D_1$.

Concluding, Def_{E_0} is isomorphic to D_1 and thus satisfies (ii) & (iii). Assertion (iv) follows from a similar assertion for D_1, since the image of the nonsmooth locus is the same for E and E/H. □

3. **Representibility theorems**

Let $\mathfrak{M}$ be the fibered category of groupoids over the category of schemes Sch taking a scheme S to the groupoid of generalized elliptic curves over S.

Lemma 3.1. *$\mathfrak{M}$ is a stack for the fpqc site.*

Proof. Let $S' \to S$ be an fpqc morphism and $p' \colon E' \to S'$ be a generalized elliptic curve equipped with descent data. By faithfully flat descent of closed subschemes, for each $n \geqslant 1$, S'^{∞}_n of Lemma 1.8 descends to a closed subscheme of S which we call S^{∞}_n. The local finiteness assertion of Lemma 1.8 lets us replace S by an open subscheme and assume that all fibers of p' are either smooth or n-gons, n fixed. The subscheme $E'[n]$ of E' is then finite over S' (c.f. Corollary 1.11) and intersects each irreducible component of each geometric fiber of p'. Hence, $\mathcal{O}_{E'}(E'[n])$ is a relatively ample invertible sheaf (c.f. Theorem B.4) which is also equipped with a descent data. We are now done by faithfully flat descent of quasi-projective schemes (c.f. Theorems B.3 and B.2). □

Let $\mathfrak{M}_*$ be the substack of $\mathfrak{M}$ which classifies generalized elliptic curves $E \to S$ such that, for every geometric point $\bar{s} \in S$, the characteristic of $\kappa(\bar{s})$ is either 0 or doesn't divide the number of irreducible components of the geometric fiber $E_{\bar{s}}$. For each $n \geqslant 1$, the standard n-gon over $\mathbb{Z}$ defines a morphism $\operatorname{Spec} \mathbb{Z}[1/n] \to \mathfrak{M}_*$ which we call f_n. This is a closed embedding of stacks.

The main theorem of this section is

Theorem 3.2. *(i) $\mathfrak{M}_*$ is a smooth Deligne-Mumford stack over* $\operatorname{Spec} \mathbb{Z}$.

(ii) The image $\mathfrak{M}_*^{\infty}$ of the non-smooth locus of the universal generalized elliptic curve over $\mathfrak{M}_*$ is the union of the images of f_n, $n \geqslant 1$.

We want to use Artin's criterion for this–

Theorem 3.3 (Artin's criterion)**.** *Let S be a scheme of finite type over a field or an excellent Dedekind domain. Let $\mathfrak{X}$ be a category fibered in groupoids over* $\mathsf{Sch}_{/S}$. *Then $\mathfrak{X}$ is a Deligne-Mumford stack locally of finite type over S if and only if the following conditions hold:*

1. *(1) $\mathfrak{X}$ is a stack for the étale site.*
2. *(2) $\mathfrak{X}$ is locally of finite presentation, that is, for every filtered inverse system of affine schemes* $\operatorname{Spec} A_i$ *in* $\mathsf{Sch}_{/S}$, *the canonical functor*

$$\operatorname{colim}_i \mathfrak{X}(\operatorname{Spec} A_i) \to \mathfrak{X}(\lim_i \operatorname{Spec} A_i)$$

is an equivalence of categories.

(3) *Suppose ξ and η are two objects in $\mathfrak{X}(U)$, where U is a finite type S-scheme. Then $\mathrm{Isom}_U(\xi,\eta)$ is an algebraic space locally of finite type over S, where $\mathrm{Isom}\colon \mathsf{Sch}^{\mathrm{fpqc}}_{/U} \to \mathsf{Sets}$ takes a U-scheme X to the set of X-isomorphisms $\xi \times_U X \to \eta \times_U X$.*

(4) *For each field k_0 of finite type over S (that is, $\operatorname{Spec} k_0 \to S$ is finite type) with a 1-morphism $\xi_0\colon \operatorname{Spec} k_0 \to \mathfrak{X}$, there exist a complete local ring R, a morphism u from the spectrum of a finite separable extension k_0' of k_0 to the closed point s of $\operatorname{Spec} R$, and a commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} k_0' & \longrightarrow & \operatorname{Spec} k_0 \\ \Big\downarrow u & & \Big\downarrow \xi_0 \\ \operatorname{Spec} R & \xrightarrow{\ \xi\ } & \mathfrak{X} \end{array}$$

with ξ formally étale at s.

(5) *If ξ is a 1-morphism from a finite type S-scheme U to $\mathfrak{X}$, and if ξ is formally étale at a point u (of U) of finite type over S, then ξ is formally étale in a neighborhood of u (in U).*

Sketch (one direction). Suppose $\mathfrak{X}$ satisfies conditions (1)-(4). Let p be a point of $\mathfrak{X}$ which is finite type over S. (4) can be interpreted as saying that we can find a p-finite point q such that there is an effective universal formal deformation (R,ξ) of $q \hookrightarrow \mathfrak{X}$, where R is a complete local ring with closed point q. Then (2) and [Art70, Theorem 1.6] tells us that (R,ξ) is algebraizable, say by (X_p, x_p, ξ_p). Also, ξ_p is formally étale at x_p by (4). Using (5), we can shrink X_p to assume that ξ_p is formally étale everywhere. In fact, this means that ξ_p is representable and étale due to (3). Finally, one verifies that $\bigsqcup_p X_p$ is an étale cover of $\mathfrak{X}$. □

Remark 3.4. (i) If the finite type residue fields of S are perfect, we may replace (4) with the following condition:

(4') Let $s \in S$ and k_0 a finite extension of $\kappa(s)$. We suppose that $u\colon \operatorname{Spec} k_0 \to S$ is of finite type. By hypothesis, $\kappa(s)$ is also perfect and $k_0/\kappa(s)$ is separable. There exists a unique complete local ring $\Lambda(k_0)$ with residue field k_0 together with a formally étale morphism $\overline{u}\colon \operatorname{Spec}\Lambda(k_0) \to S$ extending u (this comes from the theory of Witt vectors). For $\xi_0 \in \operatorname{Ob}\mathfrak{X}(k_0)$, we denote by $\underline{\mathrm{Def}}\ \xi_0$ the following 'deformation category' over the opposite category $\hat{\mathsf{C}}^{\circ}_{\Lambda(k_0)}$: for $A \in \operatorname{Ob}\hat{\mathsf{C}}_{\Lambda(k_0)}$, the objects of $\underline{\mathrm{Def}}\ \xi_0(A)$ are the objects ξ of $\mathfrak{X}(A)$, equipped with an isomorphism $\xi_0 \xrightarrow{\sim}$ (image of ξ in $\mathfrak{X}(k_0)$). Then condition (4) is the conjunction of the following:

(4'a) the objects of $\mathfrak{X}$ have no infinitesimal automorphisms; more precisely, for k_0 and ξ_0 as above, and ξ_0' the pullback of ξ_0 over $k_0[\varepsilon]/(\varepsilon^2)$, we have $\operatorname{Aut}\xi_0' \xrightarrow{\sim} \operatorname{Aut}\xi_0$. This condition, and (3), implies that objects of $\underline{\mathrm{Def}}\ \xi_0$ have no nontrivial automorphisms.

(4'b) for k_0 and ξ_0 as above, the functor

$$A \mapsto \text{set of isomorphism classes in } \underline{\mathrm{Def}}\ \xi_0(A)$$

is effectively prorepresentable (by a local complete ring R and $\xi \in \mathfrak{X}(R)$ mapping to ξ_0).

(ii) If S is of finite type over a field or an excellent Dedekind ring with an infinite number of primes, and if all possible complete local rings R appearing in condition (4'b) are normal and of same Krull dimension, then we can remove condition (5).

Proof of Theorem 3.2. (i) We verify the conditions of Theorem 3.3 and Remark 3.4 with $S = \operatorname{Spec}\mathbb{Z}$.

(1) Immediate by Lemma 3.1.
(2) Follows from standard arguments (c.f. [EGA, IV, 8.8]).
(3) By [Art70, Corollary 6.2], we know that $\mathrm{Hilb}_{X/S}$ is representable by an algebraic space locally of finite presentation over S when $X \to S$ is a morphism of algebraic spaces locally of finite presentation. There is a morphism of functors $\mathrm{Mor}_U(\xi,\eta) \to \mathrm{Hilb}_{\xi\times_U \eta/U}$ induced by associating

to each morphism its graph. It is then easily verified that this is an open embedding using [Stacks, Tag 05XD]. There is a morphism

$$\mathrm{Mor}_U(\xi,\eta)\times_U \mathrm{Mor}_U(\eta,\xi)\to \mathrm{Mor}_U(\xi,\xi)\times_U \mathrm{Mor}_U(\eta,\eta)$$

given by $(f,g)\mapsto(gf,fg)$. Then $\mathrm{Isom}_U(\xi,\eta)$ is just the preimage of the U-point $(\mathrm{id}_\xi,\mathrm{id}_\eta)$ under the above morphism. Of course, since generalized elliptic curves only have finitely many automorphisms, $\mathrm{Isom}_U(\xi,\eta)$ is also quasi-finite over U. See [Con07, Theorem 3.1.2] for another proof.

(4') Let k_0 be a finite type field over $\operatorname{Spec}\mathbb{Z}$, which in particular must be a finite field.
  (4'a) Theorem 2.1 (i).
  (4'b) Theorem 2.1 (iii).

(5) According to Theorem 2.1 (ii), the deformation functors are prorepresented by regular rings of dimension 2. By Remark 3.4 (ii), condition (5) is automatically satisfied.

(ii) Let $\mathfrak{E}\to\mathfrak{M}_*$ be the universal generalized elliptic curve. Choose an atlas $M\to\mathfrak{M}_*$ and a point $m\in M$. It suffices to show that M is $\mathbb{Z}$-smooth. Let $E_0=\mathfrak{E}\times_{\mathfrak{M}_*} m$ and $\mathscr{E}=\mathfrak{E}\times_{\mathfrak{M}_*}\operatorname{Spec}\hat{\mathscr{O}}_{M,m}$. I claim that $\mathscr{E}\to\operatorname{Spec}\hat{\mathscr{O}}_{M,m}$ is a universal formal deformation for $E_0\to m$. Indeed, if $E\to\operatorname{Spec}A$ is an infinitesimal deformation of $E_0\to m$ then of course we get a morphism $\operatorname{Spec}A\to\mathfrak{M}_*$. By formal étaleness, this lifts to a morphism $\operatorname{Spec}A\to M$. By universal property of completions, this factors through $\operatorname{Spec}\hat{\mathscr{O}}_{M,m}$. It follows that $\mathscr{E}\to\operatorname{Spec}\hat{\mathscr{O}}_{M,m}$ is a universal formal deformation. By Theorem 2.1 (iii), it follows that $\hat{\mathscr{O}}_{M,m}\cong W[\![t]\!]$ where $W=\kappa(m)$ if $\operatorname{char}\kappa(m)=0$ or the unique complete regular local ring with residue field $\kappa(m)$ otherwise. Thus, M is $\mathbb{Z}$-smooth.

By Lemma 1.9, there is a universal 'nonsmooth locus' closed substack $\mathfrak{M}_*^\infty$ of $\mathfrak{M}_*$ which is a 'disjoint union' of $\mathfrak{M}_{*,n}^\infty$, $n\geqslant 1$. □

4. Level n structure

Denote by $\mathfrak{M}_{(n)}$ the stack over $\mathbb{Z}[1/n]$ classifying generalized elliptic curves C/S, with geometric fibers smooth or n-gons.

Proposition 4.1. *$\mathfrak{M}_{(n)}$ is a Deligne-Mumford stack proper and smooth over $\operatorname{Spec}\mathbb{Z}[1/n]$. Its 'locus at infinity' $\mathfrak{M}_{(n)}^\infty=\mathfrak{M}_{(n)}\cap\mathfrak{M}_*^\infty$ is the image of f_n.*

Proof. By Lemma 1.9, $\mathfrak{M}_{(n)}$ is an open substack of $\mathfrak{M}_*$. Hence, it's a Deligne-Mumford stack smooth over $\operatorname{Spec}\mathbb{Z}[1/n]$ and f_n defines a section of $\mathfrak{M}_{(n)}\to\operatorname{Spec}\mathbb{Z}[1/n]$. Properness follows from Theorem 1.13 and the valuative criterion. □

For a generalized elliptic curve $E\to S$ with geometric fibers smooth or n-gons, n invertible on S, $E^{\mathrm{sm}}[n]$ is a finite étale S-group of rank n^2 (c.f. Corollary 1.11). It is well known that finite étale group schemes are twisted constants.

Definition 4.2. Using the above notations, a **level n structure** is an S-group isomorphism $(\mathbb{Z}/n\mathbb{Z})_S^2\xrightarrow{\sim}E^{\mathrm{sm}}[n]$.

In particular, any $E\to S$ with n invertible on S admits a level n structure étale-locally over S.

Definition 4.3. $\mathfrak{M}_n[1/n]$ be the following stack over $\mathbb{Z}[1/n]$: for S a scheme on which n is invertible, $\mathfrak{M}_n[1/n](S)$ is the groupoid of generalized elliptic curves E/S with geometric fibers either smooth or n-gon, equipped with a level n structure.

It is clear that the natural map $\mathfrak{M}_n[1/n] \to \mathfrak{M}_{(n)}$ is representable. Indeed, for any morphism from a scheme $U \to \mathfrak{M}_{(n)}$ with corresponding family $E \to U$, $U \times_{\mathfrak{M}_{(n)}} \mathfrak{M}_n[1/n]$ is essentially the functor of level n structures on E/U. This functor is an étale sheaf. So, we may assume that $E^{\mathrm{sm}}[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2_U$. Then the functor in discussion is represented by $\operatorname{Spec}\mathbb{Z}[1/n] \times I$ where I is the (index) set of all $\mathbb{Z}/n\mathbb{Z}$-basis for $(\mathbb{Z}/n\mathbb{Z})^2$. Hence, $\mathfrak{M}_n[1/n]$ is a Deligne-Mumford stack which is a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-torsor over $\mathfrak{M}_{(n)}$. By Lemma 1.9, we can form $\mathfrak{M}_n^\infty[1/n]$, the image of nonsmoothness locus, which can be also viewed as the 'pullback' of $\mathfrak{M}_{(n)}^\infty$. The following result is a direct consequence of Proposition 4.1–

Theorem 4.4. *$\mathfrak{M}_n[1/n]$ is a Deligne-Mumford stack proper and smooth over $\operatorname{Spec}\mathbb{Z}[1/n]$, and $\mathfrak{M}_n^\infty[1/n]$ is finite étale over $\operatorname{Spec}\mathbb{Z}[1/n]$.*

When $n \geqslant 3$, all automorphisms are killed. Indeed, if E/k, k algebraically closed, then any automorphism $\sigma\colon E \to E$ which fixes 5 closed points must be identity. For any rational function $f \in K(E)$, $f - \sigma^* f$ is either constant or has least 5 zeroes, which is same as saying it is constant or has at least 5 poles. If $p \in E$ is a closed point not fixed by σ, then a routine application of Riemann-Roch gives us a nonconstant rational function f which has a double pole at p and no poles elsewhere. This forces $\sigma = \mathrm{id}$. Secondly, an automorphism of the standard n-gon C_n/k which pointwise fixes the n-torsion subgroup $\mu_n \times \mathbb{Z}/n\mathbb{Z}$ is obviously just the identity. By [Stacks, Tag 04SZ], it follows that $\mathfrak{M}_n[1/n]$ is a proper smooth algebraic space over $\operatorname{Spec}\mathbb{Z}[1/n]$ of pure relative dimension 1 (c.f. Theorem 2.1 (ii)).

Theorem 4.5. *Let R be a Dedekind domain and let $X \to \operatorname{Spec} R$ be a proper flat morphism with X a regular algebraic space whose fibers are of pure dimension 1. Then X is an R-projective scheme.*

Sketch (from [MO207443]). It is sufficient to show that X is a scheme due to [Lic68, Theorem 2.8]. We first solve it for the case when R is a DVR, say with fraction field K, maximal ideal $\mathfrak{m}$, residue field k, then use standard arguments to prove the general case. Roughly, the idea is to pick any nonzero prime ideal $\mathfrak{p} \subset R$, find an ample divisor D on X such that the dual of the invertible sheaf corresponding to D is ample on $X \otimes_R R_{\mathfrak{p}}$, which by general spreading out techniques (c.f. Theorem B.4) gives us that the same invertible sheaf is ample over an open neighborhood U of $\mathfrak{p}$. Since $\dim R = 1$, the complement of U only has finitely many closed points, and for each of them we get analogous divisors D_i by using the DVR case, then the problem is solved by considering $D + \sum D_i$. By [Ols16, Exercise 7.D], we know that X_k is a scheme. Let $\{C_1, \dots, C_n\}$ be the irreducible components of X_k with reduced structure. Let $x_i \in C_i$ be a closed point and $(U_i, x_i) \to (X, x_i)$ be an affine étale neighborhood of x_i that is a scheme, which is identity modulo $\mathfrak{m}$. In particular, U_i is a regular affine curve over R. Shrink U_i so that its special fiber doesn't intersect $C_i \cap C_j$ for each $i \neq j$. By going-down, we can choose a closed point $u_i \in (U_i)_K$ whose closure in U_i contains x_i. Therefore, the image $\xi_i \in X_K$ is a closed point whose closure in X contains x_i. The closure of ξ_i, say D_i, in X has an invertible ideal sheaf $\mathscr{I}_i$ since X is regular. Consider $\mathscr{L} = \mathscr{I}_1^\vee \otimes \mathscr{I}_2^\vee \otimes \cdots \otimes \mathscr{I}_n^\vee$. Each D_i is flat over R, therefore the divisor associated to $\mathscr{L}_k$ is simply $x_1 + x_2 + \cdots + x_n$, which has to be ample. We have the exact sequence

$$0 \to \mathscr{F} \otimes \mathscr{L}^{\otimes m} \xrightarrow{\cdot \pi} \mathscr{F} \otimes \mathscr{L}^{\otimes m} \to \mathscr{F}_k \otimes \mathscr{L}_k^{\otimes m} \to 0$$

for any R-flat sheaf $\mathscr{F}$. The long exact sequence in cohomology together with Nakayama gives $\mathrm{H}^1(X, \mathscr{F} \otimes \mathscr{L}^{\otimes m}) = 0$ by cohomological criterion for ampleness for all large enough m_0. This implies that $\mathscr{L}^{\otimes m}$ is globally-generated and $\mathrm{H}^0(X, \mathscr{L}^{\otimes m}) \otimes_R k \xrightarrow{\sim} \mathrm{H}^0(X_k, \mathscr{L}_k^{\otimes m})$ is an isomorphism for all large $m \geqslant m_0$ There is an R-morphism $f\colon X \to \operatorname{Proj}_R \mathrm{H}^0(S, \mathscr{L}^{m_0})$ whose formation commutes with mod $\mathfrak{m}$ reduction. On the special fiber, it is given by $\mathscr{L}_k^{\otimes m_0}$, and hence it is quasi-finite. However, for any finite type morphism between Noetherian algebraic spaces, the quasi-finite locus is open on the source [EGA, IV_3, 8.10.5 (xi)]. Therefore, X being proper over R, we must have that f is quasi-finite. By Stein factorization, f can be factored as $X \to \mathcal{S}pec\, f_*\mathscr{O}_X \to \operatorname{Proj}_R \mathrm{H}^0(S, \mathscr{L}^{m_0})$ where the the first morphism is an open embedding. In particular, X must be a scheme. ∎

Corollary 4.6. *For $n \geqslant 3$, $\mathfrak{M}_n[1/n]$ is a smooth projective scheme over $\mathbb{Z}[1/n]$.*

Appendix A. Deformation theory

In what follows, functors are always covariant unless otherwise specified. Let Λ be a complete local ring with residue field k. It is safe to assume $\Lambda = k$. Denote by C_Λ the category of Artin local Λ-algebras with residue field k and by $\hat{\mathsf{C}}_\Lambda$ the category of Noetherian complete local Λ-algebras with residue field k. A functor $F\colon \mathsf{C}_\Lambda \to \mathsf{Sets}$ extends to $\hat{\mathsf{C}}_\Lambda$ by setting

$$\hat{F}(R) = \lim_{n\to\infty} F(R/\mathfrak{m}^n),$$

for $(R,\mathfrak{m}) \in \operatorname{Ob}\hat{\mathsf{C}}_\Lambda$. For each $(R,\mathfrak{m}) \in \operatorname{Ob}\hat{\mathsf{C}}_\Lambda$, we denote by h_R the functor $\operatorname{Hom}(R,-)$ from C_Λ to Sets.

Lemma A.1. *For a functor F on C_Λ and $R \in \operatorname{Ob}\hat{\mathsf{C}}_\Lambda$, the canonical map $\hat{F}(R) \to \operatorname{Mor}(h_R, F)$ is an isomorphism.*

Proof. Straightforward. Omitted. □

Definition A.2. A functor $F\colon \mathsf{C}_\Lambda \to \mathsf{Sets}$ is called **prorepresentable** if there exists $R \in \operatorname{Ob}\hat{\mathsf{C}}_\Lambda$ and $\hat{\xi} \in \hat{F}(R)$ that induces a natural isomorphism $\hat{\xi}\colon h_R \xrightarrow{\sim} F$. We call F **effectively prorepresentable** if $\hat{\xi}$ can be chosen to be in $\operatorname{Im}(F(R) \to \hat{F}(R))$.

Consider a functor $F\colon \hat{\mathsf{C}}_\Lambda \to \mathsf{Sets}$ and $\xi_0 \in F(k)$.

Definition A.3. An **infinitesimal deformation** of ξ_0 is an element $\eta \in F(A)$, $A \in \operatorname{Ob}\mathsf{C}_\Lambda$, that is mapped to ξ_0 under the residue map $F(A) \to F(k)$. A **formal deformation** of ξ_0 is an element $\hat{\xi} \in \hat{F}(R)$, $R \in \operatorname{Ob}\hat{\mathsf{C}}_\Lambda$, that is mapped to ξ_0 under the residue map $\hat{F}(R) \to \hat{F}(k) \xrightarrow{=} F(k)$.

Definition A.4. A formal deformation $\hat{\xi} \in \hat{F}(R)$ is called **versal** (respectively, **universal**) if it has the following property: let $A' \to A$ be a surjection of Artin local Λ-algebras in C_Λ, $\eta' \in F(A')$ any infinitesimal deformation of ξ_0, and $\eta \in F(A)$ the infinitesimal deformation of ξ_0 induced by η'. Then each map $R \to A$ that induces $\eta \in F(A)$ via $\hat{\xi} \in \hat{F}(R)$ can be lifted (respectively, uniquely lifted) to a morphism $R \to A'$ such that η' is induced by $R \to A'$.

Proposition A.5. *Let $(R,\mathfrak{m},k)$ be a complete local ring and suppose we are given a formal deformation of X_0 over R, that is, for each $n \geqslant 0$, schemes X_n flat and of finite type over $R/\mathfrak{m}^{n+1}$ and maps $X_n \to X_{n+1}$ inducing isomorphisms $X_n \xrightarrow{\sim} X_{n+1} \otimes_{R/\mathfrak{m}^{n+2}} R/\mathfrak{m}^{n+1}$. Then there is a Noetherian formal scheme $\mathfrak{X}$, flat over* $\operatorname{Spf} R$*, such that for each $n \geqslant 0$, we have $X_n \cong \mathfrak{X} \otimes_R R_n$.*

If in addition, there is a collection of coherent sheaves $\mathscr{F}_n$ on X_n (respectively locally free, respectively invertible), flat over $R/\mathfrak{m}^{n+1}$, and maps $\mathscr{F}_n \cong \mathscr{F}_{n+1} \otimes_{R/\mathfrak{m}^{n+2}} R/\mathfrak{m}^{n+1}$, then $\mathscr{F} = \lim \mathscr{F}_n$ will be a coherent (respectively locally free, respectively invertible) sheaf on $\mathfrak{X}$.

Theorem A.6 (Grothendieck's existence theorem)**.** *Let $\mathfrak{X}$ be a formal scheme, proper over* $\operatorname{Spf} R$*, where $(R,\mathfrak{m},k)$ is a complete local ring, and suppose there exists an invertible sheaf $\mathscr{L}$ on $\mathfrak{X}$ such that $\mathscr{L}_0 = \mathscr{L} \otimes_R k$ is ample on $X_0 = \mathfrak{X} \otimes_R k$. Then there exists an R-scheme X, together with an ample invertible sheaf $\mathscr{L}_1$, such that $(\mathfrak{X},\mathscr{L})$ is the formal completion of $(X,\mathscr{L}_1)$ along the closed fiber over R.*

Definition A.7. A surjection $f\colon R' \twoheadrightarrow R$ in C_Λ is a **small thickening** if $\operatorname{Ker} f \cong k$, or equivalently, $\mathfrak{m}_{R'} \operatorname{Ker} f = 0$ and $\operatorname{Ker} f$ is principal, where $\mathfrak{m}_{R'}$ is the maximal ideal of R'.

It is easily verified that all surjections in C_Λ are composites of small thickenings.

Theorem A.8 (Schlessinger's criterion)**.** *A functor $F\colon \mathsf{C}_\Lambda \to$* Sets *is prorepresentable if and only if*

(H0) *$F(k)$ is a singleton set.*
(H3) *the **tangent space** $T_F = F(k[\varepsilon]/(\varepsilon^2))$ is a finite-dimensional k-vector space.*
(H4) *$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$ is bijective for every map $A' \to A$ and small thickening $A'' \twoheadrightarrow A$.*

Lemma A.9 ([Sch68, Prop. 2.5 (i)]). *Let $R \to S$ be a map in $\hat{\mathsf{C}}_\Lambda$. Then $h_S \to h_R$ is formally smooth if and only if S is a power series algebra over R.*

Let $(A, \mathfrak{m}, k)$ be an Artin local ring and A' be a small thickening of the same. Let X_0 be a generically smooth finite-type lci k-scheme. We remark that if k is perfect then being generically smooth is equivalent to being reduced due to generic smoothness. A **lifting** of an A-scheme X, with special fiber isomorphic to X_0, to A' is a scheme X' flat over A' with a closed embedding $X \hookrightarrow X'$, which induces an isomorphism of X with $X'|_{\operatorname{Spec} A}$ over A.

The deformation theory of local complete intersections has been worked out in detail in [Vis99]. Consider the following setup: let Z' be a flat scheme over A', and set $Z = Z'|_{\operatorname{Spec} A}$. Let X be a flat lci scheme of finite type over A with a closed embedding $Z \hookrightarrow X$. Assume also that X_0 is generically smooth over κ. A lifting of X relative to Z' is a lifting X' of X with a closed embedding $Z' \hookrightarrow X'$ extending the given embedding of Z in X. An isomorphism of relative liftings is an isomorphism of lifting inducing the identity on Z'. Again, let J_0 be the ideal sheaf of Z_0 in X_0.

Theorem A.10 ([Vis99, 5.4]). *(a) There is a canonical element $\omega \in \operatorname{Ext}^2_{\mathscr{O}_{X_0}}(\Omega_{X_0/\kappa}, J_0)$, called the obstruction, such that $\omega = 0$ if and only if a lifting exists.*

(b) If a lifting exists, then there is a canonical action of the group $\operatorname{Ext}^1_{\mathscr{O}_{X_0}}(\Omega_{X_0/\kappa}, J_0)$ on the set of isomorphism classes of liftings making it into a principal homogeneous space.

Appendix B. Facts from EGA and SGA

Theorem B.1 (Topological invariance of étale site, [EGA, IV, 18.1.2], [Stacks, Tag 039R]). *Let S be a scheme, S_0 a closed subscheme of S whose underlying space is identical to that of S. Then the functor $X \mapsto X \times_S S_0$ from the category of S-étale schemes to the category of S_0-étale schemes is an equivalence.*

Theorem B.2 ([SGA1, VIII.5.2]). *Let F be the fibered category of morphisms in the category of schemes (VI 11.a). Then, every morphism $g : S' \to S$ that is faithfully flat and quasi-compact is a morphism of F-descent (or, as commonly said, a descent morphism in* Sch*).*

Theorem B.3 ([SGA1, VIII.7.8]). *Let $g : S' \to S$ be a morphism that is faithfully flat and quasi-compact. Then, g is an effective descent morphism for the fibered category of quasi-compact schemes Z over a scheme T, equipped with an ample relative invertible sheaf with respect to T. In particular, for any scheme X' over S' endowed with a descent data relative to $g : S' \to S$, and any invertible sheaf L' on X' ample relative to S' also equipped with a descent data relative to the given data on X' (i.e., equipped with an isomorphism of $q_1^*(L')$ with $q_2^*(L')$ satisfying the usual transitivity condition), the descent data on X' is effective. Moreover, the invertible sheaf L on the descended scheme X, obtained by descent from L', is ample relative to S.*

Theorem B.4 ([EGA, IV$_3$, 9.6.4]). *Let $X \to S$ be a proper morphism of finite presentation along with a given line bundle on X. The locus of points on the base for which the line bundle is ample on fibers is open, and over that open subscheme it is relatively ample.*

Theorem B.5 ([EGA, IV, 11.3.10, 17.9.1]). *Suppose X, Y are flat and of finite presentation over T and $f : X \to Y$ be a T-morphism. Then, let P be a property in {flat, smooth, etale, open immersion, isomorphism, flat and a relative complete intersection morphism}, then f has property P if and only if each $f_{\bar{t}}$ has P for all geometric points $\bar{t} \in T$.*

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