MGE062: ALGEBRAIC GEOMETRY II

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These notes were taken for the Algebraic Geometry II elective course I took in my sophomore year at Chennai Mathematical Institute in Spring 2023, taught by Prof. Krishna Hanumanthu and Dr. Nabanita Ray. I live-TEXed them using neovim for personal use, and as such there may be typos; send comments, complaints, and corrections to ayannath@cmi.ac.in. Additionally, the notes may include my own justifications and interpretations. I used quiver to make commutative diagrams.

CONTENTS

Everyone knows what a curve is, until he has studied enough mathematics to become confused... — Felix Klein

Lecture 1

Lecturer: Krishna Hanumanthu   Date: 02.01.2023

"Schemes are scary."

1.1. Syllabus. Hartshorne [Har77, IV & V].

1.2. Prerequisites. Hartshorne [Har77, II.1-8, III.1-5]; Serre duality and Kodaira vanishing without proofs.

1.3. Conventions. All fields k are algebraically closed. By "curve" we mean a regular/nonsingular integral projective[1] k-scheme of dimension 1.

1.4. *Remark.* X is a projective variety $\iff$ X can be embedded as a closed subvariety of some $\mathbb{P}^n$.

1.5. *Algebraic fact.* An affine integral scheme is regular if and only if its coordinate ring is integrally closed.

[1]Projective schemes are always proper.

1.6. *Example.* The projective line $\mathbb{P}^1$. Plane curves: $V(f) \subseteq \mathbb{P}^2$, where $f \in k[x,y,z]$ is a homogeneous irreducible polynomial. Nonsingularity is equivalent to $V(f_x, f_y, f_z) = \varnothing$[2]. Are there such f? Yes, $x+y$, $x+y+z$, $x^2+yz, \ldots$ but all these are isomorphic to $\mathbb{P}^1$. The curve $x^3+yz^2+y^3$ is not isomorphic to $\mathbb{P}^1$. Veronese embedding (n-uple embedding)

$$\varphi_n \colon [x:y] \mapsto [x^n : x^{n-1}y : \cdots : xy^{n-1} : y^n], \mathbb{P}^1 \to \mathbb{P}^n$$

is a closed embedding. Then $\varphi_n(\mathbb{P}^1)$ is **non-degenerate**, i.e., it's not contained in any hyperplane.

1.A. Question. Which of the above curves are different from $\mathbb{P}^1$?

1.B. Question. Let X be a curve. Which curves can be embedded in $\mathbb{P}^1$? $\mathbb{P}^2$?

1.7. *Fact.* Any curve can be embedded in $\mathbb{P}^3$.

1.8. Genus. The **arithmetic genus** of X, denoted $p_a(X)$, is defined as $1-P_X(0)$, where P_X is the **Hilbert polynomial** of X. See Hartshorne [Har77, Ex I.7.2]. The **geometric genus** of X, denoted $p_g(X)$, is defined to be $\dim_k H^0(X, \omega_X)$, where ω_X is the **canonical sheaf** on X.

1.9. Theorem (Hartshorne [Har77, IV.1.1]). — *Let X be a curve. Then $p_a(X) = p_g(X) = \dim_k H^1(X, \mathscr{O}_X)$.*

This common number is called the **genus** of X. It's invariant under isomorphisms. The first equality $p_a(X) = \dim_k H^1(X, \mathscr{O}_X)$ is Hartshorne [Har77, Exercise III.5.3] and the second equality $p_g(X) = \dim_k H^1(X, \mathscr{O}_X)$ is clear by Serre duality.

1.10. *Notation.* From now onwards, we write $h^i(X, \mathscr{F})$ for $\dim_k H^i(X, \mathscr{F})$.

1.C. Homework. Read Hartshorne [Har77, I.7]. Try Hartshorne [Har77, Exercise III.5.6].

Lecture 2

Lecturer: Krishna Hanumanthu Date: 05.01.2023

"The condition of your Hartshorne displays your prowess."

2.1. *Facts.* If X is a projective variety over k of dimension r. Then

- $H^0(X, \mathscr{O}_X) = k$.
- In general, $p_a(X) = (-1)^r(\chi(X, \mathscr{O}_X) - 1)$, where

$$\chi(X, \mathscr{O}_X) = h^0(X, \mathscr{O}_X) - h^1(X, \mathscr{O}_X) + h^2(X, \mathscr{O}_X) - \cdots$$

  is the **Euler characteristic**. See Hartshorne [Har77, Ex III.5.2].
- If $r = 1$, then $p_a(X) = 1 - \chi(X, \mathscr{O}_X) = h^1(X, \mathscr{O}_X)$ by Grothendieck's **dimensional cohomology vanishing**.

2.2. Definition. Let X be a curve. The **genus** of X is $g(X) := p_a(X) = p_g(X) = h^1(X, \mathscr{O}_X)$.

Note that $g(X)$ is a nonnegative integer.

2.A. Question. Is every nonnegative integer genus of some curve? Answer: yes.

2.3. *Example.* Let $Q \subseteq \mathbb{P}^3$ be a nonsingular quadric, for e.g., $Q = V(xy - zw)$. It turns out that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ via the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$.

[2] For homogeneous polynomials f, $V(f_x, f_y, f_z) \subseteq V(f)$ holds as f can be written as a linear combination of f_x, f_y, and f_z. Here, $f_x = \partial f / \partial x, \ldots$

2.4. Weil divisors on the quadric surface in $\mathbb{P}^3$. Define the divisor class group $\operatorname{Cl} Q$ by the quotient $\operatorname{Weil} Q/\{\text{linear equivalence}\}$, where $\operatorname{Weil} X$ is the free $\mathbb{Z}$-module of all Weil divisors. It turns out that the divisor class group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Using this isomorphism, we can denote any divisor class as a pair of integers.

2.B. Exercise. Show that $\operatorname{Cl} \mathbb{P}^1 = \mathbb{Z}$ and that $\operatorname{Cl} Q = \mathbb{Z} \times \mathbb{Z}$.

2.C. Exercise (See Hartshorne [Har77, Example II.7.6.2]). If $(a,b) \in \operatorname{Cl} Q$ then "(a,b) is ample $\iff$ (a,b) is very ample $\iff$ $a > 0, b > 0$".

2.D. Homework. Hartshorne [Har77, Ex III.5.6] (especially part (c)).

Consider Q. Let $a, b > 0$ and let $X \in |(a,b)|$ be a curve, where $|(a,b)|$ is the **linear system**[3] of the divisor (a,b). **Bertini's theorem** says that such an X exists. Observe that $g(X) = ab - a - b + 1$.

2.5. Corollary. — If $X \in |(g+1, 2)|$, then $g(X) = g$. In particular, $\mathbb{P}^1 \times \mathbb{P}^1$ contains a curve of every genus.

2.E. Question. Is there a different surface where you can produce curves of any given genus? Given any curve X, can it be embedded in $Q = \mathbb{P}^1 \times \mathbb{P}^1$?

2.F. Exercise. If X is a curve of genus 0, then $X \cong \mathbb{P}^1$.

2.6. Quick review of divisors. (X is not necessarily a curve in this section.) A **Weil divisor** on X is a formal expression $\sum_{i=1}^n a_i Y_i$ where $a_i \in \mathbb{Z}$ and Y_i are irreducible reduced codimension 1 subvarieties. The divisor associated to a rational function $f \in K(X)$ is

$$\operatorname{div} f := \sum_{\substack{Y \subseteq X,\ \operatorname{codim} Y = 1,\\ \text{reduced, irreducible, closed}}} n_Y [Y].$$

Such Y's are called **prime divisors**. What are a_Y? Let $U \subseteq X$ be an affine open set such that $U \cap Y$ is nonempty. Then $\mathscr{O}_{X,Y} := k[U]_{I_Y(U)}$, where $k[U]$ is the coordinate ring of U, and $I_Y(U)$ is the ideal of $U \cap Y$ in $U = \operatorname{Spec} k[U]$. We then define n_Y to be the valuation of f at the discrete valuation ring (DVR) $\mathscr{O}_{X,Y}$. It's worth noting that $\mathscr{O}_{X,Y}$ is same as the stalk of $\mathscr{O}_X$ at the generic point of Y.

Lecture 3

Lecturer: Krishna Hanumanthu Date: 09.01.2023

3.1. (continued) Quick review of divisors. Let $p \in Y$ be a closed point. Define

$$\mathscr{I}_{Y,p} := \text{"stalk of the ideal sheaf } \mathscr{I}_Y \text{ at } p \subseteq \mathscr{O}_{X,p}\text{"}.$$

Note $\mathscr{I}_{Y,p}$ is a height 1 prime in $\mathscr{O}_{X,p}$. Also, $\mathscr{O}_{X,Y} = (\mathscr{O}_{X,p})_{\mathscr{I}_{Y,p}}$. Let $U \subseteq X$ be an affine open subset. Then we have the following diagram

$$\begin{array}{ccccccc}
k[U] & \longrightarrow & k[U]_{\mathscr{I}_{Y,p}} & = & \mathscr{O}_{X,p} & \longrightarrow & (\mathscr{O}_{X,p})_{\mathscr{I}_{Y,p}} \\
\uparrow \scriptstyle{\text{ht 1 prime}} & & \uparrow \scriptstyle{\text{ht 1 prime}} & & & & \| \\
\mathscr{I}_Y(U) & \longrightarrow & \mathscr{I}_Y(U)_{\mathscr{I}_{Y,p}} & & & & \mathscr{O}_{X,Y} \\
& & \| & & & & \\
& & \mathscr{I}_{Y,p} & & & &
\end{array}$$

[3] Vakil [FOAG] calls this **linear series**.

Conclusion: $\mathcal{O}_{X,Y}$ is a DVR whose quotient field is $K(X)$. Let t be a uniformizing parameter on $\mathcal{O}_{X,Y}$, i.e., t generates the maximal ideal of $\mathcal{O}_{X,Y}$. This gives a discrete valuation $\nu_Y\colon K(X)^\times \to \mathbb{Z}$.

3.2. Definitions. Let $f \in k(X)^\times$ then

- **Divisor of zeros of f:**
$$(f)_0 := \sum_{\nu_Y(f)>0,\, Y \text{ prime divisor}} \nu_Y(f)[Y]$$
- **Divisor of poles of f:**
$$(f)_\infty := \sum_{\nu_Y(f)<0,\, Y \text{ prime divisor}} -\nu_Y(f)[Y]$$
- **Divisor of f:**
$$\operatorname{div} f := (f) = (f)_0 + (f)_\infty$$

A divisor on X is called **principal** if $D = (f)$ for some rational function $f \in K(X)^\times$. Divisors D_1, D_2 are called **linearly equivalent** (written $D_1 \sim D_2$) if $D_1 - D_2$ is principal. The **divisor class group** of X is $\operatorname{Cl} X := \operatorname{Div} X := \operatorname{Weil} X/\sim$.

3.3. *Remark.* Let $f \in K(X)^\times$, then $\nu_Y(f) \neq 0$ holds for only finitely many prime divisors Y. Let $0 \neq g \in k[U]$ for some affine open $U \subseteq X$. Let $Y \subseteq X$ be a prime divisor, then

$$\nu_Y(g) > 0 \iff g \in I(Y \cap U) \iff Y \cap U \subseteq V_U(g) \iff Y \cap U \text{ is an irred. comp. of } V_U(g).$$

Hence, for all $f \in K(X)^\times$, $\{Y \subseteq X \text{ prime divisor}\colon \nu_Y(f) \neq 0\}$ is finite.

3.4. *Example.* If X is an affine variety such that $k[X] := \Gamma(X, \mathcal{O}_X)$ is a UFD, then $\operatorname{Cl} X = 0$.

3.A. Question. What happens if we don't assume $k[X]$ is a UFD?

3.5. *Reference.* "Introduction to Algebraic Geometry" by Steve D. Cutkosky, Graduate studies in Mathematics 188, American Mathematical Society, 2018.

3.6. The sheaf associated to a divisor. Fix a nonsingular variety X. Let $D = \sum a_i Y_i$ a divisor[4] on X and $U \subseteq X$ be an open set. Define

$$\Gamma(U, \mathcal{O}_X(D)) := \{f \in K(X)^\times : (f)_U + D \cap U \geqslant 0\} \cup \{0\},$$

where $(f)_U$ is the divisor on U associated to $f \in K(U)$ and $D \cap U$ is the image of D under the natural map $\operatorname{Weil} X \to \operatorname{Weil} U$. We write $D \geqslant 0$ for a divisor D if all its "coefficients" are nonnegative.

3.B. Easy exercise. Show that $\mathcal{O}_X(D)$ is a sheaf of $\mathcal{O}_X$-modules.

3.C. Exercise. Show that $\mathcal{O}_X(0) = \mathcal{O}_X$.

3.7. Proposition. — *$\mathcal{O}_X(D)$ is an invertible sheaf (line bundle) for all D.*

3.8. Definition. The **Picard group of** X, denoted $\operatorname{Pic} X$, is the set of all isomorphism classes of line bundles on X, under tensor product.

3.9. Proposition. — *If X is a nonsingular variety, D_1 and D_2 are divisors on X, then*

$$D_1 \sim D_2 \iff \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2).$$

3.10. *Next.* Differentials and the Riemann-Roch theorem.

[4] From now onwards, we simply write Y instead of $[Y]$.

Lecture 4

Lecturer: Krishna Hanumanthu Date: 12.01.2023

"This is my brother's son. He wants to learn about divisors."

4.1. Cartier divisors "=" Weil divisors. Let X be an integral locally factorial scheme. The data

$$\{(U_i, f_i)\},\ U_i \subseteq X \text{ open},\ X = \cup U_i,\ f_i \in K(X),\ f_i/f_j \in \mathscr{O}_X(U_i \cap U_j)^\times,$$

is called a **Cartier divisor**. Let $Y \subseteq X$ be a prime divisor, then $\mathscr{I}_{Y,p} \subseteq \mathscr{O}_{X,p}$ is a height 1 prime ideal, hence it's principal, say $\mathscr{I}_{Y,p} = (f)$, as $\mathscr{O}_{X,p}$ is a UFD. We call "$f = 0$" a **local equation** of Y at p.

4.A. Exercise. There exists an open affine $W \subseteq X$ such that $p \in W$ and $\mathscr{I}_Y(Y \cap W) = (f)$.

Use these local equations to define a Cartier divisor corresponding to Y. For $p \in Y$, choose (W_p, f_p) and $(X \setminus Y, 1)$ when $p \notin Y$. We can also go in the opposite direction– given a Cartier divisor $\{(U_i, f_i)\}$, we can get a Weil divisor– given any prime divisor Y on X, choose any U_i such that $U_i \cap Y$ is nonempty, then define $n_Y \stackrel{\text{def}}{=} \nu_Y(f_i)$. This doesn't depend on the choice of U_i because f_i and f_j are same upto units. Denote the group of all Cartier divisors, without any equivalence, as $\operatorname{Cartier} X$.

4.2. Theorem (Hartshorne [Har77, Theorem II.6.11]). — *If X is integral, noetherian, and factorial then* $\operatorname{Weil} X \cong \operatorname{Cartier} X$. *This isomorphism preserves principal divisors.*

4.3. Line bundle associated to a Cartier divisor. Given $\{(U_i, f_i)\} = D$, then define $\mathscr{O}_X(D)$[5] as a $\mathscr{O}_X$-submodule of the constant sheaf $K(X)$–

$$\Gamma(U_i, \mathscr{O}_X(D)) = \mathscr{O}_X(U_i)\text{-submodule of } K(X) \text{ generated by } f_i^{-1} = f_i^{-1}\mathscr{O}_X(U_i) \subseteq K(X).$$

4.4. *Facts.*

(1) $D \mapsto \mathscr{O}_X(D)$ is a 1-1 correspondence between Cartier divisors and line sub-bundles of $K(X)$.
(2) $\mathscr{O}_X(D_1 + D_2) \cong \mathscr{O}_X(D_1) \otimes \mathscr{O}_X(D_2)$
(3) $\mathscr{O}_X(-D) \cong \mathscr{O}_X(D)^\vee := \mathscr{H}om(\mathscr{O}_X(D), \mathscr{O}_X)$
(4) $D_1 \sim D_2 \iff \mathscr{O}_X(D_1) \cong \mathscr{O}_X(D_2)$
(5) If X is projective over a field or is integral then every line bundle on X is a sub-bundle of $K(X)$.

4.5. In our situation:

$$\operatorname{Weil} X \xleftrightarrow{\sim} \operatorname{Cartier} X \xleftrightarrow{\sim} \{\text{line bundles}\}$$

$$\operatorname{Cl} X \xleftrightarrow{\sim} \operatorname{CaCl} X \xleftrightarrow{\sim} \operatorname{Pic} X$$

4.6. *Example: Projective space.* Let $X = \mathbb{P}^n_k = \operatorname{Proj} S_\bullet$, where $S_\bullet = k[x_0, \ldots, x_n]$. For a homogenous polynomial $F \in S_\bullet$, define a divisor associated to F as follows– $F = F_1^{e_1} \cdots F_r^{e_r}$ be an irreducible factorization. Then $\operatorname{div} F = e_1 V(F_1) + \cdots + e_r V(F_r)$.

4.7. Theorem. — $\operatorname{Cl} \mathbb{P}^n \cong \mathbb{Z}$.

Sketch. For a prime divisor $Y \subseteq \mathbb{P}^n$, there exists a polynomial $F \in S_\bullet$ such that $Y = V(F)$. This is because $S_\bullet$ is a UFD. Define $\deg Y := \deg F$ and $\operatorname{Cl} \mathbb{P}^n \to \mathbb{Z}, \sum a_i Y_i \mapsto \sum a_i \deg Y_i$. This is a well-defined map because principal divisors are given by fractions of homogeneous polynomials of same degree, i.e., $K(\mathbb{P}^n) = (\operatorname{Frac} S_\bullet)_0$. Furthermore, degree-$d$ homogeneous polynomials should be thought of as global sections of $\mathscr{O}(d)$. Then check that this is an isomorphism of groups. □

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[5] This is denoted $\mathscr{L}(D)$ in Hartshorne [Har77].

4.8. Divisor associated to global section of a line bundle. Fix X a projective variety and $\mathscr{L}$ a line bundle on X. Fix a nonzero $s \in \Gamma(X, \mathscr{L})$. We want to define a divisor associated to s. Hartshorne [Har77] denotes the **divisor of zeros** of s as $(s)_0$. Choose a local trivialization $\{U_i\}$ of $\mathscr{L}$. Then the **Cartier divisor associated to** s is $(s)_0 := \{(U_i, s_i)\}$. This is an **effective Cartier divisor**, i.e., s_i are regular on U_i.

4.9. *Remark.* This generalizes the above constructions of $\operatorname{div} F$ for a homogenous $F \in S_\bullet$ because such an F is a section of $\mathscr{O}_X(\deg F)$.

4.10. Proposition (Hartshorne [Har77, Proposition II.7.7]). — *Let X be a nonsingular projective variety over k. Let $D_0 \in \operatorname{Weil} X$ and $\mathscr{L} = \mathscr{O}_X(D_0) \in \operatorname{Pic} X$. Then*

(1) *For every nonzero section $s \in \Gamma(X, \mathscr{L})$, the divisor of zeros $(s)_0$ is an effective divisor which is linearly equivalent to D_0.*
(2) *For every effective divisor D' linearly equivalent to D_0, there exists nonzero $s \in \Gamma(X, \mathscr{O}_X(D_0))$ such that $D' = (s)_0$.*
(3) *When $k = \overline{k}$, if $s, s' \in \Gamma(X, \mathscr{L})$ have the same divisor of zeroes then $s = \lambda s'$ for some $\lambda \in k^\times$.*

4.B. Homework. Read about canonical sheaf.

4.11. *Next.* Linear systems; ampleness; globally-generated sheaves; differentials.

Lecture 5

Lecturer: Krishna Hanumanthu Date: 19.01.2023

5.1. Linear systems. Let $\mathscr{L} = \mathscr{O}_X(D_0) \in \operatorname{Pic} X$ and $D_0 \in \operatorname{Cl} X$. We have the following correspondence:

$$\mathbb{P}(\Gamma(X,\mathscr{L})) \overset{\sim}{\longleftrightarrow} |D_0| \overset{\text{def}}{=\!=} \left\{ \begin{array}{c} \text{effective divisors} \\ \text{linearly equivalent to } D_0. \end{array} \right\}$$

$$\mathbb{P}(\Gamma(X,\mathscr{L})) \overset{\text{def}}{=} \Gamma(X,\mathscr{L}) \setminus \{0\} / k^\times$$

$$\left\{ \begin{array}{c} \text{effective divisors} \\ \text{linearly equivalent to } D_0. \end{array} \right\} \overset{\text{def}}{=} \text{The } \textbf{complete linear system} \text{ associated to } D_0.$$

5.2. *Observation.* There is no effective divisor linearly equivalent to D_0 if and only if $\Gamma(X, \mathscr{L}) = 0$.

5.3. *Example.* Let $X = \mathbb{P}^2 := \operatorname{Proj} k[x_0, x_1, x_2]$, and $\mathscr{L} = \mathscr{O}_{\mathbb{P}^2}(2) \cong \mathscr{O}(2 \cdot V(x_0))$. Observe that $\Gamma(X, \mathscr{L})$ is the vector space of degree-2 homogeneous polynomials in $k[x_0, x_1, x_2]$. Therefore, $\mathbb{P}(\Gamma(X, \mathscr{L})) \cong \mathbb{P}^5$ in the classical sense. See **projectivization** of vector spaces (Vakil [FOAG]).

5.4. Maps to projective space. Let X be a nonsingular projective k-variety. Suppose $\varphi \colon X \to \mathbb{P}^n$ is a morphism. We know that $\Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) \cong kx_0 + kx_1 + \cdots + kx_n$, the linear homogeneous polynomials. Then we have a line bundle $\mathscr{L} := \varphi^*(\mathscr{O}_{\mathbb{P}^n}(1))$ on which we have global sections $s_i := \varphi^* x_i$, $0 \leqslant i \leqslant n$. We can attach to φ the data $(\mathscr{L}, s_0, \dots, s_n)$. This procedure is reversible. See Vakil [FOAG] or Hartshorne [Har77] for details. Therefore, we have a correspondence:

$$\operatorname{Mor}_k(X, \mathbb{P}^n_k) \longleftrightarrow \{(\mathscr{L}, s_0, \dots, s_n) : \mathscr{L} \in \operatorname{Pic} X,\ s_i \in \Gamma(X, \mathscr{L}), s_i \text{ have no common zeroes}\}.$$

If s_i have common zeroes, we only get a morphism from an open subscheme of X, i.e., a **rational map** since X is integral, in particular, irreducible, in our case.

5.5. Definition. — We call a line bundle $\mathscr{L}$ **globally-generated** if there is a finite set of global sections that generate $\mathscr{L}$.

Observe that this definition aligns with the one in Vakil [FOAG].

5.6. Proposition (local criterion for closed embedding into $\mathbb{P}^n$). — Hartshorne [Har77, Proposition II.7.3]

5.7. *Remark.* Think of the two conditions as separating a 0-dimensional subscheme of length 2. Condition 1: $P+Q$, $P \neq Q$. Condition 2: $2P$, $P \in X$. In condition 2, we have $P \in X$ and $t \in T_P(X)$, a tangent vector at P.

5.8. Definition. — Let X be a projective k-variety and $\mathscr{L}$ be a line bundle.

(1) $\mathscr{L}$ is **very ample** if the "map determined by $\mathscr{L}$" is a closed embedding, where the "map determined by $\mathscr{L}$" is the morphism given by a k-basis of $\Gamma(X, \mathscr{L})$. Note that global-generation is implicit in this definition.
(2) $\mathscr{L}$ is **ample** if $\mathscr{L}^{\otimes m}$ is very ample for some $m \geqslant 1$.

5.9. Theorem (definition of ample in Hartshorne [Har77]). — *A line bundle $\mathscr{L}$ is ample if and only if for all coherent sheaves $\mathscr{F}$ on X, $\mathscr{F} \otimes \mathscr{L}^{\otimes m}$ is globally-generated for all sufficiently large m.*

5.10. *Remark.* Note that ample divisors[6] are not necessarily effective.

5.11. *Examples.*

(1) $X = \mathbb{P}^n$, $\mathscr{L}_d = \mathscr{O}_X(d)$, $d \in \mathbb{Z}$. Therefore $\mathscr{L}_d$ is effective[7] if and only if $d \geqslant 0$. Note that $\mathscr{L}_0 = \mathscr{O}_X$ *is* globally-generated. Also, $\mathscr{L}_d$ is very ample/ample if and only if $d \geqslant 1$.
(2) (Hartshorne [Har77, Example 7.6.2]) Consider $X = V(xy - zw) \subseteq \mathbb{P}^3$. We know that

$$X \cong \mathbb{P}^1 \times \mathbb{P}^1, \operatorname{Pic} X \cong \pi_1^* \operatorname{Pic} \mathbb{P}^1 \oplus \pi_2^* \operatorname{Pic} \mathbb{P}^1 \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Let $a, b \in \mathbb{Z}$.
- $a < 0$ or $b < 0$: Restriction of a type (a, b) line bundle to the components are $\mathscr{O}_{\mathbb{P}^1}(a)$ and $\mathscr{O}_{\mathbb{P}^1}(b)$. Hence, it is not globally-generated.
- $a, b > 0$: We have

$$X \cong \mathbb{P}^1 \times \mathbb{P}^1 \xhookrightarrow{\text{Veronese}\times\text{Veronese}} \mathbb{P}^a \times \mathbb{P}^b \xhookrightarrow{\text{Segre}} \mathbb{P}^{ab+a+b}$$

This is the closed embedding determined by (a, b) on X.

In conclusion, (a, b) is very ample if and only if $a, b > 0$ if and only if (a, b) is ample.

Lecture 6

Lecturer: Krishna Hanumanthu Date: 23.01.2023

6.1. *(continued) Example.*

(2) What about $(0, b)$, $b > 0$? This is not ample. Observe that $(-1, 1) \otimes (0, b)^{\otimes m} = (-1, 1 + mb)$ is not globally-generated.

6.2. *Example: An ample line bundle which is not very ample* (Hartshorne [Har77, Example II.7.6.3]). Let $X = V(y^2 z - x^3 - xz^2) \subseteq \mathbb{P}^2$ be the smooth cubic in $\mathbb{P}^2$. Consider $P_0 = [0:1:0] \in X$ and $\mathscr{L} = \mathscr{O}_X(P_0)$. Is $\mathscr{L}$ very ample? ample? globally-generated? Set-theoretically, $X \cap V(z) = P_0$. Ideal of $X \cap V(z)$ is (z, x^3), hence

[6] A divisor D is called ample if the corresponding line bundle $\mathscr{O}(D)$ is ample.
[7] A line bundle is called effective if $\Gamma(X, \mathscr{L}) \neq 0$.

$X \cap V(z) = 3P_0$. Therefore, $\mathscr{O}_{\mathbb{P}^2}(1)|_X = \mathscr{L}^{\otimes 3}$, and so $\mathscr{L}^{\otimes 3}$ is very ample. In other words, the global section z of $\mathscr{O}_X(1)$ satisfies $\operatorname{div} z = 3P_0$. Therefore, $\mathscr{L}^{\otimes 3} \cong \mathscr{O}_X(3P_0) \cong \mathscr{O}_X(1)$. However, $\mathscr{L}$ is not even globally-generated, let alone ample. Also see https://math.stackexchange.com/questions/1504206.

6.A. Exercise. If $\mathscr{L}$ is globally-generated then there is a point $Q \in X \setminus \{P_0\}$ such that Q is linearly equivalent to P_0.

6.B. Exercise (Hartshorne [Har77, Example II.6.10.1]). If X is a nonsingular projective curve such that there are two distinct points $P, Q \in X$, linearly equivalent, then $X \cong \mathbb{P}^1$.

But X is not $\mathbb{P}^1$ as it is a genus 1 curve.

6.3. *Later.* On a curve, a divisor D is ample if and only if $\deg D > 0$.

6.4. Linear systems revisited. Let X be a nonsingular projective variety and $D \in \operatorname{Div} X$. Suppose $s \in \Gamma(X, \mathscr{O}_X(D))$ is a nonzero section. Then $\operatorname{div} s = (s)_0$ is effective and equivalent to D. Therefore, we have a correspondence

$$\{\text{Effective divisors lin. eq. to } D\} \xleftrightarrow{\sim} \mathbb{P}H^0(X, \mathscr{O}_X(D))$$

For a vector subspace $V \subseteq H^0(X, \mathscr{O}_X(D))$, we have $\mathbb{P}V \subseteq \mathbb{P}H^0(X, \mathscr{O}_X(D)) = |D|$. Then V is called a linear system.

6.5. *Example.* Degree d hypersurfaces in $\mathbb{P}^2$ is a complete linear system. Degree d hypersurfaces passing through a single point, degree d hypersurfaces passing through a single point with multiplicity three, and degree d hypersurfaces passing through two points with multiplicity three each are all linear systems.

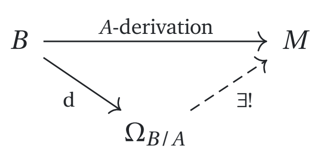
Global-geneneration and very ampleness can be expressed as properties of the corresponding linear systems. See Hartshorne [Har77, Remark II.7.8.2].

6.6. Kähler Differentials. Fix a ring map $A \to B$ and a B-module M.

6.7. Definition. An A-**derivation** of B into M is an A-module map $\mathrm{d}\colon B \to M$. such that

- d is additive
- $\mathrm{d}(bb') = b\,\mathrm{d}b' + b'\,\mathrm{d}b$
- $\mathrm{d}a = 0$ for all $a \in A$.

6.8. Module of relative differentials $\Omega_{B/A}$. There is a universal object for A-derivations of B, denoted by $\Omega_{B/A}$, called the **module of relative differentials**, with an A-derivation $\mathrm{d}\colon B \to \Omega_{B/A}$:



In other words, $\operatorname{Der}_A(B, M) \cong \operatorname{Hom}_{B\text{-Mod}}(\Omega_{B/A}, M)$. The following proposition shows that $\Omega_{B/A}$ exists.

6.9. Proposition. — *Let $f\colon B \otimes_A B \to B$ be the natural multiplication map/diagonal map. Let $I = \operatorname{Ker} f$. Then $\mathrm{d}\colon B \to I/I^2$, $b \mapsto 1 \otimes b - b \otimes 1$ is an A-derivation, and $(I/I^2, \mathrm{d})$ satisfies the universal property of $\Omega_{B/A}$.*

6.10. Sheaf of differentials. Let $f\colon X \to Y$ be a map of schemes. Consider the diagonal morphism $\Delta\colon X \to X \times_Y X$, which is known to be a locally closed embedding. Then $\Delta(X) \subseteq_{\text{closed}} W \subseteq_{\text{open}} X \times_Y X$ for some W. Let $\mathscr{I}$ be the ideal sheaf of $\Delta(X)$ in W. Define $\Omega_{X/Y}$ to be $\Delta^*(\mathscr{I}/\mathscr{I}^2)$, the **sheaf of relative differentials of** X **over** Y.

6.11. *Remark.* $\Omega_{X/Y}$ has a local description using affine opens of X and Y.

6.12. Definitions. — Let X be smooth over k. The **tangent bundle** $\mathscr{T}_{X/k}$ is defined as $\mathcal{H}om_{\mathscr{O}_X}(\Omega_{X/k}, \mathscr{O}_X)$. The **canonical bundle** $\omega_{X/k}$ is defined as the top exterior $\det \Omega_{X/k} := \wedge^n \Omega_{X/k}$, also called the **determinant bundle** of $\Omega_{X/k}$. When X is a nonsingular projective k-variety, the **geometric genus** of X is defined as $p_g(X) = h^0(X, \omega_X)$.

6.13. Definition. — A variety X is called **rational** if it is birational to $\mathbb{P}^n$, where of course, $n = \dim X$.

6.14. Useful facts.

(1) **Euler sequence.** There is an exact sequence

$$0 \to \Omega_{\mathbb{P}^n_A/A} \to \mathscr{O}_{\mathbb{P}^n_A}(-1)^{\oplus(n+1)} \to \mathscr{O}_{\mathbb{P}^n_A} \to 0$$

(2) **Smooth $\iff$ $\Omega_{X/k}$ locally free.** If X is irreducible, separated, finite-type over k, then $\Omega_{X/k}$ is locally-free of rank $\dim X$ if and only if X is smooth.

(3) **Canonical bundle of $\mathbb{P}^n$.** Taking the top exterior of the Euler sequence, we get

$$\mathscr{O}_{\mathbb{P}^n}(-n-1) \cong \det \mathscr{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \cong \det \Omega_{\mathbb{P}^n/k} \otimes \det \mathscr{O}_{\mathbb{P}^n} \cong \det \Omega_{\mathbb{P}^n/k} = \omega_X.$$

(4) **Adjunction formula.** Let $Y \subseteq X$ be nonsingular, $\operatorname{codim} Y = 1$, and $\mathscr{L} = \mathscr{O}_X(Y)$. Then

$$\omega_Y = \omega_X \otimes \mathscr{L} \otimes \mathscr{O}_Y = \omega_X \otimes \mathscr{L}|_Y.$$

(5) **Bertini's theorem.** Let $X \subseteq \mathbb{P}^n_k$ be a nonsingular closed subvariety over $k = \overline{k}$. Then there is an open subset $U \subseteq \{\text{hyperplanes in } \mathbb{P}^n\} = \mathbb{P}(H^0(\mathbb{P}^n, \mathscr{O}(1)))$ such that if $H \in U$ then $X \not\subseteq H$ and $H \cap X$ is nonsingular. If $\dim X \geqslant 2$ then we can also ensure that $H \cap X$ is connected, which means it's a nonsingular variety (remember that connected $\implies$ irreducible in our case).

(6) Geometric genus is a birational invariant.

6.15. *Next.* Examples of hypersurfaces in $\mathbb{P}^n$ and applications of Euler sequence; Riemann-Roch theorem.

Lecture 7

Lecturer: Krishna Hanumanthu Date: 27.01.2023

7.1. Examples.

(1) Let $X = \mathbb{P}^n_k$, $n \geqslant 2$, $H \subseteq \mathbb{P}^n$ hyperplane, $\mathscr{O}_X(H) = \mathscr{O}_X(1)$. Consider the complete linear system $|dH|$ on X for $d > 0$. By Bertini's theorem, there exists $Y \in |dH|$ which is regular everywhere. In fact, Y can also be chosen irreducible. Hence, for all $d \geqslant 1$, there exists a nonsingular hypersurface $Y \subseteq \mathbb{P}^n_k$ of degree k.

(2) Let $Y \subseteq \mathbb{P}^n$, $n \geqslant 2$, Y a nonsingular hypersurface of degree d. By adjunction formula,

$$\omega_Y = \omega_{\mathbb{P}^n} \otimes \mathscr{O}_{\mathbb{P}^n}(d) \otimes \mathscr{O}_Y = \mathscr{O} = \mathscr{O}_{\mathbb{P}^n}(d-n-1)|_Y = \mathscr{O}_Y(d-n-1).$$

- $n = 2, d = 2$. Then $Y \subseteq \mathbb{P}^2$ is a conic and it's the image of the 2-uple embedding $\mathbb{P}^1 \to \mathbb{P}^2$.
- $n = 2, d = 3$. Then $\omega_Y = \mathscr{O}_Y$ and $p_g(Y) = 1$. Thus, $Y \not\cong \mathbb{P}^1$. This is called an **elliptic curve**.
- $n = 2, d \geqslant 4$. Then $\omega_Y = \mathscr{O}_Y(d-3)$, $d - 3 > 0$. Thus, $p_g(Y) = (d-1)(d-2)/2$. See Hartshorne [Har77, Ex II.8.4 (f)]. Therefore, curves of different degrees in $\mathbb{P}^2$ are not isomorphic.
- $n = 3, d = 1$. Then $\omega_Y = \mathscr{O}_Y(-3)$. Of course, $Y \cong \mathbb{P}^2$.
- $n = 3, d = 2$. Then Y is the **nonsingular quadric**, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Note that $\omega_Y = \mathscr{O}_Y(-2)$. Thus, $p_g(Y) = 0$ as $\mathscr{O}_Y(-2)$ has no global sections. Another way to see this is from the fact that $\mathbb{P}^1 \times \mathbb{P}^1$ is birational to $\mathbb{P}^2$. However, $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$, as seen by comparing divisor class groups.

- $n=3$, $d=3$. Y is called the **nonsingular cubic** in $\mathbb{P}^3$. And $\omega_Y \cong \mathscr{O}_Y(-1)$, $p_g(Y)=0$. In fact, Y is rational.
- $n=3, d=4$. Then $\omega_Y = \mathscr{O}_Y$ and $p_g(Y)=1$. These are called **K3 surfaces**.
- $n=3, d \geqslant 5$. Then $\omega_Y = \mathscr{O}_Y(d-4)$ where $d-4>0$. So, $p_g(Y)>0$. These are called surfaces of **general type**[8].
- $n=4$, $d \in \{3,4\}$. Then $p_g=0$, but these **3-folds** are *not* rational.
- n arbitrary, $d \geqslant n+1$. Then $\omega_Y = \mathscr{O}_Y(d-n-1)$ where $d-n-1 \geqslant 0$. So, $p_g(Y) \geqslant 1$. Thus, Y is not rational. Hence, there are nonrational varieties in all dimensions.

7.2. Serre duality (Hartshorne [Har77, Corollary III.7.7, Remark III.7.12.1]). — *Let X be a nonsingular projective variety of dimension n over $k=\overline{k}$ and $\mathscr{F}$ a vector bundle. There is a natural isomorphism of k-vector spaces*

$$H^i(X,\mathscr{F}) \cong H^{n-i}(X,\mathscr{F}^\vee \otimes \omega_X)^\vee.$$

7.3. Curves. Let X be a curve, i.e., a nonsingular projective integral k-variety of dimension 1. We have seen that

$$g(X) := p_a(X) = p_g(X) = \dim_k H^1(X,\mathscr{O}_X) := h^1(X,\mathscr{O}_X) = h^0(X,\omega_X).$$

Weil divisors are of the form $D=\sum_{P\in X} n_P P$, where P denotes a closed point. On a curve, there's exactly one non-closed point– the generic point.

7.4. *Notation.* $\ell(D) := h^0(X,\mathscr{O}_X(D))$.

Note that $\dim|D| = \ell(D)-1$.

7.5. Lemma. — *Let X be a curve and $D \in \operatorname{Weil} X$.*

(1) $\ell(D) \neq 0 \implies \deg D \geqslant 0$.
(2) $\ell(D)=0, \deg D = 0 \implies D \sim 0$

7.6. Riemann-Roch theorem. — *Let X be a curve of genus g and $D \in \operatorname{Weil} X$. Then*

$$\ell(D) - \ell(K-D) = \deg D + 1 - g,$$

where K is a divisor associated to the canonical bundle ω_X. It's called the ***canonical divisor****.*

Note that $\ell(K-D) = h^0(\mathscr{O}_X(K-D)) = h^0(\omega_X \otimes \mathscr{O}_X(D)^\vee) = h^1(\mathscr{O}_X)$ by Serre duality. So the Riemann-Roch theorem can be rephrased as

$$\chi(X,\mathscr{O}_X(D)) := h^0(\mathscr{O}_X(D)) - h^1(\mathscr{O}_X(D)) = \deg D + 1 - g.$$

The LHS is called the **Euler characteristic of** D, also denoted $\chi(D)$.

7.7. *Next.* Riemann-Hurwitz theorem.

Lecture 8

Lecturer: Krishna Hanumanthu Date: 30.01.2023

8.1. *Proof of Riemann-Roch theorem.* The case $D=0$ is trivial. Let $D\in \operatorname{Weil} X$ and $P \in X$. Then we will show that the theorem holds for D if and only if it holds for $D+P$. To prove this, it suffices to show that $\chi(D-P) = \chi(D)+1$. Take the closed subscheme exact sequence

$$0 \to \mathscr{O}_X(-P) \to \mathscr{O}_X \to \mathscr{O}_X|_P \to 0.$$

[8] Vakil [FOAG] defines a variety to be of general type when its Kodaira dimension is maximal, i.e., equal to its (Krull) dimension.

Twist by D:

$$0 \to \mathscr{O}_X(D-P) \to \mathscr{O}_X(D) \to \mathscr{O}_X|_P(D) \to 0.$$

Taking Euler characteristics and using $\mathscr{O}_X|_P(D) \cong \mathscr{O}_X|_P$, we get the desired result. □

8.2. *Examples and remarks.*

(1) Consider $X \subseteq \mathbb{P}^n$, a curve of degree d. Let $H \subseteq \mathbb{P}^n$ be a hyperplane and $D = X \cap H$, a divisor. Hartshorne [Har77, Exercise III.5.2] gives us that $\chi(\mathscr{O}_X(D)) = d + 1 - p_a$. This is a special case of Riemann-Roch because D is very ample.

(2) *Riemann-Roch problem.* Let X be a nonsingular projective variety and $D \in \operatorname{Weil} X$. Determine $\dim|nD|$ as a function of n. And in particular, determine its behaviour as $n \to \infty$. This is equivalent to asking about $h^0(X, \mathscr{O}_X(nD))$ for $n \gg 0$. Riemann-Roch theorem solves this problem for curves by Serre vanishing.

If $\deg D < 0$ then $\dim|nD| = -1$ for all $n \geqslant 1$. If $\deg D = 0$ then if $\dim|D| \neq 0$ is basically the set of all effective divisors linearly equivalent to D, and hence $D \sim 0$. If $\deg D > 0$ then we claim that $h^1(nD) = 0$ for $n \gg 0$. By Serre duality, $h^1(nD) = h^0(K - nD)$. Note that $\deg(K - nD) < 0$ for $n \gg 0$. Thus, $h^0(K - nD) = 0$ for large enough n. Thus, Riemann-Roch tells us that $h^0(nD) = n \deg D + 1 - g$ for large enough n.

(3) Let X be a curve of genus g. Then $\deg K = 2g - 2$ by Riemann-Roch and Serre duality.

(4) A divisor $D \in \operatorname{Weil} X$ is called **special** if $\ell(K - D) = h^1(D) > 0$. **Expected dimension** of $H^0(D)$ is defined as $\deg D + 1 - g$. By Riemann-Roch, $h^0(D)$ is at least the expected dimension. So, D is special if $h^0(D)$ is strictly bigger than the expected dimension of D. A divisor D is called **non-special** if $h^0(D) = \deg D + 1 - g$.

8.3. Claim. — *If $\deg D > 2g - 2$ then D is non-special.*

Proof. Obvious by Riemann Roch. □

(5) X is rational (which is same as being isomorphic to $\mathbb{P}^1$ for curves) if and only if $g(X) = 0$. Take any two points P and Q on X. If $g(X) = 0$ then $h^0(P - Q) - h^1(P - Q) = 1$ by Riemann-Roch. Note that $h^1(P - Q) = h^0(K - P + Q) = 0$ as $\deg(K - P + Q) = -2 < 0$. Thus, $h^0(P - Q) = 1$. This means $P \sim Q$ which implies $X \cong \mathbb{P}^1$.

(6) A curve X is called **elliptic** if $g(X) = 1$. In that case, $\deg K = 0$. Also $h^0(K) = g = 1 > 0$. Therefore, $K \sim 0$.

(7) *Group law of elliptic curves.* If X is an elliptic curve and $P_0 \in X$. Define $\operatorname{Pic}^0(X) := \{D \in \operatorname{Cl} X : \deg D = 0\}$. This has the structure of an algebraic variety. There is a bijection $f : X \to \operatorname{Pic}^0(X)$, $P \mapsto \mathscr{O}_X(P - P_0)$. Take $D \in \operatorname{Pic}^0(X)$. Apply Riemann-Roch to $D + P_0$:

$$\ell(D + P_0) - \ell(K - D - P_0) = 1 + 1 - 1 = 1.$$

Also, $\deg(K - D - P_0) = -1$. Therefore, $\ell(D + P_0) = 1$. Hence, there is an effective divisor E such that $E \sim D + P_0$. Therefore, $\deg E = 1 \implies E \sim Q$ for some $Q \in X$. So, f is a surjection. It's easy to see that it's also injective because a genus 1 curve cannot have two distinct linearly equivalent points.

Lecture 9

Lecturer: Krishna Hanumanthu Date: 03.02.2023

Let X and Y be curves, $f : X \to Y$ a finite morphism[9]

[9] In the algebraically closed setting, finite morphisms are always dominant. Therefore, it makes sense to talk about the corresponding extension of function fields.

9.1. Definition. The **degree** of f is defined to be $\deg f := [K(X) : K(Y)]$.

Let $P \in X$, $Q = f(P) \in Y$ be closed points. Then we have a map of DVRs: $f^\# \colon \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$. Let $t \in \mathcal{O}_{Y,Q}$ be a uniformizing parameter at Q. Let $e_P = \operatorname{val}_P f^\# t$.

9.2. Definition. We say f is **ramified at** P if $e_P > 1$. We call P a **ramification point** of f and $Q = f(P)$ a **branch point**.

- If $\operatorname{char} k = 0$, or $\operatorname{char} k = p > 0$ and $p \nmid e_P$, we say the ramification is **tame**,
- otherwise, the ramification is **wild**.
- e_P is called the **ramification index** of f at P.
- If $e_P = 1$ then f is **unramified** at P.

We call f **separable** if the extension $K(X)/K(Y)$ is separable.

9.3. Let $f \colon X \to Y$ be a finite morphism of curves. We have a pullback map

$$f^* \colon \operatorname{Weil} Y \to \operatorname{Weil} X, \quad Q \mapsto \sum_{f(P)=Q} e_P P.$$

9.4. *Remark.* We always have a pullback map for line bundles $f^* \colon \operatorname{Pic} Y \to \operatorname{Pic} X$. In the case of finite morphism of curves, these two maps are "same": $\mathcal{O}_X(f^*D) \cong f^*\mathcal{O}_Y(D)$.

9.A. Homework. Read Hartshorne [Har77, Proposition II.6.9]. If $f \colon X \to Y$ is a finite morphism of curves and $D \in \operatorname{Weil} Y$, then $\deg f^*D = \deg f \deg D$.

9.5. Proposition (Cotangent exact sequence for curves). — *Let $f \colon X \to Y$ be finite separable morphism of curves. Then we have an exact sequence of $\mathcal{O}_X$-modules*

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

Proof. We have right exactness by the usual cotangent exact sequence. Observe that the injectivity of $f^*\Omega_{Y/k} \to \Omega_{X/k}$ can be checked at the generic point, say η. Taking the stalk of the usual cotangent right exact sequence at the generic point, we obtain

$$(f^*\Omega_{Y/k})_\eta \to \Omega_{X/k,\eta} \to \Omega_{X/Y,\eta} \to 0.$$

Since localizations commute with Ω, we have $\Omega_{X/Y,\eta} = \Omega_{K(X)/K(Y)} = 0$. Therefore, $(f^*\Omega_{Y/k})_\eta \to \Omega_{X/k,\eta}$ is a surjection. Being a map of 1-dimensional vector spaces, it must be an isomorphism. □

9.6. Proposition (Hartshorne [Har77, Proposition IV.2.2]). — *Let $f \colon X \to Y$ be a finite separable morphism of curves. Then*

(1) $\Omega_{X/Y}$ is a torsion sheaf with support equal to the ramification points of f. As a consequence, f is ramified at only finitely many points.
(2) For all $P \in X$, the stalk $\Omega_{X/Y,P}$ is a principal $\mathcal{O}_{X,P}$-module of finite length equal to $\operatorname{val}_P \frac{\mathrm{d}t}{\mathrm{d}u}$.
(3) If f is tamely ramified at P then $\operatorname{length}_{\mathcal{O}_{X,P}} \Omega_{X/Y,P} = e_P - 1$.
(4) If f is wildly ramified at P then $\operatorname{length}_{\mathcal{O}_{X,P}} \Omega_{X/Y,P} > e_P - 1$.

9.7. *Next.* Hurwitz' theorem.

Lecture 10

Lecturer: Nabanita Ray Date: 06.02.2023

10.1. Definition. Let $f\colon X \to Y$ be a finite separable morphism of curves. The **ramification divisor** is defined as

$$R = \sum_{P \in \operatorname{Supp} \Omega_{X/Y}} \operatorname{length}_{\mathscr{O}_{X,P}} \Omega_{X/Y,P}[P].$$

Observe that the structure sheaf $\mathscr{O}_R$ of R, as a closed subscheme, is isomorphic to $\Omega_{X/Y}$.

10.2. Proposition. — *It follows that $K_X \sim f^* K_Y + R$. In particular, canonical divisors pull back to canonical divisors for unramified morphisms.*

Proof. Tensor the cotangent exact sequence with $\Omega_{X/k}^\vee$ and use $\mathscr{O}_R \cong \Omega_{X/Y}$. □

10.3. Hurwitz's Theorem. — *For a separable finite morphism $f\colon X \to Y$ of curves, we have*

$$2g(X) - 2 = (\deg f)(2g(Y) - 2) + \deg R.$$

Proof. Take degrees of the cotangent exact sequence. In other words, use the preceeding proposition and that degree is stable under linear equivalence. □

10.4. *Remark.* Degree of ramification divisor is always even.

10.5. Étale morphisms. Let $f\colon X \to Y$ be a morphism. For $f(x) = y$, we have the diagram:

$$\begin{array}{ccc} \widehat{\mathscr{O}_{Y,y}} & \longrightarrow & \widehat{\mathscr{O}_{X,x}} \\ \uparrow & & \uparrow \\ k(y) & \longrightarrow & k(x) \end{array}$$

Then f is **étale** if the above square is a "tensor diagram", i.e., $\widehat{\mathscr{O}_{X,x}} \cong \widehat{\mathscr{O}_{Y,y}} \otimes_{k(y)} k(x)$, and $k(x)/k(y)$ is separable, for all $x \mapsto y$.

10.6. Proposition. *Let $f\colon X \to Y$ be a morphsim. The following are equivalent:*

(1) *f is étale.*
(2) *f is smooth of relative dimension 0.*
(3) *f is flat and $\Omega_{X/Y} = 0$.*
(4) *f is flat and unramified*[10].

10.A. Homework. Hartshorne [Har77, Exercises III.10.3-4].

10.7. Definition. A scheme Y has an **étale cover** by X if there is a finite étale morphism $f\colon X \to Y$. If f is of the form $X = \bigsqcup_{\text{finite}} Y \to Y$ then X is called a **trivial** étale cover of Y. A scheme X is called **simply connected** if X has no nontrivial étale cover.

10.8. *Example.* $\mathbb{P}^1_k$ is simply connected. Suppose $f\colon X \to \mathbb{P}^1$ is an étale cover of $\mathbb{P}^1$. Then $X \to \operatorname{Spec} k$ is smooth of relative dimension 1 as $\mathbb{P}^1 \to \operatorname{Spec} k$ is smooth of relative dimension 1. Thus, $\dim X = 1$. Let X' be an irreducible component of X. By Hurwitz's theorem, $2g(X') - 2 = -2 \implies g(X') = 0$, which implies that $X' \cong \mathbb{P}^1$, and X is a finite disjoint union of projective lines.

10.9. Definition. Let $f\colon X \to Y$ be a finite morphism between curves. Then f is called **purely inseparable** if $K(X)/K(Y)$ is purely inseparable.

[10] Warning: Vakil [FOAG] defines unramified as finite-type and $\Omega_{X/Y} = 0$.

10.10. Frobenius morphism. Let X be a scheme all of whose stalks have characteristic $p > 0$. Then we define the **Frobenius morphism** $\operatorname{Frob}\colon X \to X$ as

(1) $\operatorname{Frob} = \operatorname{id}_X$ set-theoretically.
(2) $\operatorname{Frob}^\#\colon \mathcal{O}_X \to \mathcal{O}_X$ is the pth power map. That is, at stalk level, it is the Frobenius endomorphism.

The above definition has no reference to the base field of X. Let us fix an algebraically closed base field k of characteristic $p > 0$ from now onwards. Then note that Frob defined above is *not* a k-morphism because it is not necessarily k-linear at the level of stalks. We can make it a k-morphism.

$$\begin{array}{ccc} X & \xrightarrow{\text{Frob}} & X \\ \downarrow{\scriptstyle\pi} & & \downarrow \\ \operatorname{Spec} k & \xrightarrow[\text{Frob}]{} & \operatorname{Spec} k \end{array}$$

Define X_p to be the same scheme X but with the structure map $F \circ \pi$. This is same as defining X_p as the fibered product $X \times_k k$ where the map $k \to k$ is the Frobenius endomorphism. Then k acts on stalks of X_p via pth powers. Now, $\operatorname{Frob}'\colon X_p \to X$, defined similarly as above, *is* k-linear at stalks. This is called the ***k*-linear Frobenius morphism**.

10.11. *Observation.* X_p *is* isomorphic to X as a scheme over $\operatorname{Spec}\mathbb{Z}$, but they are not isomorphic as schemes over $\operatorname{Spec} k$.

10.12. Proposition. — $K(X_p) = K(X)^{1/p}$.

Proof. We know that $X_p = X \times_{k,\operatorname{Frob}} k$, where $\operatorname{Frob}\colon k \to k$ is $x \mapsto x^p$. Therefore, the function field of X_p is $K(X) \otimes_{k,\operatorname{Frob}} k$. Now, $K(X) \otimes_{k,\operatorname{Frob}} k \to K(X)^{1/p}$ given by $a \otimes \ell \mapsto \ell a^{1/p}$ is an isomorphism. □

10.13. *Observation.* $K(X)^{1/p}$ and $K(X)$ are isomorphic as fields, however, they are not isomorphic as k-algebras.

10.14. Proposition (Hartshorne [Har77, Proposition IV.2.5]). — *If $f\colon X \to Y$ is a purely inseparable finite morphism of curves, then $X \cong Y_{p^n}$ for some $n \geqslant 1$, and f is a repeated iteration of the k-linear Frobenius morphism. In particular, $g(X) = g(X_p)$.*

10.15. *Next.* Proof of the above proposition.

Lecture 11

Lecturer: Nabanita Ray Date: 10.02.2023

11.1. *Proof of Proposition 10.14.* We have $[K(X) : K(Y)] = p^n$ for some positive integer n. This comes from the fact that $K(X)$ and $K(Y)$ are algebraic extensions of $k(t)$. Hence, $K(X)^{p^n} \subseteq K(Y)$, which implies $K(X) \subseteq K(Y)^{1/p^n}$. Now, $K(X)$ and $K(Y_{p^n}) = K(X)^{1/p^n}$ are p^n-dimensional vector spaces over $K(Y)$. Therefore, $K(X)$ is forced to be equal to $K(Y)^{1/p^n}$. Thus, $X \cong Y_{p^n}$ follows due to the equivalence of category of curves with dominant morphisms and the category of function fields of curves. □

11.2. Inseparable morphisms are ramified everywhere. Let $f\colon X \to Y$ be an inseparable morphism between two curves. By field theory, such a map factors as

$$\begin{array}{ccccc} X & & \xrightarrow{\quad f \quad} & & Y \\ & \searrow_{\text{sep.}} & & \nearrow_{\text{purely insep.}} & \\ & & Y_{p^n} & & \end{array}$$

Therefore, it is enough to show that the k-linear Frobenius twist $\mathrm{Frob}'\colon Y = X_p \to X$ is ramified everywhere. We claim that $\Omega_X \cong \Omega_{X/Y}$. It suffices to check this at stalks. Consider the following diagram:

$$\begin{array}{ccccccc} f^*\Omega_{Y,y} & \longrightarrow & \Omega_{X,x} & \longrightarrow & \Omega_{X/Y,x} & \longrightarrow & 0 \\ \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & & & \\ f^*\mathscr{O}_{Y,y} & \longrightarrow & \mathscr{O}_{X,x} & & & & \end{array}$$

Let t be a local parameter at y. Then $f^*(\mathrm{d}t) = \mathrm{d}(f^*t) = \mathrm{d}t^p = pt^{p-1}\mathrm{d}t = 0$. Thus, $\Omega_{X,x} \cong \Omega_{X/Y,x}$, which completes the proof.

11.3. *Remark.* If $f\colon X \to Y$ is a nonconstant (finite) morphism between two curves then $g(X) \geqslant g(Y)$. By factoring the morphism into separable followed by purely inseparable morphisms, we can assume f is separable because purely inseparable morphisms don't change genera. Now apply Hurwitz's theorem.

11.4. Embeddings in Projective Space. Fix a curve X. The goals of this section is to show that X can be embedded in $\mathbb{P}^3$ and that there exists a morphism, birational onto its image, $\phi\colon X \to \mathbb{P}^2$. Additionally, $\phi(X)$ has at most finitely many nodes as singularities.

11.A. Exercise. Hartshorne [Har77, Exercise I.5.1,5.3,5.4,5.6].

11.5. Proposition (Criteria for base-point-freeness and very ampleness). — *Let D be a divisor on a curve X. Then*

(1) $\mathscr{O}_X(D)$ is base-point-free $\iff \dim|D-P| = \dim|D|-1$ for each $P \in X$.
(2) $\mathscr{O}_X(D)$ is very ample $\iff \dim|D-P-Q| = \dim|D|-2$ for each $P, Q \in X$.

Proof. See Hartshorne [Har77, Proposition IV.3.1], Vakil [FOAG, 20.2.7-10]. □

11.6. Useful Proposition. — *Let X be a curve, D be a divisor, and $g(X) = g$. Then*

(1) $\deg D \geqslant 2g \implies D$ is base-point-free.
(2) $\deg D \geqslant 2g+1 \implies D$ is very ample.

Proof. Apply Riemann-Roch and use the previous theorem. □

11.7. *Remarks.*

(1) $\deg D > 0 \iff D$ is ample.
(2) Let X, a curve, be embedded in $\mathbb{P}^n$ via the very ample divisor D. Then $\deg X = \deg D$.

11.B. Exercise. Hartshorne [Har77, Exercise II.6.2].

11.8. *Example.* Let X be a degree-4 curve in $\mathbb{P}^2$. Then observe that $\deg H|_X = 4$, for any line $H \subset \mathbb{P}^2$. Also, $g(X) = 3$. Therefore, a divisor $H|_X$ of degree less than $2g+1 = 7$ can give an embedding in projective space.

11.C. Exercise. If $g(X) = 1$ then $\deg D \geqslant 3 \iff D$ is very ample.

Lecture 12

Lecturer: Nabanita Ray Date: 13.02.2023

12.1. Any curve can be embedded in $\mathbb{P}^3$. Fix a curve X in $\mathbb{P}^n$.

12.2. Definition. For any two distinct points $P, Q \in X$, we call the line ℓ_{PQ} joining points P and Q as **secant line**. The union of all secant lines of X is called the **secant variety** and it is denoted $\operatorname{Sec} X$. There is a *unique* line ℓ_P in $\mathbb{P}^n$ tangent to X at a given point P. The union of all lines tangent to X is called the **tangent variety** and it is denoted $\operatorname{Tan} X$.

12.3. *Warning.* These are not varieties in the sense of Hartshorne. They are only locally closed.

12.4. Proposition. — *Let $\varphi \colon X \to \mathbb{P}^{n-1}$, $n \geqslant 2$, be a projection from $O \in \mathbb{P}^n \setminus X$. Then φ is a closed embedding if and only if $O \notin \operatorname{Tan} X \cup \operatorname{Sec} X$.*

Proof. The linear system corresponding to the projection map $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ is $\{H \in |\mathscr{O}_{\mathbb{P}^n}(1)| : O \in H\}$. So the linear system giving φ is the pullback of this linear system along the embedding $X \hookrightarrow \mathbb{P}^n$. It is routine to verify that this linear system separates points and tangent vectors if and only if the given hypothesis holds. □

12.5. Lemma. — $\dim(\operatorname{Tan} X \cup \operatorname{Sec} X) \leqslant 3$.

Proof. There are continuous surjections of topological spaces

$$(X \times X \setminus \Delta) \times \mathbb{P}^1 \to \operatorname{Sec} X, \quad (P, Q, t) \mapsto t \in \ell_{PQ},$$
$$X \times \mathbb{P}^1 \to \operatorname{Tan} X, \quad (P, t) \mapsto t \in \ell_P.$$

Therefore, $\dim \operatorname{Sec} X \leqslant 3$ and $\dim \operatorname{Tan} X \leqslant 2$. □

12.6. Corollary. — *Any curve can be embedded in $\mathbb{P}^3$.*

The next proposition studies projection of a curve X in $\mathbb{P}^3$ to $\mathbb{P}^2$.

12.7. Proposition (Hartshorne [Har77, Proposition IV.3.7]**). —** *Let $X \subset \mathbb{P}^3$ which is not contained in any plane. Let $O \in \mathbb{P}^3 \setminus X$ and $\varphi \colon X \to \mathbb{P}^2$ be the morphism given by projection from O. Then φ is a birational morphism and the image of φ has only finitely many nodes as singularities if and only if*

(1) *O belongs to at most finitely many secant lines.*
(2) *$O \notin \operatorname{Tan} X$.*
(3) *O doesn't belong to any multisecant of X. A **multisecant** is a line which intersects X in more than two distinct points set-theoretically.*
(4) *O doesn't belong to any **secant with coplanar tangents**. A secant with coplanar tangent lines is a secant joining two points P, Q of X, whose tangent lines ℓ_P and ℓ_Q lie in the same plane, or equivalently, ℓ_P and ℓ_Q intersect.*

Proof. (1) just ensures that φ is a birational morphism. If $P, Q \in X$ and O lies on the secant ℓ_{PQ}, then tangent lines ℓ_P and ℓ_Q get mapped to tangents to $\varphi(X)$ at $\varphi(P) = \varphi(Q)$. Hence, (2), (3), and (4) ensure that every line from O which intersects X cuts X in exactly two points, it is not tangent to X at either point, and tangent lines at P and Q are mapped to distinct lines. □

12.8. Proposition. — *Let X be a curve in $\mathbb{P}^3$ not contained in any plane. If every secant is a multisecant then any two tangents are coplanar.*

Proof. See Hartshorne [Har77, Proposition IV.3.8]. □

12.9. *Next.* Any curve is birationally equivalent to a plane curve with only nodes as singularities.

Lecture 13

Lecturer: Nabanita Ray Date: 17.02.2023

13.1. Proposition. — *Let X be a curve in $\mathbb{P}^3$ not contained in any plane. If either*

(1) every secant is a multisecant
(2) any two tangents are coplanar.

then there exists a point $A \in \mathbb{P}^3$ such that all tangents pass through A.

Proof. We proved (1) $\implies$ (2) in the last lecture. Pick $P, Q \in X$. Then $\ell_P, \ell_Q \subseteq H \subseteq \mathbb{P}^3$, where H is a plane. Let $\ell_P \cap \ell_Q = \{A\}$. As X is not contained in H, we must have $X \cap H$ is finite. Pick $R \in X \setminus (X \cap H)$. Let $\ell_P \cap \ell_R = \{B_1\}$ and $\ell_Q, \cap \ell_R = \{B_2\}$. As $\ell_R \not\subset H$ we have $B_1 = B_2$. Therefore, $B_1 = B_2 \in \ell_P \cap \ell_Q = \{A\}$. Hence, $U := \{P \in X : A \in \ell_P\}$ is clopen in X. Thus, $U = X$. □

13.2. Definition. A curve $X \subseteq \mathbb{P}^n$ is called **strange** if all tangents pass through a unique point $A \in \mathbb{P}^n$.

13.3. *Example.* Suppose our base field is of characteristic 2. Any conic $X \subseteq \mathbb{P}^2$ can be written as $V(y - x^2)$ in some affine patch. Then $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for all $P \in X$. This implies that slope of the tangent line is zero everywhere. Hence, all tangents to X pass through $A = [0:0:1]$.

13.A. Exercise. Is $y = x^p$ strange in characteristic $p > 0$? Show that this curve is not regular at $[0:1:0]$.

13.4. Theorem (Samuel). — *Only strange curves are line and conics in characteristic* 2.

Proof. Omitted. See Hartshorne [Har77, Theorem IV.3.9]. □

13.5. Theorem. — *Let X be a curve in $\mathbb{P}^3$ which is not contained in any plane. Then there exists a point $O \in \mathbb{P}^3 \setminus X$ such that X is birational to $\varphi(X)$, where $\varphi\colon X \to \mathbb{P}^2$ is the projection from O. Further, $\varphi(X)$ has only finitely many nodes as singularities.*

Proof. We do a Bertini-type dimension counting argument and apply Hartshorne [Har77, Proposition IV.3.7]. By our assumptions, X is not strange. Hence, there exists a pair $(P, Q) \in X \times X$ such that ℓ_{PQ} is not a multisecant. Also, there exists a pair $(P', Q') \in X \times X$ such that ℓ_P and ℓ_Q are not coplanar. Define

$$U = \{(P, Q) \in X \times X : \ \ell_{PQ} \text{ is not a multisecant}\},$$
$$V = \{(P, Q) \in X \times X : \ \ell_P \text{ and } \ell_Q \text{ are not coplanar}\}.$$

These sets are open and nonempty. Therefore, $\dim U^c \leqslant 1$ and $\dim V^c \leqslant 1$. Also,

$$A = \{t \in \mathbb{P}^3 : \ t \in \ell_{PQ}, (P, Q) \in U^c\} \text{ and } B = \{t \in \mathbb{P}^3 : \ t \in \ell_{PQ}, (P, Q) \in V^c\}$$

have dimensions at most 2. We have the following fact–

"(Hartshorne [Har77, Exercise II.3.7]) If $f\colon X \to Y$, $\dim X = \dim Y$, is a dominant morphism of integral finite-type k-schemes, then there exists an open set $U \subseteq Y$ such that $f^{-1}(U) \to U$ is finite[11]."

Consider the local morphism to the secant variety $\operatorname{Sec} X = \mathbb{P}^3$–

$$(X \times X \setminus \Delta) \times \mathbb{P}^1 \to \mathbb{P}^3, \quad (P, Q, t) \mapsto t \in \ell_{PQ}.$$

Using the generic finiteness fact, we get points of the desired type. □

13.B. Exercise (Hartshorne [Har77, Exercise IV.1.8])[12]. If C is a degree-d nodal curve with r nodes in $\mathbb{P}^2$ then its arithmetic genus is

$$p_a(C) = \frac{(d-1)(d-2)}{2} - r.$$

Hence, number of nodes is at most $(d-1)(d-2)/2$.

13.6. We define $V_{d,r} := \{C \subseteq \mathbb{P}^2 : \deg C = d,\ C \text{ has } r \text{ nodes}\}$. Then there is a correspondence, not necessarily one-one,

$$\{\text{all smooth curves}\} \longleftrightarrow \bigcup_{d\in\mathbb{N},\, 0\leqslant r\leqslant (d-1)(d-2)/2} V_{d,r}.$$

The map from right to left is given by blowing up at nodes.

13.7. *Next.* Elliptic curves.

Lecture 14

Lecturer: Krishna Hanumanthu Date: 27.02.2023

14.1. Elliptic curves. A nonsingular curve of genus 1 is called an **elliptic curve**. Fix an elliptic curve X.

14.2. *Example.* Any degree-3 plane curve in $\mathbb{P}^2$ is elliptic.

14.3. *Later.* Every elliptic curve can be embedded in $\mathbb{P}^2$.

We will assume $\operatorname{Char} k \neq 2$. Choose $P_0 \in X$. Consider $D = 2P_0$. By Riemann-Roch, $h^0(D) = 2$ as D is nonspecial because $\deg D \geqslant 2g-2$. Hence, $\dim |D| = 1$. As $\deg D \geqslant 2g$, D is also base-point-free. Therefore, $|D|$ gives a morphism $f\colon X \to \mathbb{P}^1$. Degree of f is 2 because $f^*\mathscr{O}_{\mathbb{P}^1}(1) = \mathscr{O}_X(D) \implies f^*Q \sim 2P_0$, where Q is any point in $\mathbb{P}^1$. By Hurwitz' theorem $\deg R = 4$, where R is the ramification divisor. Observe that $e_P = 1$ if the inverse image of $f(P)$ is two distinct points and $e_P = 2$ if the inverse image of $f(P)$ is a single point. Since we have assumed that characteristic of the base field is not 2, all ramification must be tame– $p \mid e_P$ is not possible. Therefore, R is four distinct points and P_0 is one of them. WLOG, $f(P_0) = 0 = [1:0]$. By performing a linear automorphism, we may assume that the other branch points are $0, 1, \lambda \in \mathbb{P}^1$, $\lambda \in k \setminus \{0, 1\}$.

14.4. *Remark.* λ is the cross-ratio of the branch points.

14.5. Definition. The **j-invariant** of X is defined as

$$j = j(\lambda) := 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}.$$

Our goal is to prove the following theorem:

14.6. Theorem. — Hartshorne [Har77, Theorem IV.4.1].

[11] This will be an assignment problem.

[12] This will be an assignment problem.

14.7. Lemma. — *Let $P, Q \in X$, not necessarily distinct. Then there exists an automorphism $\sigma \in \operatorname{Aut} X$ such that*

(1) $\sigma^2 = \mathrm{id}$.
(2) $\sigma P = Q$.
(3) *For all* $R \in X$, $R + \sigma R \sim P + Q$.

Proof. Note that $\dim|P+Q| = 1$ and $P+Q$ is base-point-free. So it gives a morphism $g\colon X \to \mathbb{P}^1$. Its degree is 2. So $K(X)/K(\mathbb{P}^1)$ is separable as $K(X)/K(\mathbb{P}^1)$ being purely-inseparable would mean genera of X and $\mathbb{P}^1$ are same. Therefore, $K(X)/K(\mathbb{P}^1)$ is Galois. Let $\sigma\colon K(X) \to K(X)$ be the nontrivial $K(\mathbb{P}^1)$-automorphism. Clearly $\sigma^2 = \mathrm{id}$. Observe that for all $R \in X, \sigma R \in g^{-1}(gR)$ because σ gives a $\mathbb{P}^1$-automorphism of X. Therefore, $\sigma(P) = Q$ because σ is nontrivial (see the following addendum for a justification). If $R \in X$ then $R + \sigma R$ is a fiber of g. Therefore, $R + \sigma(R) \sim P + Q$ because all fibers of g are linearly equivalent. □

14.8. *Addendum.* We embed X in $\mathbb{P}^2$ so that the map $X \to \mathbb{P}^1$ given by $|P+Q|$ is "projection on the x-axis". Consider the sequence

$$H^0(P) \subseteq H^0(P+Q) \subseteq H^0(2P+Q) \subseteq H^0(2P+2Q) \subseteq H^0(3P+2Q) \subseteq H^0(3P+3Q) \subseteq H^0(4P+3Q)$$

with the following sequence of bases

$$\{1\} \subseteq \{1,x\} \subseteq \{1,x,y\} \subseteq \{1,x,y,x^2\} \subseteq \{1,x,y,x^2,xy\} \subseteq \{1,x,y,x^2,x^3,xy\} \subseteq \{1,x,y,x^2,x^3,xy,x^2y,y^2\}.$$

Here, $x, y \in K(X)$ is such that $\mathrm{val}_Q\, x = -1$, $\mathrm{val}_P\, x = -1$, $\mathrm{val}_P\, y = -2$, and $\mathrm{val}_Q\, y = -1$. By Riemann-Roch, $h^0(D) = \deg D$ when $\deg D \geqslant 1$. Therefore, the image of X in $\mathbb{P}^2$ given by the closed embedding determined by the global sections $1, x, y$ of $H^0(2P+Q)$ is of the form

$$f(x,y,1) := y^2 + y(a_1x^2 + a_2x + a_3) + (a_4x^3 + a_5x^2 + a_6x + a_7) = 0,$$

for $a_i \in k$, with x and y scaled if need be. Removing the line at infinity and looking at (affine) coordinate rings, we have the commutative diagram

$$\begin{array}{ccc} k[x,y]/(f) & \xrightarrow{\sigma^*} & k[x,y]/(f) \\ & \nwarrow_{x \mapsto x} \quad \nearrow^{x \mapsto x} & \\ & k[x] & \end{array}$$

Therefore, σ^* must fix x and send y to either x or $a_1x^2 + a_2x + a_3 - y$ (this is obtained from Vieta's relations). These are the only two possibilities. Thus, σ either fixes everything or swaps each fiber of $X \to \mathbb{P}^1$.

14.9. Corollary. — $\operatorname{Aut} X$ *is transitive.*

14.10. Lemma. — *Given $f_1, f_2\colon X \to \mathbb{P}^1$ morphisms of degree 2, there exist automorphisms $\sigma \in \operatorname{Aut} X$ and $\tau \in \operatorname{Aut} \mathbb{P}^1$ such that $f_2 \circ \sigma = \tau \circ f_2$.*

$$\begin{array}{ccc} X & \overset{\sigma}{\dashrightarrow} & X \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ \mathbb{P}^1 & \overset{\tau}{\dashrightarrow} & \mathbb{P}^1 \end{array}$$

Proof. Let $P_i \in X$ be a ramification point of f_i, $i = 1, 2$. By the previous lemma there exists an automorphism $\sigma \in \operatorname{Aut} X$ such that $\sigma P_1 = P_2$. Note that f_1 is determined by $2P_1$ and f_2 is determined by $2P_2$. Also $f_2 \circ \sigma$ is given by $2P_1$. Therefore f_1 and $f_2 \circ \sigma$ differ by a linear automorphism $\tau\colon \mathbb{P}^1 \to \mathbb{P}^1$. □

14.11. Lemma. — Hartshorne [Har77, Lemma IV.4.5].

Lecture 15

Lecturer: Krishna Hanumanthu Date: 03.03.2023

15.1. Proposition. *Let X be an elliptic curve over k, a field of characteristic not equal to 2. Fix $P_0 \in X$. Then there exists a closed embedding $X \to \mathbb{P}^2$ such that the image is*

$$y^2 = x(x-1)(x-\lambda)$$

for $\lambda \in k$; P_0 maps to $[0:1:0]$, and the above λ is same as the λ defined earlier, upto an element of S_3 as in Hartshorne [Har77, Lemma IV.4.5].

Proof. First embed X in $\mathbb{P}^2$ using $|3P_0|$. Consider

$$k = H^0(\mathscr{O}_X) \subseteq H^0(\mathscr{O}_X(P_0)) \subseteq H^0(\mathscr{O}_X(2P_0)) \subseteq \cdots$$

By Riemann-Roch, $h^0(\mathscr{O}_X(nP_0)) = n$ for $n > 0$. Choose a basis $\{1, x\}$ of $H^0(\mathscr{O}_X(2P_0))$, and a basis $\{1, x, y\}$ of $H^0(\mathscr{O}_X(3P_0))$. Consider $1, x, y, x^2, xy, x^3, y^2 \in H^0(\mathscr{O}_X(6P_0))$. We have a linear dependence relation between them as $h^0(6P_0) = 6$. This k-linear relation must involve both x^3 and y^2 with nonzero coefficients. We may also assume by scaling that the coefficients of x^3 and y^2 are 1. So, the relation is of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

With standard (linear) manipulations, we can transform it to $y^2 = x(x-1)(x-\lambda)$. Now, the original embedding with $|3P_0|$ has to be done using the sections $\{1, x, y\}$. The first part is done. The image of P_0 is $[0:1:0]$ as x and y have poles at P_0 and $y^2 = x(x-1)(x-\lambda)$ has a unique point at infinity– $[0:1:0]$.

Consider $X \to \mathbb{P}^1$ defined by $[x:y:z] \mapsto [x:y]$. One can check that this is a degree-2 map sending P_0 to ∞ and it is branched at $0, 1, \lambda, \infty$. The proof is complete. □

15.2. Theorem. — Hartshorne [Har77, Theorem IV.4.1].

There is a bijective correspondence between elliptic curves over k modulo isomorphisms and k.

Lecture 16

Lecturer: Krishna Hanumanthu Date: 07.03.2023

16.1. Proof of Hartshorne [Har77, Theorem IV.4.1]**.**

(a) Let $P_1, P_2 \in X$ and the corresponding maps are $f_1, f_2 \colon X \to \mathbb{P}^1$. Then there is a commutative square

$$\begin{array}{ccc} X & \overset{\sigma}{\dashrightarrow} & X \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ \mathbb{P}^1 & \overset{\tau}{\dashrightarrow} & \mathbb{P}^1 \end{array}$$

where $\tau(\infty) = \infty$. Also, τ maps $\{0, 1, \lambda_1\}$ to $\{0, 1, \lambda_2\}$. Therefore, λ_1 and λ_2 are in the same orbit for the action of S_3. What remains to check is that the j-invariants are the same. This is a routine calculation.

(b) Let X and X' be elliptic curves with equal j-invariant. Let λ and λ' be the corresponding elements of $k^\times$. Think of $j(\lambda)$ as a morphism $\mathbb{P}^1 \to \mathbb{P}^1$. This is a finite morphism of degree-6. In fact, the extension $K(\mathbb{P}^1)/K(\mathbb{P}^1)$ is a degree 6 Galois extension with Galois group S_3. Hence, $j(\lambda) = j(\lambda') \iff \lambda, \lambda'$ are in the same orbit under S_3-action.

Now consider the embeddings of X, X' in $\mathbb{P}^2$. Their images are $y^2 = x(x-1)(x-\lambda)$ and $y^2 = x(x-1)(x-\lambda')$. Since λ, λ' are in the same orbit, there is a linear change of variable in x such that $\lambda = \lambda'$, which completes the proof.

(c) This is trivial because $j\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a nonconstant morphism, which in turn, has to be surjective. □

16.2. *Examples.*

(a) $y^2 = x^3 - x$. This is an elliptic curve when $\operatorname{char} k \neq 2$. The j-invariant comes out to be $2^6 \cdot 3^3 = 1728$.

(b) *"Fermat curve"* $x^3 + y^3 = z^3$. This is nonsingular when $\operatorname{char} k \neq 3$. After change of variables, completion of squares, we get $\lambda \in \{-\omega, \omega^2\}$, where ω is the primitive cube root of unity. Therefore, the j-invariant is 0.

16.3. Corollary (Hartshorne [Har77, Corollary IV.4.7]). — *Let X be an elliptic curve and $P_0 \in X$. Put $G = \operatorname{Aut}(X, P_0)$, the automorphisms of X fixing P_0. Then G is a finite group of order*

- 2 *if* $j \notin \{0, 1728\}$.
- 4 *if* $j = 1728$ *and* $\operatorname{char} k \neq 3$.
- 6 *if* $j = 0$ *and* $\operatorname{char} k \neq 3$.
- 12 *if* $j = 0$ *and* $\operatorname{char} k = 3$.

16.4. *Remark.* Curves of genus at least 2 have finite automorphism group.

16.5. Group structure on an elliptic curve. Let X be an elliptic curve, $P_0 \in X$. The map

$$X \to \operatorname{Pic}^0 X, \quad P \mapsto \mathscr{O}_X(P - P_0)$$

can be verified to be an bijection. So X inherits the group structure of $\operatorname{Pic}^0 X$. The identity element is P_0. Consider $X \xrightarrow{|3P_0|} \mathbb{P}^2$. Pick points $P, Q, R \in X \subseteq \mathbb{P}^2$. If P, Q, R are collinear, then observe that $P + Q + R \sim 3P_0$. Therefore, $P + Q + R = 0$ in the group structure. The converse is also easy to see.

16.6. Proposition. — *Let X be an elliptic curve and $P_0 \in X$ be the identity for the group structure. Then the maps $\rho\colon X \to X, P \mapsto -P$ and $\mu\colon X \times X \to X, (P, Q) \mapsto P + Q$ are morphisms of varieties.*

Proof. Apply Lemma 14.7 with $P = Q = P_0$, we get an automorphism $\sigma\colon X \to X$ such that $R + \sigma R \sim 2P_0$ $\implies \sigma R = -R$. Now σ is our ρ. For μ, first show that translations are morphisms: apply (4.2) with P, P_0 so we get an automorphism σ such that $R + \sigma R \sim P + P_0$. This gives that $\sigma R = P - R$. So, $\sigma \circ \rho$ is same as "translating by P".

16.A. Homework. Read the rest of the proof from Hartshorne [Har77]. □

16.7. *Next.* Canonical embedding.

Lecture 17

Lecturer: Krishna Hanumanthu Date: 10.03.2023

"You don't always get what you want. Sometimes, you have to compromise..."

17.1. *Some remarks.* Let X be an elliptic curve, $n \in \mathbb{Z}$. Then $n_X\colon X \to X, X \mapsto nX$ is a morphism of varieties and groups.

- If $n \neq 0$ then n_X is a finite morphism of degree n^2.
- $$\operatorname{Ker} n_X \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}, & \text{if } \gcd(n,p) = 1 \, or \operatorname{char} k = 0 \\ \mathbb{Z}/p\mathbb{Z} \text{ or } 0, & \text{if } n = p = \operatorname{char} k. \end{cases}.$$

In the latter cases, the "or" depends on the **Hasse invariant**.

- $n = 2$. If $P \in X$ has order 2, i.e., $2P = 0 = 2P_0$, then consider the morphism $X \to \mathbb{P}^1$ given by $2P_0$, assume $\operatorname{char} k \neq 2$, then P is a ramification point. Therefore, 2_X is a finite morphism and its kernel is $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$.
- $n = 3$. If $3P = 0$ then $3P = 3P_0$. Consider the closed embedding $X \to \mathbb{P}^2$ given by $|3P_0|$. Then P is an **inflection point** of X. Hartshorne [Har77, Exercise IV.2.3] says that X has exactly 9 inflection points when $\operatorname{char} k \neq 2, 3$. Thus, $\operatorname{Ker} 3_X = (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$. The line joining two inflection points intersects X at another inflection point.

17.2. Canonical embedding. If the canonical divisor K_X gives a closed embedding then we call it the canonical embedding. The morphism, if any, given by K_X is called the canonical morphism.

17.3. Lemma. — *If $g \geqslant 2$ then $|K_X|$ has no base points.*

Proof. Apply the numerical criterion for base-point-freeness and Riemann-Roch. □

17.4. Definition. A curve X of genus at least 2 is called **hyperelliptic** if there is a degree-2 finite morphism $X \to \mathbb{P}^1$.

If $g = 2$ then $|K_X|$ cannot be very ample. However, the canonical morphism gives a degree-2 finite morphism to $\mathbb{P}^1$. Thus, any genus-2 curve is hyperelliptic.

17.5. *Notation.* The symbol g^r_d denotes a linear system of dimension r and degree d.

17.6. Proposition. — *Let X be a curve of genus $\geqslant 2$. Then the canonical morphism is a closed embedding if and only if X is not hyperelliptic.*

Proof. Use the numerical criterion for very-ampleness and Riemann-Roch. □

17.A. Trivial exercise. If X is a curve, D is a divisor, then $\dim|D| \leqslant \deg D$. Equality holds if and only if $D = 0$ or $g = 0$.

17.7. Definition. If X is a non-hyperelliptic curve of genus $g \geqslant 3$. The embedding given by $|K_X|$ is the canonical embedding of X. The image $X' \subseteq \mathbb{P}^{g-1}$ has degree $2g-2$ and is called a **canonical curve**.

17.8. *Example.* Let $X \subseteq \mathbb{P}^2$, a curve, $\deg X = 4$. Then $\omega_X = \mathscr{O}_X(1)$. Thus, X is a **canonical curve**. Consequently, it's not hyperelliptic.

17.9. *Useful Example (Hartshorne [Har77, Example IV.5.2.2]).* Let X be a non-hyperelliptic curve of genus $g = 4$. The canonical embedding is, let's say, $f\colon X \to \mathbb{P}^3$. It has degree 6. The goal is to show

- X is contained in a unique quadric hypersurface $Q \subseteq \mathbb{P}^3$.
- There exists a cubic surface $F \subseteq \mathbb{P}^3$ such that X is the complete intersection of Q and F.

We have $X \subseteq \mathbb{P}^3$, degree 6, genus 4. Let $\mathscr{I}_X$ be the ideal sheaf. Then we have an exact sequence

$$0 \to \mathscr{I}_X \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_X \to 0.$$

The fact that X is contained in a unique quadric hypersurface $Q \subseteq \mathbb{P}^3$ translates to saying $\mathscr{I}_X(2)$ has a global section. Twist the exact sequence by 2 and take global sections:

$$0 \to \Gamma(\mathbb{P}^3, \mathscr{I}_X(2)) \to \Gamma(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)) \to \Gamma(X, \mathscr{O}_X(2)).$$

Lecture 18

Lecturer: Krishna Hanumanthu Date: 13.03.2023

18.1. *(continued) Useful Example.* We have $h^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2)) = 10$, and by Riemann-Roch, $h^0(X, \mathscr{O}_X(2)) = 9$. Therefore, $\mathscr{I}_X(2)$ has a nonzero global section, i.e., there exists a quadric $Q \subseteq \mathbb{P}^3$ containing Q. It must be irreducible and reduced. Further, this Q is unique because if it is contained in another integral quadric Q' then X would be contained in the degree-4 curve $Q \cap Q'$, a contradiction. Similarly, we can show that $h^0(\mathbb{P}^2, \mathscr{I}_X(3)) \geqslant 5$. The cubic forms in $\Gamma(\mathbb{P}^2, \mathscr{I}_X(3))$ that are divisible by the quadratic form above form a subspace of dimension 4. Thus, there is a cubic surface F such that X is the complete intersection of Q and F.

18.A. Easy exercise. Every non-hyperelliptic curve of genus 3 is a quartic in $\mathbb{P}^2$.

18.2. Proposition . — *Let X be a hyperelliptic curve of genus $g \geqslant 2$. Then X has a unique g^1_2. If $f_0 \colon X \to \mathbb{P}^1$ is the corresponding morphism of degree 2, then the canonical morphism $f \colon X \to \mathbb{P}^{g-1}$ consists of f_0 followed by the $(g-1)$-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^{g-1}$. In particular, the image $X' = f(X)$ is a rational normal curve of degree $g-1$, and f is a morphism of degree 2 onto X'. Finally every effective canonical divisor on X is a sum of $g-1$ divisors in the unique g^1_2, we so write $|K| = \sum_1^{g-1} g^1_2$.*

Proof. See [Har77, Proposition IV.5.3] ☐

18.3. *Alternative explanation for uniqueness of g^1_2.* Let $\mathscr{L}$ be a line bundle corresponding to a hyperelliptic map $f \colon C \to \mathbb{P}^1$. I claim that $\mathscr{L}^{\otimes(g-1)} \cong K$. The morphism corresponding to $\mathscr{L}^{\otimes(g-1)}$ is

$$C \xrightarrow{|\mathscr{L}|} \mathbb{P}^1 \xrightarrow{|\mathscr{O}_{\mathbb{P}^1}(g-1)|} \mathbb{P}^{g-1}.$$

Note that $\mathscr{L}^{\otimes(g-1)}$ has degree $2g-2$. Also, it must have at least g sections because the image of the above composition is nondegenerate. By Riemann-Roch, $K \otimes (\mathscr{L}^{\otimes(g-1)})^\vee$ is a degree 0 line bundle with at least one section. Thus, $K \otimes (\mathscr{L}^{\otimes(g-1)})^\vee \cong \mathscr{O}_C$ and it follows that $\mathscr{L}^{\otimes(g-1)} \cong K$. Now if there are two g^1_2's, let's say $\mathscr{L}_1$ and $\mathscr{L}_2$. Then we can 'reconstruct' the hyperelliptic cover by considering the canonical map. It is a double cover of a degree $g-1$ rational normal curve. Thus, the maps corresponding to $\mathscr{L}_1$ and $\mathscr{L}_2$ are same, which means $\mathscr{L}_1 \cong \mathscr{L}_2$, that is, the g^1_2 is unique.

Lecture 19

Lecturer: Krishna Hanumanthu Date: 17.03.2023

"My daughter had a math exam. She calculated the greatest common divisor of two natural numbers, say $8 = 2 \times 2 \times 2$ and $28 = 2 \times 2 \times 7$, as 2 instead of 2×2."

"Scheme-theoretic intersection of Weil divisors in $\operatorname{Spec}\mathbb{Z}$*!"*

19.A. Question. Is g^1_2 unique at the level of linear system of divisors or at the level of divisor classes?

19.1. *Hartshorne's proof for $(g-1)g^1_2 \subseteq |K|$.* Any $g-1$ points of X' determine a hyperplane section in $\mathbb{P}^{g-1}$. So, $|K| = \sum_{i=1}^{g-1} g^1_2$. ☐

19.2. Clifford's Theorem. — *Let D be an effective special divisor on X. Then we have*

$$\dim|D| \leqslant \frac{1}{2}\deg D.$$

Furthermore, equality holds if and only if $D \in \{0, K\}$ or X is hyperelliptic and D is a multiple of the unique g^1_2.

19.3. Lemma. — *Let D, E be effective divisors on X. Then*

$$\dim|D| + \dim|E| \leqslant \dim|D+E|.$$

Proof. The standard inclusion $H^0(X, \mathcal{O}_X(D)) \times H^0(X, \mathcal{O}_X(E)) \hookrightarrow H^0(X, \mathcal{O}_X(D+E))$ gives a morphism $|D| \times |E| \to |D+E|$ with finite fibers. □

19.B. Exercise. When is the above inequality strict?

19.4. *Proof of Clifford's Theorem.* Equality obviously holds when $D \in \{0, K\}$. Let D is effective and special so $K-D$ is effective. We have from our lemma that

$$\dim|D| + \dim|K-D| \leqslant \dim|K| = g-1.$$

and Riemann-Roch gives

$$\dim|D| - \dim|K-D| \leqslant \deg D - g + 1.$$

Adding, we get the desired bound. Now, assume $\dim|D| = \frac{1}{2}\deg D$, $D \neq 0, K$. We induct on $\deg D$. Base case is $\deg D = 2$, where D is of course the unique g^1_2. Now let $\deg D \geqslant 4$. Then $\dim|D| \geqslant 2$. Pick $E \in |K-D|$ and fix $P, Q \in X$ such that $P \in \operatorname{Supp} E$ and $Q \notin \operatorname{Supp} E$. Since $\dim|D| \geqslant 2$ there exists $D_1 \in |D|$ such that $P, Q \in \operatorname{Supp} D_1$. For $\dim|D-P-Q| \geqslant \dim|D|-2 \geqslant 0$. Now, let $D' = D \cap E$, the scheme-theoretic intersection. We have $Q \in \operatorname{Supp} D, Q \notin \operatorname{Supp} E$, hence $\deg D' < \deg D$. We claim that $\dim|D'| = \frac{1}{2}\dim D'$. We have the exact sequence

$$0 \to \mathcal{O}_X(D') \to \mathcal{O}_X(D) \oplus \mathcal{O}_X(E) \to \mathcal{O}_X(D+E-D') \to 0.$$

Therefore, $\dim|D| + \dim|E| \leqslant \dim|D'| + \dim|D+E-D'|$ by left-exactness of global section functor. But, the LHS is just $g-1$ because equality holds everywhere. The RHS is at most $\dim|D+E| = \dim|K| = g-1$. Therefore, equality holds everywhere and D' is a multiple of g^1_2. Consequently, X is hyperelliptic by induction hypothesis. Consider the linear system $|D| + (g-1-\dim|D|)g^1_2$. It has degree $2g-2$ and dimension at least $g-1$ by the lemma. Hence, it must be same as the canonical system. This finishes the proof. □

19.5. *Next.* Classification of curves.

Lecture 20

Lecturer: Krishna Hanumanthu Date: 20.03.2023

20.1. Classification of curves. Classification problem of curves of genus g.

- $g = 0$. Only $\mathbb{P}^1$.
- $g = 1$. Parameter space is k.
- $g \geqslant 2$. Much more difficult.

Subdividing $\mathcal{M}_g$, the moduli space of curves of genus g, according ot whether the curve admits linear systems of certain degrees and dimensions is useful. For instance, whether the curve admits a g^1_2, i.e., whether the curve is hyperelliptic. More generally, we may ask which curves admit a g^1_d for some $d \geqslant 2$. A curve X is called **trigonal** if it admits a g^1_3.

20.2. *Facts (Kleiman-Laksov).* Let X be a curve of genus g.

- For any $d \geqslant \frac{1}{2}g + 1$, X has a g^1_d.
- For any $d < \frac{1}{2}g + 1$, there are curves without any g^1_d.

20.3. *Examples.* We consider $g = 3, 4$. Every curve of genus 3 or 4 has a g^1_3; if X is hyperelliptic then there's nothing to do, otherwise, use the canononical embedding when X is nonhyperelliptic. Also, there are nonhyperelliptic curves of these genera.

20.A. EXERCISE. Prove that there are non-hyperelliptic curves of every genus.

In fact, there are nonhyperelliptic curves of every $g \geqslant 3$.

- For $g = 3$, use the map given by $\mathscr{O}_X(K - P)$ for some $P \in X$.
- For $g = 4$, consider the canonical embedding $X \hookrightarrow \mathbb{P}^3$, degree 6. Then X is contained in a quadric Q. It is well known that if Q is nonsingular then $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. In this case, X corresponds to the $(3,3)$ divisor on Q. We know that Q has two families of lines, each parametrized by $\mathbb{P}^1$. Intersecting each family with X gives a g^1_3. Therefore, X has two g^1_3. Secondly, if Q is singular, Q is a quadric cone. Then Q has a family of lines parametrized by $\mathbb{P}^1$. This family will restrict to a unique g^1_3 to X.
- $g = 5$. For $d \geqslant 4$, every genus 5 curve has a g^1_4. There are curves of genus 5 which do not have a g^1_2 or g^1_3. Let X be a nonhyperelliptic curve of genus 5. Let us assume X is non-hyperelliptic. Then the canonical embedding gives a degree 8 map $X \hookrightarrow \mathbb{P}^4$.

  Claim. *X has a g^1_3 if and only if X has a trisecant for a canonical embedding.*

  Proof. Let $P, Q, R \in X$. By Riemann-Roch,
  $$\dim|P + Q + R| = \dim|K - P - Q - R| - 1.$$
  On the other hand, $|K - P - Q - R|$ is exactly the linear system of hyperplane sections containing P, Q, R. The dimension of $|K - P - Q - R|$ is equal to dimension of hyperplanes in $\mathbb{P}^4$ containing P, Q, R. So, the $\dim|P + Q + R| = 1$ happens if and only if P, Q, R are collinear. □

  Claim. *There exists $X \subseteq \mathbb{P}^4$ of degree 8 with no trisecants.*

  Proof. Take X to be the complete intersection of three quadric hypersurfaces, $X = Q_1 \cap Q_2 \cap Q_3$. The canonical bundle of X is $\mathscr{O}_X(1)$. Therefore, X is a canonical curve and genus of X is 5. We claim that X has no trisecants. If a line L intersects X at three points, then L interects each of Q_i at three points, then $L \subseteq Q_1 \cap Q_2 \cap Q_3 = X$ (Bezout). Contradiction! Thus, X has no g^1_3. □

  Claim. *The above X has a g^1_4.*

  Proof. Pick $P \in X$. Consider the projection $X \hookrightarrow \mathbb{P}^4 \to \mathbb{P}^3$ from point P. The degree of the image, say X', is 7. Then X' is nonsingular because X has no trisecants. Now, X' has trisecants because otherwise we can project it again to $\mathbb{P}^2$ violating degree-genus formula. Let Q, R, S lie on a trisecant of X'. Their inverse images along with P form four points which lie on a plane (2-dimensional linear subspace) in $\mathbb{P}^4$. This gives a g^1_4 by considering $P + Q + R + S$. □

20.B. QUESTION. Is every non-hyperelliptic curve of genus 5 a complete intersection of nonsingular quadrics in $\mathbb{P}^4$? [13]

20.C. HOMEWORK. Read Hartshorne [Har77, Proposition IV.6.1] and Hartshorne [Har77, Pages 346-367].

[13] No! Blow up a plane nodal quintic.

Lecture 21

Lecturer: Krishna Hanumanthu Date: 21.03.2023

21.1. Classification of curves in $\mathbb{P}^3$. Given (d,g), is there a curve $X \subseteq \mathbb{P}^3$ of degree d and genus g? Or in other words, given a curve of genus g, is there a linear system $\mathscr{L}$ which has $\dim \geqslant 3$ and which is very ample.

21.2. Halphen's Theorem. — *A curve X of genus $g \geqslant 2$ has a nonspecial very ample D of degree d if and only if $d \geqslant g+3$.*

Proof. Suppose D is a nonspecial very ample divisor of degree d. By Riemann-Roch, $\dim|D| = d-g$. Therefore, $d-g \geqslant 2$. If $d-g=2$ then X is a embeds as a plane curve of degree d. We know that $\omega_X = \mathscr{O}_X(d-3)$. This contradicts the fact that D is nonspecial.

The rest of the proof in Hartshorne's text is sloppy and unclear. See https://mathoverflow.net/a/410071. □

Lecture 22

Lecturer: Krishna Hanumanthu Date: 22.03.2023

22.1. Corollary. — *There exists a curve X of degree d and genus g in $\mathbb{P}^3$ whose hyperplane section D is nonspecial, if and only if*

- $g=0$ *and* $d \geqslant 1$,
- $g=1$ *and* $d \geqslant 3$, *or*
- $g \geqslant 2$ *and* $d \geqslant g+3$.

Proof. The first two parts are obvious. The third one is same as Halphen's Theorem. □

22.2. Proposition. — *If X is a nondegenerate curve in $\mathbb{P}^3$ for which the hyperplane section D is special, then $d \geqslant 6$ and $g \geqslant \frac{1}{2}d+1$. Furthermore, the only such curve with $d=6$ is the canonical curve of genus 4.*

Proof. See Hartshorne [Har77, Proposition IV.6.3]. □

22.3. Castelnuovo's Theorem. — *Let X be curve of degree d and genus g in $\mathbb{P}^3$, which is not contained in any plane. Then $d \geqslant 3$, and*

$$g \leqslant \begin{cases} \frac{1}{4}d^2 - d + 1, & \text{if } d \text{ is even} \\ \frac{1}{4}(d^2-1) - d + 1, & \text{if } d \text{ is odd.} \end{cases}$$

Furthermore, the equality is attained for every $d \geqslant 3$, and any curve for which equality holds lies on quadric surface.

22.4. *Example.* Let $d=10$. For every $0 \leqslant g \leqslant 7$, we have a curve $X \subseteq \mathbb{P}^3$ of genus g and degree 10. If $g=0$ then we can do $\mathbb{P}^1 \xrightarrow{\text{Veronese}} \mathbb{P}^{10} \xrightarrow{\text{repeated projections}} \mathbb{P}^3$. If $g=1$, then we can use a degree 10 point to get an embedding $X \to \mathbb{P}^9 \xrightarrow{\text{repeated projections}} \mathbb{P}^3$. Halphen's theorem gives $2 \leqslant g \leqslant 7$. However, Castelnuovo's bound says $g \leqslant 16$ and that $g=16$ is attained. Consider a $(7,3)$ type curve $X \subseteq Q := \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ then we get a degree $7+3=10$ and genus $7 \cdot 3 - 7 - 3 + 1 = 12$ curve.

22.5. *Proof of Castelnuovo's theorem.* If $d = 2s$ then take $X = (s,s) \subseteq Q \subseteq \mathbb{P}^3$ and if $d = 2s+1$ we take $X = (s, s+1) \subseteq \mathbb{P}^3$. These curves show that equality can be attained. Let $D = P_1 + \cdots + P_d$ be a hyperplane section of X such that all these points are distinct and no three of the P_i are collinear (Hartshorne [Har77, Exercise IV.3.9]). We will estimate $\dim|nD| - \dim|(n-1)D|$.

Claim. *Let* $n \geqslant 1$. *For each* $i = 1, \ldots, \min(d, 2n+1)$, P_i *is not a base point of* $|nD - P_1 - \cdots - P_{i-1}|$.

It suffices to construct a degree-n surface that contains $P_1, \ldots, P_{i-1}$ but not P_i. We find a union of n planes as follows– take the first plane to contain P_1 and P_2 but no other P_i; take the second plane to contain P_3 and P_4 but no other P_j, and so on... From this, it follows that $\dim|nD| - \dim|(n-1)D| \geqslant \min(d, 2n+1)$ because we can delete at least $\min(d, 2n+1)$ non-base-points from nD to get to $(n-1)D = nD - P_1 - \cdots - P_d$. By Riemann-Roch, we have $\dim|nD| = nd - g$ for all large n. Telescoping the difference inequality, we get $nd - g = \dim|nD| \geqslant r(r+2) + (n-r)d$, where $r = \lfloor \frac{1}{2}(d-1) \rfloor$. Simplifying, this gives us the desired bound.

When equality holds, equality must hold everywhere, so we have $\dim|2D| \leqslant 8$ in particular. Now one can verify that $H^0(\mathbb{P}^3, \mathscr{I}_{X/\mathbb{P}^3}(2))$ is nonzero by twisting and taking cohomology of the closed subscheme exact sequence $0 \to \mathscr{I}_{X/\mathbb{P}^3} \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_X \to 0$. □

22.6. *Remark.*

- For plane curves, $g = (d-1)(d-2)/2$.
- A complete intersection $F_1 \cap F_2 = X \subseteq \mathbb{P}^3$ of degrees (a,b) satisfies $\deg X = ab$ and $g(X) = \frac{1}{2}ab(a+b-4)$.
- For every (a,b)-type curve on $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$, we have $d = a+b$ and $g = ab - a - b + 1$.
- Let Q be a singular quadric in $\mathbb{P}^3$. If $d = 2a$, we may take $X = Q \cap X$, where X is a degree-a hypersurface, then the genus will be $g(X) = a^2 - 2a + 1$. If $d = 2a+1$ and $X \subseteq Q$, we can achieve $g = a^2 - a$.

22.7. Classification of curves of degree $\leqslant 7$ in $\mathbb{P}^3$.

- $d = 1$. $\mathbb{P}^1$
- $d = 2$. conic in $\mathbb{P}^2$
- $d = 3$. elliptic curve, twisted cubic .
- $d = 4$. plane quartic, rational quartic curves, elliptic quartic curves (complete intersection of two quadrics).
- $d = 5$. plane quintic, and there are curves with nonspecial $\mathscr{O}_X(1)$ with $g = 0, 1, 2 \leqslant d - 3$.
- $d = 6$. plane sextic, and there are $\mathscr{O}_X(1)$ nonspecial curves with $g \leqslant d - 3 = 3$, canonical curve of genus 4 (this is equal to the complete intersection of a quadric and a cubic surface).
- $d = 7$. plane septic, and there are nonspecial $\mathscr{O}_X(1)$ curves of genera $0, 1, 2, 3, 4$. There is a curve of type $(3,4)$ on a smooth quadric, which has $g = 6$. By Castelnuovo, this is the maximum possible genus for a degree 7 curve.

22.A. Question. Does there exist a a curve of degree 7 with genus 5 in $\mathbb{P}^3$? It does! Read Hartshorne [Har77, Page 353].

Lecture 23

Lecturer: Nabanita Ray Date: 27.03.2023

23.1. Surfaces. A **surface** is a projective, smooth, 2-dimensional k-variety, where k is algebraically closed. Examples: $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, nonsingular hypersurfaces in $\mathbb{P}^3$. By **curve**, we mean an effective Cartier divisors on a surface. Goals of the upcoming few lectures:

- Intersection theory on a surface

- Riemann-Roch for surfaces
- Hodge index theorem, etc.
- Ruled surfaces
- Monoidal transformations (projective bundles, blow-ups,...)

23.2. Intersection theory on surfaces. For C, D are two curves in $\mathbb{A}^2$, then we defined the intersection multiplicity of $P \in C \cap D$ as $(C \cdot D)_P = \operatorname{length} \mathscr{O}_P/(f, g)$, where $C = V(f)$ and $D = V(g)$. We then define

$$C \cdot D = \sum_{P \in C \cap D} (C \cdot D)_P.$$

If the local equations for C and D at point $P \in C \cap D$ generate the maximal ideal $\mathfrak{m}_P$ of the stalk, then the intersection is called **transversal**.

23.3. *Remark.* If two curves intersect transversally at a point P then they are regular at P.

X always denotes a surface from now onwards.

23.4. Lemma. — *Let C be a smooth curve and D be any curve. Moreoever, C and D intersect transversally. Then $\#(C \cap D) = \deg_C \mathscr{O}_X(D) \otimes \mathscr{O}_C = \deg_C D|_C$*

Proof. Consider $0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$. Tensor by $\mathscr{O}_C$ and use that $\mathscr{O}_D \otimes_{\mathscr{O}_X} \mathscr{O}_C = \mathscr{O}_{C \cap D}$ to get $\mathscr{O}_X(-D) \otimes \mathscr{O}_C \cong \mathscr{O}_X(-C \cap D)$. This gives the desired result by taking degrees. □

23.5. Lemma. — *Let $C_1, \ldots, C_r$ be curves on surface X and D be a very ample divisor on X. Then almost all $D' \in |D|$ are irreducible, nonsingular and meet each of the C_i transversally.*

Proof. Apply Bertini to X and each of the curves. □

23.6. Definition. Let C and D be two very ample divisors on X. Define $C \cdot D = \#(C' \cap D')$ where we take $C' \in |C|$ and $D' \in |D|$ such that D' and C' intersect transversally. This is well-defined by Lemma 23.4.

23.7. Intersection product for arbitrary curves. Define $\operatorname{Vamp}(X) = \{D \in \operatorname{Weil} X : D \text{ is very ample}\}$. Consider $\operatorname{Vamp} X \times \operatorname{Vamp} X \to \mathbb{Z}$, $(C, D) \mapsto C \cdot D$. One can verify that this is symmetric, invariant under linear equivalence, and additive in both arguments. We can generalize this notion to arbitrary curves. Let C and D be any curves on X and H be an ample divisor. Then $C + nH, D + nH, nH$ are very ample for $n \gg 0$ (Vakil [FOAG, Exercise 17.6.C]). Choose

- $C' \in |C + nH|$ smooth and irred.
- $D' \in |D + nH|$ smooth and irred and transversally to C'.
- $E' \in |nH|$ smooth, irred, and transversal to D'.
- F' in nH smooth, irred, transversal to C', E'.

Then $C \sim C' - E'$ and $D \sim D' - F'$. Now finally, define $C \cdot D = C' \cdot D' - C' \cdot F' - D' \cdot E' + E' \cdot F'$. One can check that this is a well-defined map. Thus, we have an extended map $\operatorname{Weil} X \times \operatorname{Weil} X \to \mathbb{Z}$.

23.8. *Remark.* Any divisor can be written as the difference two very ample divisors.

23.A. Homework. Riemann-Roch for singular curves (Hartshorne [Har77, Exercise IV.1.9]).

23.9. *Remark.* Let C and D be any curves, possibly singular. Then $C \cdot D = \deg \mathscr{O}_X(D) \otimes \mathscr{O}_C$. Write $D \sim D' - F'$ where D', F' are very ample and transversal to C. Then

$$\deg D = \deg[(\mathscr{O}_X(D') \otimes \mathscr{O}_C) \otimes_{\mathscr{O}_C} (\mathscr{O}_X(-F') \otimes \mathscr{O}_C)] = \#(D' \cap C) - \#(F' \cap C) = \deg(\mathscr{O}_X(C) \otimes \mathscr{O}_{D'}) - \deg(\mathscr{O}_X(C) \otimes \mathscr{O}_{F'})$$

Lecture 24

Lecturer: Nabanita Ray Date: 31.03.2023

24.1. *Recall.* If $p \in C \cap D$ and C, D don't have any common component, then we can define

$$(C \cdot D)_P = \dim_k \mathscr{O}_{X,p}/(f_p, g_p).$$

24.2. Theorem. — *If C and D don't have any common component, then $C \cdot D = \sum_{p \in C \cap D} (C \cdot D)_P$.*

Proof. Take Euler characteristics of $0 \to \mathscr{O}_X(-D) \otimes \mathscr{O}_C \to \mathscr{O}_C \to \mathscr{O}_{C \cap D} \to 0$ and use the fact that $\mathscr{O}_{C \cap D}$ has finite support. From this, we observe that $C \cdot D$ depends only on the linear equivalence class of D, and by symmetry, on the linear equivalence class of C. We now replace C and D by nonsingular transversal curves and finish the proof. ☐

24.A. Exercise. Let $C \subseteq X$ be a curve and $D \in \operatorname{Weil} X$. Then $\deg C|_D = C \cdot D$.

The self intersection number of a curve C is $C^2 = C \cdot C = \deg \mathscr{O}_X(C) \otimes \mathscr{O}_C = \deg \mathscr{N}_{C/X}$, the degree of the normal sheaf. If C is nonsingular then $\mathscr{N}_{C/X}$ is a line bundle of rank $\operatorname{codim}_X C$.

24.3. *Example.* Take $X = \mathbb{P}^1 \times \mathbb{P}^1$, the nonsingular quadric surface. Any curve C can be denoted by bidegree $(a, b) \in \operatorname{Pic} X = \mathbb{Z} \oplus \mathbb{Z}$. And D be another curve of bidegree (a', b'). Then $C \cdot D = ab' + a'b$. To see this, consider the two rulings on X.

24.4. *Example.* If H is ample on X and C is any curve. Then $H \cdot C > 0$. This is seen by considering the closed embedding given by nH, for some large enough n. Then $\deg C$ in the closed embedding, which is a positive integer, is $\deg nH|_C = (nH) \cdot C = n(H \cdot C)$.

24.5. Genus formula. Adjunction formula says that $\omega_Y \cong \omega_X \otimes \det \mathscr{N}_{Y/X}$ for a closed embedding $Y \hookrightarrow X$. When Y is an effective Cartier divisor, $\omega_Y \cong \omega_X \otimes \mathscr{O}_X(Y)|_Y$. Taking degrees,

$$2g(C) - 2 = (K_X + C) \cdot C.$$

This is the genus formula.

24.B. Exercise. Let $C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of bidegree (a, b). Using the genus formula, compute $g(C)$.

24.6. Riemann-Roch Theorem. — *Let $D \in \operatorname{Weil} X$. Then $\chi(D) = \frac{1}{2} D \cdot (D + K_X) + 1 + p_a(X)$.*

Proof. Write $D \sim C - E$ where C and E are very ample. Then $0 \to \mathscr{O}_X(-E) \to \mathscr{O}_X \to \mathscr{O}_E \to 0$ and $0 \to \mathscr{O}_X(-C) \to \mathscr{O}_X \to \mathscr{O}_C \to 0$. Twist both by C, take Euler characteristics, and apply Riemann-Roch for curves to compute $\chi(\mathscr{O}_X(C) \otimes \mathscr{O}_C)$ and $\chi(\mathscr{O}_X(C) \otimes \mathscr{O}_E)$. Finally, apply the genus formula. ☐

24.7. Lemma. — *Let H be any ample on X. Denote $n_{X,H} = H \cdot K_X$. If $D \cdot H > n_{X,H}$ then $h^2(D) = 0$.*

Proof. Apply Serre duality to see that $K - D$ is effective. Then use one of the above example. ☐

Lecture 25

Lecturer: Nabanita Ray Date: 03.04.2023

25.1. Checking effectivity. On curves, if $\deg D > 0$ then nD is effective for $n \gg 0$.

25.2. Theorem. — *On a surface X, if $H\cdot D>0$ and $D^2>0$ then nD is effective for $n\gg 0$.*

Proof. Recall that $D\cdot H > K_X\cdot H$ implies $H^2(D)=0$. For $n\gg 0$, we have $(nD)\cdot H > K_X\cdot H$, hence $H^2(nD)=0$. By Riemann-Roch, $h^0(nD)\geqslant \frac{1}{2}n^2D^2 - nD\cdot K_X + 1 - p_a(X)$. Sending $n\to\infty$, we get the result. ☐

25.3. Numerical equivalence. A divisor $D\in \operatorname{Weil}X$ is called **numerically trivial** or numerically equivalent to zero if $D\cdot C=0$ for each curve $C\subseteq X$. We denote

$$\operatorname{Pic}^0 X := \{D\in \operatorname{Cl}X\colon D\cdot C=0 \text{ for all curves } C\subset X\}.$$

Denote $N'(X):=\operatorname{Pic}X/\operatorname{Pic}^0 X$. It is a nontrivial fact that this is a finitely-generated free abelian group. This is called the **Néron-Severi group**. We also define

$$\operatorname{nef}X = \{D\in \operatorname{Weil}X\colon D\cdot C\geqslant 0 \text{ for each } C\subseteq X\}.$$

Then $N'(X)_{\mathbb{R}} := N'(X)\otimes_{\mathbb{Z}}\mathbb{R}$ is a finite-dimensional $\mathbb{R}$-vector space called the **real Néron-Severi group**. There is a natural map $\operatorname{Pic}X\to N'(X)\to N'(X)_{\mathbb{R}}$. We may then look at the "cone" generated by ample/nef line bundles in $N'(X)_{\mathbb{R}}$. It turns out that the cone generated by ample line bundles and nef line bundles in $N'(X)_{\mathbb{R}}$ are same.

25.4. Algebraic equivalence (Hartshorne [Har77, III.9.8.5, Exercise V.1.7]). Let X be a surface and C a smooth curve. Also, D be an effective Cartier divisor on $X\times C$, flat over C. We have a projection $\pi\colon X\times C\to C$. Then $\pi^{-1}(t)=X_t\cong X$. We then get a family of divisors $\{D|_{X_t}=D_t=D\times_X X_t\colon t\in C\}$. Pick closed points $t_1,t_2\in C$. Then D_{t_1} and D_{t_2} are called **pre-algebraically equivalent**. Now, $D_1,D_2\in\operatorname{Weil}X$ are pre-algebraically equivalent if $D_1-D_2=D_{t_1}-D_{t_2}$ for some curve C, closed points t_1,t_2, divisor D. We say $D',D''\in\operatorname{Weil}X$ are **algebraically equivalent** if there exists a sequence of divisors such that $D'=D_1,D_2,\dots,D_n=D''$, where D_i and D_{i+1} are pre-algebraically equivalent. Denote $D'\sim_{\text{alg}} D''$. It can be verified that

- $\{D\in\operatorname{Div}X\colon D\sim_{\text{alg}}0\}$ is a subgroup of $\operatorname{Weil}X$.
- Linear equivalence $\implies$ algebraic equivalence $\implies$ numerical equivalence.

25.5. Hodge Index Theorem. — *Let H be an ample divisor on the surface X, and suppose that D is a divisor, $D\not\sim 0$, with $D\cdot H=0$. Then $D^2<0$.*

Proof. Suppose $D^2\geqslant 0$. Consider two cases

- $D^2>0$. Then $H'=nH+D$ is ample for $n\gg 0$. So $H'\cdot D=D^2>0$. Therefore, nD is effective by the first theorem of this lecture, which contradicts the fact that $D\cdot H=0$.
- $D^2=0$. Since $D\not\sim 0$, hence there exists E with $D\cdot E\neq 0$. Replacing E by $E'=(H^2)\cdot E-(E\cdot H)\cdot H$, we may assume $H\cdot E=0$. Now let $D'=nD+E$. Then $D'\cdot H=0$ and $(D')^2=2nD\cdot E+E^2$. Since $D\cdot E\neq 0$, we have $(D')^2>0$ for large n. We are now in the first case. ☐

25.6. Nakai-Moishezon Criterion. — *A divisor D on the surface X is ample if and only if $D^2>0$ and $D\cdot C>0$ for all irreducible curves $C\subseteq X$.*

25.7. Sheaf Proj. Let X be a Noetherian scheme. Let $\mathscr{F}=\oplus_{d\geqslant 0}\mathscr{F}_d$ is a graded sheaf of $\mathscr{O}_X$-algebras. Also, assume $\mathscr{F}_0=\mathscr{O}_X$ and $\mathscr{F}_1$ is a coherent $\mathscr{O}_X$-module. For any affine open $U=\operatorname{Spec}A\subseteq X$, $\mathscr{F}(U)$ is a graded A-algebra. There is a map $\operatorname{Proj}\mathscr{F}(U)\xrightarrow{\pi} U$. We can then "glue" these to define $\mathcal{P}roj\,\mathscr{F}$; of course, we must have compatibility conditions.

25.8. *Remarks.*

- $\mathcal{P}roj\,\mathscr{O}_X[T_0,\dots,T_n]=\mathbb{P}^n_X$.

- Let $\mathscr{E}$ be a vector bundle of rank n. Denote $\mathscr{F} = \operatorname{Sym}\mathscr{E} = \oplus_{d\geqslant 0} S^d(\mathscr{E})$. Then $Y = \mathcal{P}roj\,\mathscr{F}\, X \xrightarrow{\pi} X$ is a **projective bundle**. Note that $\dim Y = \dim X + n - 1$. Also, $\pi_*\mathscr{O}_Y(\ell) = S^\ell\mathscr{E}$ for $\ell \geqslant 0$. (Hartshorne [Har77, II.7])
- $\operatorname{Pic} Y = \pi^* \operatorname{Pic} X \times \mathbb{Z}\mathscr{O}_Y(1)$ (Hartshorne [Har77, Exercise II.7.9]).
- The projective bundle $\mathbb{P}\mathscr{E}$ of a vector bundle $\mathscr{E}$ is characterized by the universal property: given a morphism $f\colon T \to X$, to factorize f through the projection map $\mathbb{P}\mathscr{E} \to X$ is to specify a line sub-bundle of $f^*\mathscr{E}$.

25.9. *Example.* Let $\mathscr{L}$ be a line bundle on X. Then $\mathbb{P}\mathscr{L} \to X$, the **projectivisation** of the line bundle, is an isomorphism.

Lecture 26

Lecturer: Nabanita Ray Date: 05.04.2023

26.1. Right derived sheaves. Let $f\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf on X. Define $R^i f_*\mathscr{F}$ to be the sheafification of of the presheaf $U \mapsto H^i(f^{-1}(U), \mathscr{F}_{f^{-1}(U)})$. These are the right derived functors of the pushforward $f_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_Y$.

- When $Y = \operatorname{Spec} A$, $R^i f_*\mathscr{F} = \widetilde{H^i(X, \mathscr{F})}$.

26.2. Grauert's theorem. — *Let $f\colon X \to Y$ be a morphism of schemes, $\mathscr{F}$ a coherent sheaf on X which is flat over Y. Fix y. Define $h^i(y, \mathscr{F}_y) = \dim_{\kappa(y)} H^i(X_y, \mathscr{F}_y)$. If h^i is constant for all y then $R^i f_*\mathscr{F}$ is locally free and moreoever, $R^i f_*\mathscr{F} \otimes \kappa(y) \xrightarrow{\sim} H^i(X_y, \mathscr{F}_y)$ is an isomorphism.*

26.A. Homework. Solve the exercises in Hartshorne [Har77, III.8].

26.3. Ruled surfaces. A **ruled surface** is a surjective morphism $\pi\colon X \to C$, X is a surface and C is a (smooth) curve, such that every fiber is isomorphic to $\mathbb{P}^1$ and there is a section $\sigma\colon C \to X$ of π. Here, X is called a **ruled surface**.

26.4. *Example.* $\mathbb{P}^1 \times \mathbb{P}^1$ is a ruled surface which has two rulings given by the two projections.

26.5. Lemma. — *If $\pi\colon X \to C$ is a ruled surface, D a divisor on X, with $D \cdot f = n \geqslant 0$, then $\pi_*\mathscr{O}_X(D)$ is locally free of rank $n+1$. Here, f denotes the generic fiber of π. In particular, $\pi_*\mathscr{O}_X = \mathscr{O}_C$.*

Proof. See Hartshorne [Har77, Lemma V.2.1]. □

26.6. Corollary. — $R^i\pi_*\mathscr{O}_X(D) = 0$ for $i > 0$.

26.7. *Reference.* Hartshorne [Har77, III.9, Exercise V.1.7].

26.8. Theorem. — *Let $\pi\colon X \to Y$ be a ruled surface if and only if $X \cong \mathbb{P}\mathscr{E}$ where $\mathscr{E}$ is a rank 2 vector bundle on C.*

Proof. See Hartshorne [Har77, Proposition V.2.2]. □

26.9. *Remark.* Unramified, flat, bijective $\implies$ isomorphism.

Lecture 27

Lecturer: Nabanita Ray Date: 10.04.2023

27.A. Assignment Problems.

(1) Hartshorne [Har77, Exercise III.8.1].
(2) Let $C \subset X$ be a smooth curve on a surface and $D \in \operatorname{Weil} X$ be a divisor. Then $C \cdot D = \deg D|_C$.
(3) Hartshorne [Har77, Exercise V.1.4 (a), V.1.7, V.1.9 (a) (b)].
(4) Hartshorne [Har77, Exercise V.2.3, V.2.8 (a) (b)].
(5) Show that torsion-free sheaves on a smooth curve are locally free.
(6) $q(X) = h^1(X, \mathscr{O}_X)$ called the **irregularity** of X and $p_n(X) = h^0(X, \omega_X^{\otimes n})$ is called the n**th-plurigenera**. Observe that $p_a = p_g - q(X)$. Show that $p_a, p_g, q(X), p_n$ are birational invariants for smooth surfaces. If $X \to C$ is a ruled surface then $q(X) = g(C)$, the genus of the curve, and $p_g(X) = 0$, $p_n(X) = 0$, for $n \geqslant 2$.
(7) If $\pi \colon X \to C$ is a ruled surface, D is any section and f is any fiber then D and f intersect transversally.

27.1. Consider a ruled surface $\pi \colon \mathbb{P}\mathscr{E} \to C$. Then one can easily show that $\operatorname{Pic} \mathbb{P}\mathscr{E} = \mathbb{Z}C_0 \oplus \pi^* \operatorname{Pic} C$ for some section C_0. This gives that $N^1(\mathbb{P}\mathscr{E}) \cong \mathbb{Z}C_0 \oplus \mathbb{Z}f$ where N^1 is denotes the Néron-Severi group. Also, there is a surjective map $\operatorname{Pic} \mathbb{P}\mathscr{E} \to N^1(\mathbb{P}\mathscr{E})$.

27.2. Proposition. — *If $X = \mathbb{P}\mathscr{E} \to C$ is a ruled surface with section σ. Denote $\sigma(C) = C_0$. By universal property, this means there is a line bundle $\mathscr{L}$ corresponding to σ such that $\mathscr{E} \to \mathscr{L} \to 0$ on C. Then $\operatorname{Ker}(\mathscr{E} \to \mathscr{L}) := \mathscr{N}$ is a line bundle. Also, $\pi^*\mathscr{L} = \mathscr{O}_X(1) \otimes \mathscr{O}_X(-C_0)$ and $\mathscr{N} = \pi_*(\mathscr{O}_X(1) \otimes \mathscr{O}_X(-C_0))$.*

Proof. $\mathscr{N}$ is of course a line bundle because it's torsion-free. We have a sequence $0 \to \mathscr{O}_X(-C_0) \to \mathscr{O}_X \to \mathscr{O}_{C_0} \to 0$. Twist by $\mathscr{O}_X(1)$ and push it forward–

$$0 \to \pi_*(\mathscr{O}_X(1) \otimes \mathscr{O}_X(-C_0)) \to \pi_*(\mathscr{O}_X(1) \otimes \mathscr{O}_X) \to \pi_*(\mathscr{O}_{C_0} \otimes \mathscr{O}_X(1)) \to 0.$$

We know that the middle term is just $\mathscr{E}$. Also, $\pi_*(\mathscr{O}_X(1) \otimes \mathscr{O}_{C_0}) = \pi_*\mathscr{O}_{C_0}(1) = \mathscr{L}$. As $\pi^*\pi_*(\mathscr{O}_X(1) \otimes \mathscr{O}_{C_0}) = \mathscr{O}_X(1) \otimes \mathscr{O}_{C_0}$ so the proof is complete. □

27.3. Proposition. — *Let $\pi \colon X \to C$ be a ruled surface. Then there exists a vector bundle $\mathscr{E}$ such that $X = \mathbb{P}\mathscr{E}$ with*

- $H^0(\mathscr{E}) \neq 0$,
- $H^0(\mathscr{E} \otimes \mathscr{L}) = 0$ *for all line bundles $\mathscr{L}$ with* $\deg \mathscr{L} < 0$.

Such a vector bundle $\mathscr{E}$ is called ***normalized****. Also,*

- $-e = \deg \mathscr{E} := \deg \det \mathscr{E}$ *is invariant on X.*
- *There exists a section $\sigma \colon C \to X$, $\sigma(C) = C_0$ such that $\mathscr{O}_X(1) = \mathscr{O}_X(C_0)$.*

Proof. See Hartshorne [Har77, Proposition V.2.8]. □

27.4. *Remark.* Normalization is not unique.

Lecture 28

Lecturer: Nabanita Ray Date: 12.04.2023

28.1. *Recall.* We saw that if $\pi\colon \mathbb{P}\mathscr{E} \to C$ is a ruled surface with section σ, with image $C_0 \subset \mathbb{P}\mathscr{E}$, and $\mathscr{E}$ is normalized then $\mathscr{O}_{\mathbb{P}\mathscr{E}}(1) \cong \mathscr{O}_X(C_0)$. This C_0 is called a **normalized section**. From now onwards, C_0 always denotes a normalized section and $\mathscr{E}$ is normalized.

28.2. Proposition. — *Let $\sigma\colon C \to X$ be a section with $\sigma(C) = D$. We can write a sequence $\mathscr{E} \to \mathscr{O}_C(E) \to 0$ on C. Then $D \sim C_0 + (E - \det \mathscr{E}) \cdot f$ and $\deg E = C_0 \cdot D$.*

Proof. Observe that

$$C_0 \cdot D = \deg(\mathscr{O}_X(C_0) \otimes \mathscr{O}_D) = \deg(\mathscr{O}_{\mathbb{P}\mathscr{E}}(1) \otimes \mathscr{O}_D) = \deg(\mathscr{O}_D(1)) = \deg \mathscr{O}_C(E) = \deg E.$$

Taking degrees of the sequence $0 \to \mathscr{O}_C \to \mathscr{E} \to \mathscr{O}_C(E) \to 0$ we get $\deg \mathscr{E} = \deg E$. There is a kernel bundle $\mathscr{N}$ with $0 \to \mathscr{N} \to \mathscr{E} \to \mathscr{O}_C(E) \to 0$. Taking determinants, $\deg \mathscr{E} = \mathscr{N} \otimes \mathscr{O}_C(E) \implies \mathscr{N} = \det \mathscr{E} \otimes \mathscr{O}_C(E)^\vee$. Also, we have shown that $\pi^* \mathscr{N} = \mathscr{O}_{\mathbb{P}\mathscr{E}}(1) \otimes \mathscr{O}_X(-D)$. ☐

28.3. If $\mathscr{E} = \mathscr{O}_C \oplus \mathscr{O}_C(-nP)$ then $C_0^2 = \deg \det \mathscr{E} = \deg \mathscr{O}_C(-nP) = -n$ using the above proposition. So we can have self-intersection as any negative integer.

28.4. Proposition. — $K_X \sim -2C_0 + (K_C + \det \mathscr{E}) \cdot f$

Proof. Clear by adjunction formula and the fact that $\operatorname{Pic} X = \mathbb{Z}C_0 \oplus \pi^* \operatorname{Pic} C$. ☐

28.5. Review of blow-ups. Let X be any Noetherian scheme and $\mathscr{I}$ be an ideal sheaf. Denote the graded algebra $\bigoplus_{d \geqslant 0} \mathscr{I}^d$ by $\mathscr{F}$. Then $\tilde{X} = \operatorname{Proj} \mathscr{F} \to X$ is the blow-up of X with respect to $\mathscr{I}$. There is an obvious way to state this definition in terms of closed subschemes due to the duality between closed subschemes and quasicoherent ideal sheaves. If Y is the closed subscheme corresponding to $\mathscr{I}$ then $\pi^{-1}(Y) \cong \mathbb{P}\mathscr{N}_{Y/X}^\vee \to Y$ is called the **exceptional divisor**. Further, $\mathscr{N}_{\pi^{-1}(Y)/\tilde{X}} \cong \mathscr{O}_{\pi^{-1}(Y)}(-1)$. Let Z be any closed subscheme of X not contained in Y. Then the scheme-theoretic closure of $\pi^{-1}(X \setminus Z)$ is called the **strict transform** of Z and it is denoted $\tilde{Z}$. This is same as the blow up of Z with respect to $Y \cap Z$. If P is a closed point of codimension n in X, then $\mathscr{N}_{P/X} = \mathfrak{m}_P/\mathfrak{m}_P^2 = \kappa(p)^{\oplus n}$. Thus, $\mathbb{P}\mathscr{N}_{P/X} \cong \mathbb{P}^{n-1}$.

28.6. Monoidal transformations. Let X be a surface and $P \in X$ be any closed point. Denote by $\pi\colon \tilde{X} \to X$ the blow up of X along P. Such point blow-ups are called monoidal transformations. We write E for the exceptional divisor $\pi^{-1}(P)$. Note that

$$E^2 = \deg \mathscr{O}_{\tilde{X}}(E) \otimes \mathscr{O}_E = \deg \mathscr{N}_{E/\tilde{X}} = \deg \mathscr{O}_{\mathbb{P}^1}(-1) = -1.$$

We also have $\operatorname{Pic} \tilde{X} \cong \mathbb{Z}E \oplus \pi^* \operatorname{Pic} X$. This is always true regardless of whether X is a surface or if blow-up locus is a point (but we do need nonsingularity).

Lecture 29

Lecturer: Nabanita Ray Date: 17.04.2023

29.1. Picard group of blow-up. Recall that we mentioned $\operatorname{Pic} \tilde{X} \cong \mathbb{Z}E \oplus \pi^* \operatorname{Pic} X$. As P has high codimension, it follows that $\operatorname{Pic} X \cong \operatorname{Pic} X \setminus P \cong \operatorname{Pic}(\tilde{X} \setminus E)$. We have the excision exact sequence

$$\mathbb{Z} \to \operatorname{Pic} \tilde{X} \to \operatorname{Pic}(\tilde{X} \setminus E) \to 0.$$

The left map is actually injective. Indeed, $nE \cdot nE = -n^2 \neq 0$. As $\operatorname{Pic}(\tilde{X} \setminus E) = \operatorname{Pic} X \to \operatorname{Pic} \tilde{X}$ splits using π^*, it follows that $\operatorname{Pic} \tilde{X} \cong \mathbb{Z}E \oplus \pi^* \operatorname{Pic} X$. We denote the projection map $\operatorname{Pic} \tilde{X} \to \operatorname{Pic} X$ as π'. Also, $\pi^* C \cdot D = C \cdot \pi' D$. This can be checked by using $E^2 = -1$, $\pi^* C \cdot E = 0$, and $\pi^* C \cdot \pi^* D = C \cdot D$.

29.2. Proposition. —

- $\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$,
- $R^i f_* \mathcal{O}_{\tilde{X}} = 0$,
- $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^i(X, \mathcal{O}_X)$.

Proof. Omitted. □

29.3. Proposition. — $K_{\tilde{X}} = \pi^* K_X + E$.

Proof. Let $K_{\tilde{X}} = a\pi^* \mathscr{L} + bE$. Restricting, $K_{\tilde{X}}|_{\tilde{X}\setminus E} = (a\pi^* \mathscr{L} + bE)|_{\tilde{X}\setminus E} = (a\pi^* \mathscr{L})|_{\tilde{X}\setminus E} = a\mathscr{L}|_{X\setminus P}$. Therefore, $a\mathscr{L} = K_X$ as blow-ups are isomorphisms away from P. Also, by multiplying both sides by E, we get $b = 1$. □

29.4. Consider a blow-up $\pi \colon \tilde{X} \to X$. Let C be a curve on X and $p \in C$ a closed point. Obesrve that $\pi^{-1}C = E \cup \tilde{C}$. Write $\pi^* C = \tilde{C} + xE$. We wish to determine x.

One of the most important results of this discussion is–

29.5. Theorem. — *If the multiplicity of C at P is r then $\pi^* C = \tilde{C} + rE$.*

Proof. Let $\mathfrak{m}$ be the ideal of P. We know $\tilde{X} = \mathcal{P}roj_X \bigoplus_{d \geqslant 0} \mathfrak{m}^d$. Choose local parameters $x, y \in \mathfrak{m} \subset \mathcal{O}_{X,P}$. Replace X by some affine open neighborhood $P \in \operatorname{Spec} A$ such that $x, y \in \mathfrak{m}$ and $\mathfrak{m} \subset A$ be the ideal of P. Put $Au \oplus At = A^{\oplus 2}$. We have the sequence

$$0 \to (uy - xt)A \to A^{\oplus 2} \xrightarrow{u \mapsto x,\, t \mapsto y} \mathfrak{m} \to 0$$

Thus, $\mathfrak{m} = A^{\oplus 2}/(uy - xt)$ and $\bigoplus_{d \geqslant 0} \mathfrak{m}^d \cong A[u,t]/(uy - xt)$ as graded A-algebras. So, $\tilde{X} = \operatorname{Proj}_A A[u,t]/(uy - xt) \subset \mathbb{P}^1_A$. Let $f = f_r(x,y) + g$ be a local equation of C where $f_r \in \mathfrak{m}^r \setminus \mathfrak{m}^{r+1}$, $g \in \mathfrak{m}^{r+1}$. Restrict to the distinguished open set $D(t) \subset \mathbb{P}^1_A$. Then $\tilde{X} \cap D(t) = \operatorname{Spec} A[u]/(uy - x) \to \operatorname{Spec} A$. It is easy to check that the exceptional divisor in $\tilde{X} \cap D(t)$ is cut out by y. The pullback of f along the blow up gives the local equation $\pi^* f = f_r(uy, y) + g(uy, y) = y^r(f_r(u,1) + yh)$. This completes the proof. □

29.6. *Remark.* From the above proof, we can also see that blow-ups of curves are finite.

29.7. One can show using the adjunction formula that $p_a(\tilde{C}) = p_a(C) - \frac{1}{2}r(r-1)$. Indeed,

$$2g(\tilde{C}) - 2 = \tilde{C}(\tilde{C} + K_X) = (\pi^* C - rE)(\pi^* C - rE + \pi^* K_X + E) = 2p_a(C) - 2 - r(r-1).$$

Thus, we see that one can resolve all singularities by repeatedly blowing-up at singularities.

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