# An introduction to Intersection Theory

Lectures by Prof. Barbara Fantechi

These are notes from the lecture series on Intersection theory given by Prof. Fantechi at Chennai Mathematical Institute during January, 2014. This is only a preliminary draft of the notes I took. This is far from being complete and self-contained. Since I never revised or checked the notes thoroughly, there might be some mistakes. So please proceed with caution. :)

CHAPTER 1

# Lecture 1

**Contents.** The group of algebraic cycles associated to a scheme. Fundamental cycle associated to a pure-dimensional scheme. Definition of proper push-forward and flat pullback. Functoriality and compatibility among them. Rational equivalence.

**Conventions and Notations:** All schemes considered are of *finite type* over a field and morphisms are also *finite type*. We mean by a *variety* a reduced and irreducible scheme over a field. A *point* of a scheme refers to a closed point and by a *subvariety* of a scheme we mean a closed subscheme which is a variety. Let $X$ be a scheme and $V \subset X$ a subvariety, then $\mathcal{O}_{V,X}$ denotes the local ring $\mathcal{O}_{X,\eta_V}$ where $\eta_V$ is the generic point of $V$. The function field of $V$ a variety is denoted by $R(V)$.

**Definition 1.1** (Group of Algebraic Cycles) Let X be a scheme. For $d \in \mathbb{N}$, the group of algebraic $d$-cycles, denoted by $Z_d(X)$, is the free abelian group generated by the $d$-dimensional subvarieties of $X$.
If $V$ is a $d$-dimensional subvariety we write $[V]$ for the cycle corresponding to $V$. For any $d$-cycle $\alpha$, we have $\alpha = \sum_{i=1}^n a_i[V_i]$ for some $d$-dimensional subvarieties $V_1, \ldots, V_n \subset X$.

**Remark 1.1** (i) If $d > \dim(X)$ then $Z_d(X) = 0$.
(ii)If $\dim(X) = d$, then $Z_d(X) \cong \mathbb{Z}$.
(iii)If $V \subset X$ is a closed subscheme with inclusion map $i : Y \hookrightarrow X$, then we have a natural injection of abelian groups

$$i_* : Z_d(Y) \hookrightarrow Z_d(X).$$

(iv) If $U \hookrightarrow X$ is an open subscheme, then we have a natural homomorphism (surjective)

$$Z_d(X) \to Z_d(U); \;\; [V] \mapsto [V \cap U].$$

It is surjective since for all $Z \in Z_d(U)$ we have the closure $\overline{Z} \in Z_d(X)$ and $\overline{Z} \cap U = Z$. Therefore we have a short exact sequence‘

$$0 \to Z_d(X/U) \to Z_d(X) \to Z_d(U) \to 0.$$

**Definition 1.2** (Proper pushforward of cycles) Let $f : Y \to X$ a proper morphism of schemes. The pushforward homomorphism

$$f_* : Z_d(Y) \to Z_d(X)$$

is defined as follows

$$f_*[V] = \begin{cases} [R(V) : R(f(V))].[f(V)] & \text{if dim(f(V)) = d} \\ 0 & \text{Otherwise} \end{cases}$$

Note that, since $V$ is a subvariety and $f$ is proper, we have $f(V)$ is a subvariety with the reduced induced subscheme structure. As $f : V \longrightarrow f(V)$ is dominant and of finite type, $[R(V) : R(f(V))]$ is well-defined.

The following lemma shows that proper push forward is functorial.

**Lemma 1.1** For $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$ with $f$ and $g$ proper morphisms of schemes, we have

$$(g \circ f)_* = g_* f_* : Z_d(X_1) \to Z_d(X_3).$$

Proof. Let $V \subset X_1$ a $d$-dimensional variety. If $\dim((g \circ f)(V)) < d$, then clearly either

$$\dim(f(V)) < d \text{ or } \dim(g(f(V))) < d.$$

Hence in either case we have $(g \circ f)_*([V]) = 0 = g_*(f_*([V]))$. Now consider the case when $\dim((g \circ f)(V)) = \dim(f(V)) = \dim(g(f(V))) = d$. We then have

$$l = [R((g \circ f)(V)) : R(V)] = [R((g \circ f)(V)) : R(f(V))][R(f(V)) : R(V)] = mn.$$

and hence

$$(g \circ f)_*([V]) = l[(g \circ f)(V)] = mn[(g \circ f)(V)] = g_*(m[f(V)]) = g_* f_*([V])$$

□

Let $X$ be a scheme with irreducible components $\{X_1, \ldots, X_r\}$. Set

$$m_i = \text{length}_{\mathcal{O}_{X_i,X}} \mathcal{O}_{X_i,X} = l_{\mathcal{O}_{X_i,X}} \mathcal{O}_{X_i,X}$$

which is called the multiplicity of $X$ along $X_i$. The ring $\mathcal{O}_{X_i,X}$ is an artinian local ring and $m_i$ is it's length as a module over itself (See Fulton, Appendix A, Def.A.1. for the definition of length).

**Definition 1.3** (Fundamental cycle of $X$) The element

$$\Sigma_i m_i [X_i] \in Z_*(X) = \bigoplus_d Z_d(X)$$

is called the fundamental cycle of $X$, denoted by $[X]$.

If $Z \subset X$ is a closed subscheme of pure dimension $d$ with natural inclusion $i : Z \to X$, then by abuse of notation we denote the cycle $i_*([Z]) \in Z_d(X)$ by $[Z]$.

**Definition 1.4** (Flat pull back of cycles) Let $f : X \to Y$ a flat morphism of relative dimension $r \geq 0$ (i.e all the fibers are of pure dimension $r$). We define

$$f^* : Z_d(Y) \to Z_{d+r}(X).$$

by setting for $V \subset Y$ a $d$-dimensional subvariety,

$$f^*([V]) = [f^{-1}(V)]$$

where $f^{-1}(V)$ denote the scheme theoretic inverse $X \times_Y V$.

Note that $f^{-1}(V)$ has pure dimension $d + r$ and $[f^{-1}(V)]$ is its cycle. The following Lemma shows that flat pull-back makes sense for any closed subscheme.

**Lemma 1.2** Let $f : X \to Y$ be a flat map of relative dimension $r$. Consider $Z \subset Y$ a closed subscheme of pure dimension $d$. Then $f^{-1}(Z) \subset X$ is a closed subscheme of pure dimension $d + r$ and we have

$$f^*([Z]) = [f^{-1}(Z)].$$

PROOF. Let $W$ be an irreducible component of $f^{-1}(Z)$ and $V := \overline{f(W)}$ be the closure of the image of $W$ in $Y$. It is enough to show that $V$ is an irreducble component of $Z$ with the same multiplicity as $m := l_{\mathcal{O}_{W,f^{-1}(Z)}}(\mathcal{O}_{W,f^{-1}(Z)})$. Recall that, if $A \longrightarrow B$ is a flat local homomorphism of rings then the induced map $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is surjective. Moreover,

$$l_B(B) = l_A(A).l_B(B/m_A B).$$

Suppose $V$ not an irreducible component of $Z$ and it is contained in the component $V'$. Let $A = \mathcal{O}_{V,Z}$ and $B = \mathcal{O}_{W,f^{-1}(Z)}$. By the result mentioned above, there must be an irreducible subvariety $W'$ of $f^{-1}(Z)$ which contains $W$ and dominates $V'$. But $W$ is an irreducible component, hence we must have $W = W'$ and $V = V'$. Therefore $V$ is an irreducible component of $Z$. Now $f^*[Z] = \sum l_{\mathcal{O}_{V_i,Z}} \mathcal{O}_{V_i,Z}[f^{-1}(V_i)]$ and for each $i$, we have $[f^{-1}(V_i)] = \sum l_{W_{ij},f^{-1}(V_i)}[W_{ij}]$. The equality follows from the equation above since the lengths correspond to the factors $l_B(B/m_A B)$. $\square$

The following topological property of flat morphisms is very useful and follows from the proof of the Lemma.

**Fact:** Let $X \longrightarrow Y$ be flat and $Z$ a closed subset of $Y$. Then every irreducible component of $f^{-1}(Z)$ dominates an irreducible component of $Z$.

**Corollary 1.1** Flat pull-back is functorial, i.e. if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are flat, then

$$(g \circ f)^* = f^* \circ g^*.$$

Proof. Note that, $g \circ f$ is flat, and hence pullback is defined. Let $V \subset Z$ be a subvariety of $Z$, then by Lemma 2,

$$(g \circ f)^*[V] = [(g \circ f)^{-1}(V)] = [f^{-1} \circ g^{-1}(V)] = f^*[g^{-1}(V)] = f^* \circ g^*[V].$$

□

**Proposition 1.1** Flat pull-back commutes with proper push forward, i.e. given a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

where $f$ is proper and $g$ is flat of relative dimension $r$. Then,

$$f'_* \circ g'^* = g^* \circ f_* : Z_d(X) \longrightarrow Z_{d+r}(Y').$$

Proof. Let $V \subset X$ be a $d$-dimensional subvariety of $X$ and then $W = f(V)$ is also a subvariety of $Y$. We would like to show that $f'_* \circ g'^*[V] = g^* \circ f_*[V]$. Since flatness and properness are preserved under base change, first we base change by $W \longrightarrow Y$ and then by $V \longrightarrow X$ to assume $V = X$, $W = Y$, i.e. $X$ and $Y$ are varieties and $f$ is surjective.

Now suppose $f_*[X] = d[Y]$. Then $g'^*[X] = [X']$ and $g^*[Y] = [Y']$, hence we need to show that $f'_*[X'] = d[Y']$. Suppose $f'_*[X'] = \sum_i l_{X'_i,X'} f'_*[X'_i]$. Note that we only need to consider those components $X'_i$ which dominate a component of $Y'$. Let $Y'_j$ be a component of $Y'$ and $A$ be its local ring. Since $g$ is flat, $Y'_j$ dominates $Y$ by the Fact mentioned above. As $X'$ is the fibre product, we see that the preimage of $Y'_j$ is given by $B = A \otimes_K L$, where $K = R(Y)$ and $L = R(X)$. As $[L : K] = d$, we have $B = A^d$ as an $A$-module and $l_A(B) = rd$ where $r = l_A(A)$. Since $B$ is also Artinian, we may decompose $B = \oplus B_i$ into a direct sum of local Artinian rings. Then, by additivity of length, $rd = \sum l_A(B_i)$. Now $l_A(B_i) = l_{B_i}(B_i)d_i$, where $d_i$ is the degree of the residue field extension. Let $X'_{ij}$ be the components which dominate $Y'_j$. Then $l_{B_i}(B_i) = l_{X'_{ij},X'}$ and for $d_{ij} = [R(X'_{ij}) : R(Y'_j)]$, we have $\sum_i l_{X'_{ij},X'} d_{ij}[Y'_j] = rd[Y'_j]$. Then summing over all the components of $Y'$ we get, $f'_*[X'] = \sum_i l_{X'_i,X'} f'_*[X'_i] = \sum_j rd[Y'_j] = d[Y']$. □

**Definition 1.5** (Rational equivalence) Let $X$ be a scheme. For any $(d+1)$ dimensional subvariety $W$ of $X$, and for any $r \in R(W)^*$, define a $d$-cycle $[\mathrm{div}(r)]$ on $X$ as

$$[\mathrm{div}(r)] = \sum \mathrm{ord}_V(r)[V],$$

the sum over all codimension one subvarieties $V$ of $W$. Here $\mathrm{ord}_V$ is the order function on $R(W)^*$ defiend by the local ring $\mathcal{O}_{V,W}$.

The subgroup $\mathrm{Rat}_d(X)$ inside $Z_d(X)$ of cycles rationally equivalent to 0 is defined to be the subgroup generated by $[\mathrm{div}(r)]$ for all $W \subset X$ a $(d+1)$-dimensional subvariety and all $r \in R(W)^*$.
Two $d$-cycles are said to be rationally equivalent if their difference is rationally equivalent to 0. We define the Chow groups as follows

$$A_d(X) := \frac{Z_d(X)}{\mathrm{Rat}_d(X)}.$$

CHAPTER 2

# Lecture 2

**Contents.** Rational equivalence, alternative definitions of rational equivalence. The Chow group of algebraic cycles modulo rational equivalence. Proper pushforward and flat pullnack preserve rational equivalence, Degeneration to the normal cone.

Recall that for any scheme $X$ , we defined the Chow groups as

$$A_d(X) := \frac{Z_d(X)}{\mathrm{Rat}_d(X)}$$

We define

$$A_*(X) = \oplus_{d\geq 0} A_d(X)$$

as a graded group.

**Proposition 2.1** Proper pushforward of cycles preserve rational equivalence.

Proof. Let $f : X \to Y$ be a proper map and $\alpha$ a $d$-cycle which is rationally equivalent to zero. We have to show that $f_*(\alpha)$ is rationally equivalent to zero on $Y$. We may assume that there is a subvariety $W \subseteq X$ of dimension $d+1$ and $r \in R(W)^*$ such that $\alpha = \mathrm{div}(r)$. Since proper pushforward is functorial, we may assume that $W = X$ and $f(X) = Y$, so that $X$ and $Y$ are varieties and $f$ is surjective.

The proposition follows from the following result and for the proof we refer to Fulton, Prop.1.4.

Let $f : X \longrightarrow Y$ is a proper, surjective morphism of varieties and let $r \in R(X)^*$. Then

(a) $f_*[\mathrm{div}(r)] = 0$ if $\dim(Y) < \dim(X)$.

(b) $f_*[\mathrm{div}(r)] = [\mathrm{div}(N(r))]$ if $\dim(Y) = \dim(X)$, where $N(r)$ is the norm of $r$, i.e., the determinant of the $R(Y)$-homomorphism of $R(X)$ given by multiplication by $r$.

Let us discuss a proof of part (a) slightly diffferent from the one in Fulton. Suppose $\dim(Y) < \dim(X) - 1$. Now $f_*[\mathrm{div}(r)] \in Z_d(Y)$. Since $\dim(Y) < d$, we have $Z_d(Y) = 0$. Now suppose $\dim(Y) = \dim(X) - 1$. Then we have $f_*[\mathrm{div}(r)] = n.[Y]$ where $n$ is the number of points in a general fibre of $f$ counted with multiplicities with which they occur in the restriction of $r$ to this fibre, i.e., we have $n = \sum_{P:f(P)=Q} \mathrm{ord}_P(r|_{f^{-1}(Q)})$ where $Q \in Y$ is a general point. Hence $n$ is equal to the degree of $r|_{f^{-1}(Q)}$ on the complete curve $f^{-1}(Q)$, which is zero.

$\square$

**Alternate definition of Rational equivalence:** Let $X$ be a scheme and $V$ be a $d+1$ dimensional subvariety of $X \times \mathbb{P}^1$ such that the projection to the second factor induces a dominant morphism $f$ from $V$ to $\mathbb{P}^1$. For any point $P \in \mathbb{P}^1$ the scheme theoretic fibre $f^{-1}(P)$ is a subscheme of $X \times \{P\}$, which is isomorphically mapped to $X$ by the first projection $p_1$. We denote this subscheme by $V(P)$ and we have $p_{1*}[f^{-1}(P)] = [V(P)]$ in $Z_k(X)$. The morphism $f : V \longrightarrow \mathbb{P}^1$ determines a rational function $f \in R(V)^*$. Therefore

$$[f^{-1}(0)] - [f^{-1}(\infty)] = [\mathrm{div}(f)].$$

Therefore

$$[V(0)] - [V(\infty)] = p_{1*}[\mathrm{div}(f)]$$

which is rationally equivalent to zero.

Conversely, let $W \subset X$ be a $d+1$ dimensional subvariety and $r \in R(W)^*$. Then we have a rational map induced by $r$

$$r : W \dashrightarrow \mathbb{P}^1.$$

Denote by $\overline{W}$, the closure of the graph of $r$ in $W \times \mathbb{P}^1$. We then have a natural morphism

$$p_1 : \overline{W} \to W.$$

which is proper,birational and $r \in R(\overline{W})^*$ and defines a morphism (dominant)

$$r : \overline{W} \to \mathbb{P}^1$$

which also agrees with the restriction of the second projection $p_1 : W \times \mathbb{P}^1 \to \mathbb{P}^1$ to $\overline{W}$. We also have

$$div(r) = (p_1)_*([p_2^{-1}(0)] - [p_2^{-1}(\infty)]).$$

Therefore we have the following proposition.

**Proposition 2.2** A cycle $\alpha$ in $Z_k(X)$ is rationally equivalent to zero if and only if there are $k+1$ dimensional subvarieties $V_1, \ldots, V_t$ of $X \times \mathbb{P}^1$, such that the projections from $V_i$ to $\mathbb{P}^1$ are dominant with

$$\alpha = \sum [V_i(0)] - [V_i(\infty)]$$

in $Z_k(X)$.

**Algebraic equivalence.** As a digression we briefly discuss another equivalence relation on the group of cycles. To define algebraic equivalence, we will replace $\mathbb{P}^1$ by any smooth curve and $\{0, \infty\}$ by any two rational points on such a curve. More precisely consider the following subgroup of $Z_d(X)$
K $:= < p_{1*}(p_2^{-1}(c_1) - p_2^{-1}(c_2)) \mid \overline{W} \subset W \times C,\ p_2 : \overline{W} \to C$ is surjective, $c_1, c_2 \in C >$. The the algebraic chow group is defined as

$$A_d^{alg}(X) := Z_d(X)/K.$$

Like proper push forwards, flat pull backs respect algebraic equivalence too. But the algebraic Chow groups may not be finitely generated always.

Let us move back to the study of rational equivalence. The next result shows that flat pullbacks also respect rational equivalence like proper pushforwards do.

**Proposition 2.3** Let $f : X \longrightarrow Y$ be a flat morphism of relative dimension $n$, and $\alpha$ a $k$-cycle on $Y$ which is rationally equivalent to zero. Then $f^*(\alpha)$ is rationally equivalent to zero in $Z_{k+n}(X)$.

PROOF. We may assume that $\alpha = [V(0)] - [V(\infty)]$, where $V$ is a subvariety of $Y \times \mathbb{P}^1$ and teh projection $g$ from $Y$ to $\mathbb{P}^1$ is dominant, hence flat. Let $W = (f \times 1)^{-1}(V) \subset X \times \mathbb{P}^1$ and $h$ be the morphism induced by the projection to $\mathbb{P}^1$. Let $p : X \times \mathbb{P}^1$, $q : Y \times \mathbb{P}^1$ be the projections to the first component. Then

$$f^*\alpha = f^* q_*([g^{-1}(0)] - [g^{-1}(\infty)]) = p_*(f \times 1)^*([g^{-1}(0)] - [g^{-1}(\infty)])$$

and this equals $p_*([h^{-1}(0)] - h^{-1}(\infty)])$. Let $W_1, \ldots, W_t$ be the irreducible components of $W$, $h_i$ the restriction of $h$ to $W_i$. Let $m_i$ be the multiplicities along $W_i$, i.e. $[W] = \sum m_i[W_i]$. Since

$$[h_i^{-1}(0)] - [h_i^{-1}(\infty)] = [\mathrm{div}(h_i)]$$

and since $p_*$ preserves rational equivalence, it is enough to show that $[h^{-1}(P)] = \sum m_i[h_i^{-1}(P)]$, for $P = 0$ and $P = \infty$. To show this we use the following result from Fulton, Lemma. 1.7.2.

Let $X$ be a pure $n$-dimensional scheme, with irreducible components $X_1, ]dots, X_r$ and geometric multiplicities $m_1, \ldots, m_r$. Let $D$ be an effective Cartier divisor on $X$, i.e. a closed subscheme of $X$ whose ideal sheaf is locally generated by one non-zero divisor. Let $D = D \cap X_i$ be the restriction of $D$ to $X_i$. Then

$$[D] = \sum m_i[D_i]$$

in $Z_{n-1}(X)$.

The required equality follows from the result above by setting $D = h^{-1}(P)$. $\square$

**Lemma 2.1** Let $X$ be a scheme and $Y \subset X$ a closed subscheme and $U$ the complement (open) subscheme $X - Y$, with natural inclusion maps, $i : Y \to X$ and $j : U \to X$. Then the sequence

$$A_d(Y) \xrightarrow{i_*} A_d(X) \xrightarrow{j_*} A_d(U) \to 0.$$

is exact.

Proof. We have the following commutative diagram,

$$\begin{array}{ccccccc} Z_d(Y) & \xrightarrow{i_*} & Z_d(X) & \xrightarrow{j^*} & Z_d(U) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A_d(Y) & \xrightarrow[i_*]{} & A_d(X) & \xrightarrow[j^*]{} & A_d(U) & \longrightarrow & 0 \end{array}$$

The first row is exact as we have seen earlier. Since the vertical arrows are surjective, we have $j^* : A_d(X) \longrightarrow A_d(U)$ is surjective and $j^* \circ i_* = 0$. Let $\alpha \in A_d(X)$ such that $j^*\alpha = 0$. We need to show that $\alpha \in \mathrm{Im}(i_*)$. We may assume that $\alpha = \sum m_i[V_i]$. Then there exists $W \subset U$ a $(d+1)$-dimensional subvariety and $r \in R(W)^*$, such that

$$\sum m_i[V_i \cap U] = \mathrm{div}(r).$$

Let $\overline{W}$ be the closure of $W$ in $X$. Then $R(\overline{W}) = R(W)$. Therefore, we have

$$\mathrm{div}_{\overline{W}}(r) \cap U = \sum m_i[V_i \cap U]$$

as a cycle. Hence

$$\sum m_i[V_i] - \mathrm{div}_{\overline{W}}(r) \in i_* Z_d(Y).$$

Therefore $\alpha \in i_* A_d(Y)$. $\square$

**Exercise 2.1** Let $\pi : E \to X$ be a rank $r$ vector bundle, then

$$\pi^* : A_d(X) \to A_{d+r}(E)$$

is surjective.

**Solution.** Let $Y$ be a closed subscheme of $X$ such that $U = X/Y$ is affine and $E$ is trivial over $U$. We have a commutative diagram,

$$\begin{array}{ccccccc} A_*(Y) & \longrightarrow & A_*(X) & \longrightarrow & A_*(U) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A_*(\pi^{-1}(Y)) & \longrightarrow & A_*(E) & \longrightarrow & A_*(\pi^{-1}U) & \longrightarrow & 0 \end{array}$$

where the vertical maps are flat pull-backs and the rows are exact. If the first and the third vertical arrows are surjective then so is the second one. Therefore we need to show it for $U$ and $Y$. By Noetehrian induction, we repeat the process on $Y$, we may assume $X = U$. Therefore we are reduced to the case when $E$ is trivial. Suppose $E = X \times \mathbb{A}^n$. The projection $\pi$ factors as

$$X \times \mathbb{A}^n \longrightarrow X \times \mathbb{A}^{n-1} \longrightarrow X$$

Therefore, by induction, it suffices to show when $E = X \times \mathbb{A}^1$.

Let $V \in X \times \mathbb{A}^1$ be a $(d+1)$-dimensional subvariety. We want to show that $[V] \in \mathrm{Im}(\pi^*)$. Let $W = \overline{\pi(V)}$. We may assume that $X = W$ and $\pi$ maps $V$ dominantly to $X$.

If $\dim(X) = d$, then $V = E$. So $[V] = \pi^*[X]$. Otherwise let $\dim(X) = d+1$. Then $V$ defines a prime ideal in $R(X)[t]$ generated by $r$. Then

$$[V] - [\mathrm{div}(r)] = \sum n_i[V_i]$$

for some $(d+1)$-dimensional subvarieties $V_i$ of $E$ whose projections to $X$ are not dominant. Therefore we have $V_i = \pi^{-1}(W_i)$, with $W_i = \pi(V_i)$, and hence

$$[V] = [\mathrm{div}(r)] + \sum n_i \pi^*[W_i].$$

Therefore $[V]$ is a pull-back. □

**Exercise 2.2** Let $X$ be a $d$-dimensional variety, then

(a) $A_d(X) = \mathbb{Z}.[X] \cong \mathbb{Z}$.
(b) If X is non-singular, Then $A_{d-1}(X) = Pic(X)$. In general we have a homomorphism

$$Pic(X) \to A_{d-1}(X).$$

(c) Let $X = \mathbb{A}^n$. Then we have $A_i(X) = 0, \ \forall i < n$ and $A_n(X) \cong \mathbb{Z}$.

**Solution.** Parts (a) and (b) are immediate consequences of the definition. We only prove part (c) here. For $i = n$, the statement follows from part (a). Let $d < n$, then $A_0(\mathbb{A}^{n-d}) \longrightarrow A_0(\mathbb{A}^n)$ is surjective, by the previous Exercise. Therefore it is enough to show that $A_0(\mathbb{A}^n) = 0$ for all $n$. Let $P \in \mathbb{A}^1$ be a point and let $W \simeq \mathbb{A}^1 \subset \mathbb{A}^n$ be a line passing through $P$. Suppose $r$ is a linear function on $W$ that vanishes at $P$ only. Then $[\mathrm{div}(r)] = [P]$. Therefore $[P] \sim 0$ in $A_0(\mathbb{A}^n)$. □

**Exercise 2.3** Compute $A_*(\mathbb{P}^n)$ and $A_*(Grass(n,m))$ using cell (Bruhat) decomposition.

**Solution.** We use the following general fact. Let $X$ be a scheme with a cell decomposition, i.e., a filtration $\emptyset = X_{-1} \subset X_0 \subset \ldots X_{n-1} \subset X_n = X$, by closed subschemes such that $X_i/X_{i-1}$ is a disjoint union of affine schemes $U_{ij} \cong \mathbb{A}^{n_{ij}}$. Then $A_*(X)$ is finitely generated by $\{[V_{ij}]\}$, where $V_{ij}$ is the closure of $U_{ij}$ in $X$. For $X = \mathbb{P}^n$, we have a cell decomposition such that $X_i = \mathbb{P}^i$ and $X_i/X_{i-1} \cong \mathbb{A}^i$. Therefore, we have $A_k(\mathbb{P}^n) \cong \mathbb{Z}$ for all $0 \leq k \leq n$. Similarly, we can compute the Chow ring of the Grassmannians using a cell-decomposition.

**Remark 2.1** X a smooth projective variety.

- If $\dim(X) = 0$, we have $A_*(X) \cong \mathbb{Z}$.
- If $\dim(X) = 1$, then $A_*(X) = \mathrm{Pic}(X) \oplus \mathbb{Z}$.
- For $\dim(X) \geq 2$, it is difficult to compute $A_*(X)$. In fact Mumford showed that if $H^0(X, \Omega_X^2) \neq 0$, then $A_0(X)$ is infinite dimensional. The converse is conjectured to hold true.

**Intersection theory.** We showed that for a rank $r$ vector bundle $\pi : E \to X$, the map

$$\pi^* : A_d(X) \to A_{d+r}(E)$$

is surjective. In the next lecture we will see that $\pi^*$ is actually an isomorphism. Now the inverse map

$$(\pi^*)^{-1} : A_{d+r}(E) \to A_d(X)$$

should be viewed as obtained by intersecting cycles on $E$ with the zero section of $\pi$, $s : X \to E$. Intersection theory is concerned with intersecting cycles (and cycle classes) with subschemes. As we have seen above the map $(\pi^*)^{-1}$ solves the intersection problem for the closed subscheme of $E$ given by the zero section. This will form the basis for general intersection theory.

**Degeneration to the normal cone.** Let $X \hookrightarrow Y$ be a closed embedding of schemes. We define the normal cone of $X$ in $Y$ as

$$C := C_{X/Y} = \operatorname{Spec} \oplus_{n\geq 0} \frac{I^n}{I^{n+1}}$$

where $I$ is the ideal sheaf of $X$ in $Y$. Note that we have a surjective morphism $\oplus_{n\geq 0}\frac{I^n}{I^{n+1}} \longrightarrow \mathcal{O}_X$ which is zero for $n \geq 1$ and is the canonical isomorphism to $\mathcal{O}_X$ fo r$n = 0$. This determines a closed embdding $X \hookrightarrow C$, whcih we call the zero-section embedding. Let $M$ be the blow-up of $Y \times \mathbb{P}^1$ along $X \times \{\infty\}$. Now the normal cone of $X \times \{\infty\}$ in $Y \times \mathbb{P}^1$ is $C \oplus 1$ and the exceptional divisor in the blow-up is $P(C \oplus 1)$.

We have a sequence of embeddings

$$X = X \times \{\infty\} \hookrightarrow X \times \mathbb{P}^1 \hookrightarrow Y \times \mathbb{P}^1.$$

Now the blow-up of $X \times \mathbb{P}^1$ along $X \times \{\infty\}$ is canonically isomorphic to $X \times \mathbb{P}^1$. Therefore we have an embedding

$$X \times \mathbb{P}^1 \hookrightarrow M.$$

Similarly the blow-up $\widetilde{Y}$ of $Y$ along $X$ has a closed embedding into $M$. We have a flat morphism

$$M \longrightarrow Y \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

where the first map is the blow-down map. We have a commutative diagram

$$\begin{tikzcd} X \times \mathbb{P}^1 \arrow[rr, hook] \arrow[dr, "\pi_2"] & & M \arrow[dl, "\rho"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

such that:

(1) Over $\mathbb{P}^1 - \{\infty\} = \mathbb{A}^1$, we have $\rho^{-1}(\mathbb{A}^1) = Y \times \mathbb{A}^1$ and the embedding is the trivial embedding

$$X \times \mathbb{A}^1 \hookrightarrow Y \times \mathbb{A}^1.$$

(2) Over $\infty$, the divisor $M_\infty = \rho^{-1}(\infty)$ is the sum of two effective Cartier divisors

$$M_\infty = P(C \oplus 1) + \widetilde{Y}.$$

The embedding of $X = X \times \{\infty\}$ in $M_\infty$ is the zero-section embedding $X \hookrightarrow C$. The divisors $P(C \oplus 1)$ and $\widetilde{Y}$ intersect in the scheme $P(C)$, which is embedded as the hyperplane at infinity in $P(C \oplus 1)$ and as the exceptional divisor in $\widetilde{Y}$.

Let $\widetilde{M} = M \setminus \widetilde{Y}$. Then we have a family of embeddings

$$\begin{tikzcd} X \times \mathbb{P}^1 \arrow[rr, hook] \arrow[dr, "\pi_2"] & & \widetilde{M} \arrow[dl, "\widetilde{\rho}"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

which deforms the given embedding $X \hookrightarrow Y$ to the zero-section embedding $X \hookrightarrow C_{X/Y}$. Note that if $X$ is a $d$-dimensional variety, then $X \times \mathbb{P}^1$ is a $(d+1)$ -dimensional variety and $C_{X/Y}$ has pure dimension $d$.

CHAPTER 3

# Lecture 3

In the last lecture we discussed degeneration to the normal cone. For a closed embedding $X \hookrightarrow M$, let $I = \mathcal{I}_{X/M}$ be the ideal sheaf of $X$ in $M$. We defined an affine scheme over $X$, as

$$C_{X/M} = \operatorname{Spec} \oplus_{n \geq 0} \frac{I^n}{I^{n+1}} \longrightarrow X,$$

called the normal cone of $X$ in $M$. The normal cone plays an important role in Intersection theory. Let us discuss cones in general.

Recall the property (†) from Hartshorne:
Let $X$ be a scheme and $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_X$-modules which has the structure of a sheaf of graded $\mathcal{O}_X$-algebras. We say that $\mathcal{A}$ satisfies (†) if

- $\mathcal{A} = \oplus_{n \geq 0} \mathcal{A}_n$ and $\mathcal{A}_0 = \mathcal{O}_X$,
- $\mathcal{A}_1$ is a coherent $\mathcal{O}_X$-module,
- $\mathcal{A}_\infty$ generates $\mathcal{A}$ as an $\mathcal{O}_X$-algebra. (It follows that $\mathcal{A}_n$ is coherent for all $n \geq 0$.)

**Definition 3.1** A cone over $X$ is $\pi : C \longrightarrow X$ together with a $\mathbb{G}_m$-action, where $C = \operatorname{Spec}\mathcal{A}$ and $\mathcal{A}$ satisfies (†), i.e. there is an action of $\mathbb{G}_m$ on $C$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[dr, "\pi"'] & C \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

for all $g \in \mathbb{G}_m$.

**Note:**

(1) Cones should be thought of as a generalization of vector bundles. We know that for a vector bundle $E \longrightarrow X$, there is a locally free sheaf $\mathcal{E}$ on $X$ such that $E = \operatorname{Spec}\operatorname{Sym}^*\mathcal{E}$.

(2) We can recover the sheaf $\mathcal{A}$ as the pushforward, i.e., $\mathcal{A} = \pi_* C$.
(3) The $\mathbb{G}_m$-action corresponds to the grading. Let $C = \mathrm{Spec}\mathcal{A} \longrightarrow X$ be a cone. We have,

$$\mathbb{G}_m \times C = \mathrm{Spec}\mathcal{A}[t, t^{-1}]$$

and $\mathbb{G}_m$ acts on $C$ by the map

$$\mathrm{Spec}\mathcal{A}[t, t^{-1}] \longrightarrow \mathrm{Spec}\mathcal{A}$$

induced by $\phi : \mathcal{A} \longrightarrow \mathcal{A}[t, t^{-1}]$, where $\phi$ is defined as follows

$$\phi(a_0 + \cdots + a_n) = a_0 + a_1 t + \cdots + a_n t^n$$

for $a_i \in \mathcal{A}_i$. Note that, $\mathcal{A}_i$ is the eigensheaf in $\mathcal{A}$ corresponding to the character $t \mapsto t^i$.

**Definition 3.2** An *Abelian cone* is a cone $\mathrm{Spec}\mathcal{A}$ such that the natural map (given by the product)

$$\mathrm{Sym}^* \mathcal{A}_1 \to \mathcal{A}$$

is an isomorphism. Note that the map is surjective by assumption. A morphism of cones $C_1 \to C_2$ is a morphism of $X$-schemes which is $\mathbb{G}_m$-equivariant.

**Notation.** We denote by $Cone_X, Abcone_X, Vb_X$ the categories of cones over $X$, abelian cones over $X$ and vector bundles over $X$ respectively. Similarly $Shf_X^\dagger, Coh_X, locf_X$ the categories of sheaf of $\mathcal{O}_X$ algebras satisfying (†), coherent sheaves on $X$, locally free sheaves on $X$ respectively.

**Lemma 3.1** We have natural equivalence of categories

$$Shf_X^\dagger \leftrightarrow (Cone_X)^o, \;\; Coh_X \leftrightarrow (Abcone_X)^o, \;\; locf_X \leftrightarrow (Vb_X)^o.$$

where $(Cone_X)^o$ denotes the opposite category of $Cone_X$ and similarly for others.

PROOF. We have functor $\mathrm{Spec} : Shf_X^\dagger \longrightarrow (Cone_X)^o$ with the inverse given by $(C \xrightarrow{\pi} X) \mapsto \pi_* \mathcal{O}_C$. We have a natural functor $locf_X \longrightarrow Shf_X^\dagger$ given by $\mathcal{E} \mapsto \mathrm{Sym}^* \mathcal{E}$. We have a commutative diagram,

$$\begin{tikzcd} locf_X \arrow[r, "\mathrm{Spec}"] \arrow[d, "\mathcal{E} \mapsto \mathrm{Sym}^*\mathcal{E}"'] & Vb_X \arrow[d, "\subset"'] \\ Shf_X^\dagger & Cone_X \arrow[l, "\mathrm{Spec}"'] \end{tikzcd}$$

□

**Example 3.1** Let $X$ be a scheme over a field $k$ such that $\Omega_{X/k} \in \mathrm{Coh}_X$. Then $T_X = \mathrm{Spec}\mathrm{Sym}\Omega_{X/k}$.

**Lemma 3.2** The inclusion $Abcone_X \hookrightarrow Cone_X$ has an adjoint

$$C = \mathrm{Spec}\mathcal{A} \hookrightarrow \mathrm{Spec}\mathrm{Sym}^*\mathcal{A}_1 := A(C)$$

and if $C_1$ is an abelian cone, then

$$\mathrm{Mor}_{Cone_X}(A(C), C_1) \longrightarrow \mathrm{Mor}(C, C_1)$$

is a bijection.

**Remark 3.1** If $C_1 \longrightarrow X$ and $C_2 \longrightarrow X$ are cones, then so is $C_1 \times_X C_2$. This is easy to see, since the condition (†) is closed under tensor product.

**Remark 3.2** Let $C = \mathrm{Spec}\mathcal{A} \longrightarrow X$ be cone. Then we have a map $\mathcal{A} \longrightarrow \mathcal{O}_X$ given by $\mathcal{A}_i \mapsto 0$ for $i > 0$. Hence we get an unique $\mathbb{G}_m$-invariant zero section $s_0 : X \longrightarrow C$.

**Remark 3.3** If $f : \widetilde{X} \longrightarrow X$ is a morphism, it induces

$$f^* : \mathrm{Cone}_X \longrightarrow \mathrm{Cone}_{\widetilde{X}}$$

which is given by $C \mapsto C \times_X \widetilde{X}$. Note that this preserves being abelian and vector bundles.

Recall, that for $X \hookrightarrow M$ a closed closed subscheme, defined by an ideal sheaf $\mathscr{I}$. The *normal cone* $C_{X/M}$ to $X$ in $M$ is defined by the sheaf of graded $\mathcal{O}_X$-algebras $\oplus \mathscr{I}^n/\mathscr{I}^{n+1}$:

$$C_{X/M} = \mathrm{Spec}(\oplus \mathscr{I}^n/\mathscr{I}^{n+1}).$$

Then $N_{X/M} := A(C_{X/M})$ is an abelian cone called the normal sheaf of $X$ in $M$. Therefore we have a natural closed embedding $C_{X/M} \hookrightarrow N_{X/M}$ (as seen in Lemma 0.5).

**Question.** Suppose we have a commutative diagram,

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, hook, "i"'] & P \arrow[d, "\rho"] \\ & M \end{tikzcd}$$

where $i$, $j$ are closed embeddings and $\rho$ is smooth (For example, $P = \mathbb{P}^n$ and $\rho$ is projection from a point). Is there a relation between $C_{X/P}$ and $C_{X/M}$ ? between $N_{X/P}$ and $N_{X/M}$ ?

Let $I = \mathcal{I}_{X/M}$ and $J = \mathcal{I}_{X/P}$ be the ideal sheaves of $X$ in $M$ and $P$. Then the isomorphism $\rho^*\mathcal{O}_M \xrightarrow{\sim} \mathcal{O}_P$ in $\mathrm{Coh}_P$ induces a map $\phi : \rho^* I \hookrightarrow J$, since the composition $\rho^* I \longrightarrow \rho^*\mathcal{O}_M \xrightarrow{\sim} \mathcal{O}_P \longrightarrow j_*\mathcal{O}_X$ is zero and $J = \ker(\mathcal{O}_P \longrightarrow j_*\mathcal{O}_X)$. Pulling back by $j$, we get an injective map $j^*\rho^* I \longrightarrow j^* J$. Note that, $j^* J = J/J^2$ and $j^*\rho^* I = I/I^2$, as locally on $\mathrm{Spec} A$ we have $I \otimes A/I = I/I^2$.

**Lemma 3.3** We have an exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow J/J^2 \longrightarrow j^*\Omega_{P/M} \longrightarrow 0.$$

Proof. Recall that, for a map $P \longrightarrow M$ and $X$ a closed subscheme of $P$, we have an exact sequence

$$J/J^2 \longrightarrow \Omega_{P/M} \otimes \mathcal{O}_X \longrightarrow \Omega_{X/M} \longrightarrow 0.$$

Since $i : X \hookrightarrow M$ and $\rho j = i$, we have $\Omega_{X/M} = 0$. Therefore, we have a surjective map $J/J^2 \longrightarrow j^*\Omega_{P/M}$. It remains to check exactness in the middle. For that we pass to formal completions as follows. First we note that it is enough to check exactness at the stalks, i.e. after base extension to $\mathrm{Spec}(\mathcal{O}_{x,X})$ for all points $x \in X$. We know that a sequence of sheaves is exact iff it is exact after faithfully flat base change. Recall that $\mathrm{Spec}(\hat{\mathcal{O}}_{x,X}) \longrightarrow \mathrm{Spec}(\mathcal{O}_{x,X})$ is faithfully flat. Therefore after base change we may assume that $M = \mathrm{Spec}(A)$, $A$ a complete local ring and $P = \mathrm{Spec}(B)$, $B$ a formal power series ring over $A$. Let $B = A[[X_1, \ldots, X_n]]$, then $\Omega_{B/A}$ is the free $B$-module generated by $dX_1, \ldots dX_n$. Now we have the map $\delta : J/J^2 \longrightarrow \Omega_{B/A} \otimes_B (B/J)$ given by $\delta\bar{b} = db \otimes 1$ for a class $\bar{b} \in J/J^2$. A little computation shows that $\ker(\delta) = I/I^2$. $\square$

Note that $j^*\Omega_{P/M}$ is locally free since $P \longrightarrow M$ is smooth,. Therefore, the exact sequence above is Zariski locally split exact (as the Ext-groups of free modules are trivial). Therefore by dualizing the exact sequence above, we obtain the following.

**Corollary 3.1** There are natural morphisms

$$0 \longrightarrow j^* T_{P,M} \longrightarrow N_{X/P} \longrightarrow N_{X/M} \longrightarrow 0$$

such that Zariski locally on $X$ the sequence splits.

**Lemma 3.4** We have a sequence of cones

$$0 \longrightarrow j^* T_{P/M} \longrightarrow C_{X/P} \longrightarrow C_{X/M} \longrightarrow 0$$

which locally splits.

Proof. The proof is similar to the one of Lemma 3.3. For details we refer to Fulton. $\square$

Note that $j^*T_{P/M}$ is a vector bundle and the sequence is Zariski locally split. This motivates the following definition.

**Definition 3.3** An exact sequence of cones on a scheme $X$ is

$$0 \longrightarrow E \longrightarrow C \longrightarrow C_1 \longrightarrow 0$$

where $E$ is a vector bundle, $C$, $C_1$ are cones and Zariski locally on $X$, there exists a splitting $C_1 \longrightarrow C$ such that $C \simeq C_1 \times_X E$.

**Remark 3.4** Recall that, if $M$ is $d$-dimensional variety, then $C_{X/M}$ is a purely $d$-dimensional scheme. As we showed in the last lecture, $C_{X/M}$ is a Cartier divisor in the blow-up which is a $(d+1)$-dimensional variety.

If $X \hookrightarrow M$ is a closed embedding where $X$ and $M$ are non-singular, then $C_{X/M} = N_{X/M}$ is a bundle. More generally, this is true if $X \hookrightarrow M$ is a regular embedding, i.e. $\mathcal{I}_{X/M,x}$ is generated by a regular sequence in $\mathcal{O}_{M,x}$.

**Example 3.2** Let $M = \mathbb{A}^3$ and $\mathcal{I}_X = (xy, yz)$. Then $C_{X/M} = N_{X/M}$ has two irreducible components both of dimension 3.

**Example 3.3** Let $M = \mathbb{A}^2$ and $\mathcal{I}_X = (y^2, yx)$. Then $C_{X/M} = N_{X/M}$ has two irreducible components of dimension 2.

**Example 3.4** Let $Y \subset \mathbb{P}^{n-1}$ be a closed subscheme, $M \subset \mathbb{A}^n$ be the cone over $Y$ and $X = 0 \subset M$. Then $C_{X/M} \simeq M$, but $N_{X/M} \neq M$.

**Example 3.5** Let $M = \mathbb{A}^2_{\mathbb{C}}$ and $\mathcal{I}_X = (x^2, y^2, xy)$. Let $R = \mathbb{C}[x, y]/(x^2, y^2, xy)$. Then $C_{X/M} = \mathrm{Spec}R[a, b, c]/(ab - c^2)$ and $N_{X/M} = \mathrm{Spec}R[a, b, c]$.

**Definition 3.4** A morphism of schemes $f : X \longrightarrow Y$ is said to admit a factorization if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, hook, "i"'] & Y \\ & M \arrow[u, "\pi"'] \end{tikzcd}$$

where $i$ is a closed embedding and $\pi$ is a smooth morphism. A factoriazation $X \xrightarrow{i} M \xrightarrow{\pi} Y$ is said to dominate another factroization $X \xrightarrow{i_1} M_1 \xrightarrow{\pi_1} Y$ if there exists $\phi : M \longrightarrow M_1$ such that TFDC

$$\begin{tikzcd} & M_1 \arrow[dr, "\pi_1"] & \\ X \arrow[ur, "i_1"] \arrow[r, "i"] & M \arrow[u, "\phi"] \arrow[r, "\pi"] & Y \end{tikzcd}$$

**Remark 3.5** (1) Factorizations exist if $X$ and $Y$ are quasi-projective.

(2) It always exists locally on $X$.

(3) If $f$ admits two factorizations $X \xrightarrow{i_a} M_a \xrightarrow{\pi_a} Y$ for $= 1, 2$. Then there is a factorization that dominates both of them. We can take $M = M_1 \times_Y M_2$, then TFDC

$$\begin{tikzcd} & M_1 \arrow[dr, "\pi_1"] & \\ X \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & M \arrow[u, "\phi_1"] \arrow[r, "\pi"] \arrow[d, "\phi_2"] & Y \\ & M_2 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

**Definition 3.5** A closed embedding of schemes $i : X \longrightarrow Y$ is called a regular embedding of codimension $d$ if every point in $X$ has an affine neighbourhood $U$ in $Y$, such that if $A$ is the coordinate ring of $U$, $I$ the ideal of $A$ defining $X$, then $I$ is generated by a regular sequence of length $d$.

A morphism of schemes $f : X \longrightarrow Y$ is called a local complete intersection (in short, l.c.i) if it has a factorization $X \xrightarrow{i} M \xrightarrow{\pi} Y$ where $i$ is a regular embedding.

Note that this implies for any factorization $X \xrightarrow{i_1} M_1 \xrightarrow{\pi_1} Y$, the map $i_1$ is a regular embedding. We define the relative dimension of such an l.c.i. morphism $f : X \longrightarrow Y$ to be $r = \mathrm{rk}(T_{M/Y}) - \mathrm{codim}(i)$.

The following result is a key construction in intersection theory. We will discuss it in detail later.

Given $f : X \longrightarrow Y$ an l.c.i. morphism of relative dimension $r$ and given a cartesian diargam

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

we can associate to it a map $f^! : A_d(\widetilde{Y}) \longrightarrow A_{d+r}(\widetilde{X})$ which is functorial and compatible with proper push forward, flat pull back, Chern classes etc. This map is called the refined Gysin homomorphism. The idea of the construction is as follows. We take a factorization $f : X \xrightarrow{i} M \xrightarrow{\pi} Y$ and base change it by $\widetilde{Y} \longrightarrow Y$. Note that this gives a factorization $\widetilde{X} \longrightarrow \widetilde{M} \longrightarrow \widetilde{Y}$, but it's no longer an l.c.i morphism.

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{M} \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d] \\ X \arrow[r] & M \arrow[r] & Y \end{tikzcd}$$

We know that $C_{X/M} = N_{X/M}$ is a vector bundle since $i : X \longrightarrow M$ is a regular embedding. We can understand $C_{\widetilde{X}/\widetilde{M}}$ as follows.

**Lemma 3.5** For any cartesian diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{M} \arrow[d] \\ X \arrow[r, "i"'] & M \end{tikzcd}$$

with $i$ a closed embedding, we have a natural closed embedding $C_{\widetilde{X}/\widetilde{M}} \hookrightarrow f^*C_{X/M}$.

Now if we start with a cycle class in $A_d(\widetilde{Y})$ we can pull it back to $\widetilde{M}$ as $\widetilde{M} \longrightarrow \widetilde{Y}$ is smooth, in particular flat. Then we degenerate it to the normal cone $C_{\widetilde{X}/\widetilde{M}}$ and consider it as an element in $f^*C_{X/M}$, which is a vector bundle as $C_{X/M}$ is so. Then we can intersect with the zero-section to obtain an element in $A_{d+r}(\widetilde{X})$.

Now we go back to our previous situation. Assume that we have a commutative diagram

$$\begin{tikzcd} & P \arrow[dr, "\pi_1"] & \\ X \arrow[ur, "j"] \arrow[r, "i"] & M \arrow[u, "\rho"] \arrow[r, "\pi"] & Y \end{tikzcd}$$

such that $i$,$j$ are closed embeddings and $\pi$ and $\rho$ are smooth. Let $I$, $J$ be in Lemma 3.3. Then there is a natural commutative diagram with exact rows in $\mathrm{Coh}(X)$.

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & \ker\alpha \arrow[r] & I/I^2 \arrow[r] \arrow[d] & i^*\Omega_\pi \arrow[r] \arrow[d] & \Omega_{X/Y} \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & \ker\beta \arrow[r] & J/J^2 \arrow[r] \arrow[d] & i^*\Omega_{P/Y} \arrow[r] \arrow[d] & \Omega_{X/Y} \arrow[r] & 0 \\ & & j^*\Omega_\rho \arrow[d] & j^*\Omega_\rho \arrow[d] & & \\ & & 0 & 0 & & \end{tikzcd}$$

The first vertical column is exact by Lemma 3.3 and the second column is exact by definition of $j^*\Omega_\rho$. This induces an isomorphism on $\ker\alpha \xrightarrow{\sim} \ker\beta$. Therefore we have the following result.

**Corollary 3.2** If $f : X \longrightarrow Y$ has a factorization then we can associate to each factorization $X \xrightarrow{i} M \xrightarrow{\pi} Y$, a complex with objects in $\mathrm{Coh}(X)$,

$$0 \longrightarrow I/I^2 \longrightarrow i^*\Omega_{M/Y} \longrightarrow 0 \longrightarrow 0$$

where $I/I^2$ is in degree $-1$ and $i^*\Omega_{M/Y}$ is in degree 0. This defines an object in the derived category $D(\mathrm{Coh}(X)$. We define $\widetilde{L}_f$ to be this object in $D(\mathrm{Coh}(X)$.

**Remark 3.6** The complex $\widetilde{L}_f$ is the truncation of the cotangent complex of $f$, denoted $L_f$, i.e. $\widetilde{L}_f = \tau_{\geq -1}L_f$. The morphism $f : X \longrightarrow Y$ is l.c.i. then $\widetilde{L}_f$ is perfect in $[-1, 0]$, which means Zariski locally we can write it as a morphism of locally free sheaves of finite rank conecentrated in the degrees $-1$ and 0. Furthermore $f$ is smooth if and only if $\widetilde{L}_f$ is perfect in $[0,0]$, i.e. there is no cohomology in degree $(-1)$ and the cohomology in degree 0 is locally free.

CHAPTER 4

# Lecture 4

Let $\pi : E \longrightarrow X$ be a rank $r$ vector bundle on a scheme $X$. Recall, we proved that $\pi^* : A_d(X) \longrightarrow A_{d+r}(E)$ is surjective. We want to show that it is injective. Before we go into the proof, we take a detour into Chern classes and Segre classes.

Recall that is $X$ is a $d$-dimensional variety, there is a natural group homomorphism $\mathrm{div} : \mathrm{Pic}(X) \longrightarrow A_{d-1}(X)$. This motivates the following definition.

**Definition 4.1** Let $X$ be a scheme, $L$ a line bundle on $X$, and we denote $A_*(X) = \oplus_{d\geq 0} A_d(X)$. We define $c_1(L)$ to be an endomorphism of degree $(-1)$ in $A_*(X)$, i.e.

$$c_1(L) : A_d(X) \longrightarrow A_{d-1}(X)$$

for all $d \geq 1$, as follows :
if $V \subset X$ is a $d$-dimensional subvariety and $L|_V \in \mathrm{Pic}(V)$, we define

$$c_1(L).[V] := i_*\mathrm{div}(L|_V) \in A_{d-1}(X),$$

where $i : V \hookrightarrow X$ is the inclusion. This $c_1(L)$ respects rational equivalence and descends to a map $c_1(L) : A_d(X) \longrightarrow A_{d-1}(X)$. This is called the first Chern class map and we denote $c_1(L).\alpha$ by $c_1(L) \cap \alpha$.

**Proposition 4.1** First Chern class of line bundles commute with proper push forward, flat pull back and each other, i.e.

(1) Let $f : \overline{X} \longrightarrow X$ be proper and $L \in \mathrm{Pic}(X)$, $\alpha \in A_d(\overline{X})$. Then

$$f_*(c_1(f^*(L)) \cap \alpha) = c_1(L) \cap f_*\alpha \in A_{d-1}(X).$$

(2) Let $f : \overline{X} \longrightarrow X$ be a flat morphism of relative dimension $r$, and $L \in \mathrm{Pic}(X)$, $\alpha \in A_d(X)$. Then

$$f^*(c_1(L) \cap \alpha) = c_1(L) \cap f^*\alpha.$$

(3) Given $L_1$, $L_2 \in \mathrm{Pic}(X)$,

$$c_1(L_1) \cap (c_1(L_2) \cap \alpha) = c_1(L_2) \cap (c_1(L_1) \cap \alpha)$$

for all $\alpha \in A_d(X)$.

PROOF. (1) By functoriality of pull-back and push-forward we may assume $\alpha = [V]$, $V = X$ and $f(V) = \overline{X}$. Let $D$ be a Cartier divisor such that $\mathcal{O}(D) = L$ and then $\mathrm{div}(L) = [D]$. Then we need to prove that

$$f_*([f^*D]) = \deg(\overline{X}/X)[D].$$

Since this question is local on $X$, we may assume $D = \mathrm{div}(r)$ for some rational function $r$ on $X$. Let $d = \deg(\overline{X}/X)$. Then we have,

$$f_*[\mathrm{div}(f^*r)] = [\mathrm{div}(N(f^*r))] = \mathrm{div}(r^d) = d[\mathrm{div}(r)].$$

(2) As above, we may assume $\alpha = [V] = [X]$ and let $D$ be as above. The we need to prove

$$[f^*D] = f^*[D]$$

as cycles on $\overline{X}$. This is again local on $X$, so we may assume $D$ to be a difference of effective divisors. Then it is enough to prove for effective divisors, which follows from Lemma 1.2. For (3) we refer to Fulton. The general argument involves pseudo-divisors.

□

Let $E = \mathrm{SpecSym}^*\mathcal{E}$ be a rank $r$ vector bundle on $X$. Consider $P(E) = \mathrm{Proj}(\mathrm{Sym}^*\mathcal{E})$ and let $f : P(E) \longrightarrow X$ be the canonical map, which is both proper and flat. On $P(E)$ we have a line bundle $\mathcal{O}(1)$.

**Definition 4.2** For all $i \geq -r+1$, define $s_i(E) \in \mathrm{End}_{-i}(A_*(X))$ i.e. an endomorphism of degree $-i$, as

$$s_i(E) \cap \alpha := f_*(c_1(\mathcal{O}(1))^{r-1+i} \cap f^*\alpha)$$

**Proposition 4.2** Segre classes commute with proper push forward, flat pullback and each other.

PROOF. Refer to Fulton. Prop 3.1. □

**Lemma 4.1** $s_0(E) = \mathrm{id}$ and $s_i(E) = 0$ for $i < 0$. ($s_i$'s are called Segre classes)

PROOF. As seen earlier, by Noetherian induction we reduce to $E$ being trivial. Let $P(E) = \mathbb{P}^{r-1} \times X$. Let $V \subset X$ be a subvariety. Then,

$$f^*[V] = [\mathbb{P}^{r-1} \times V]$$

and

$$\begin{aligned}
&c_1(\mathcal{O}(1)) \cap [\mathbb{P}^{r-1} \times V] = [\mathbb{P}^{r-2} \times V] \\
&c_1(\mathcal{O}(1))^{r-1} \cap [\mathbb{P}^{r-1} \times V] = [p \times V], \quad p \in \mathbb{P}^{r-1}(K) \\
&f_*[p \times V] = [V].
\end{aligned}$$

As $s_i$ commtes with proper push forward, we may assume $V = X$ and then $s_i(E)[V] \in A_{n-i}(X) = 0$ as $i < 0$ ($n = \dim(X)$).

$\square$

**Remark 4.1** $A_*(X)$ is complete as a graded $\mathbb{Z}$-module.

**Definition 4.3** We define the total Segre class to be

$$s(E) = s_0(E) + s_1(E) + \cdots + s_n(E).$$

Consider the power series

$$s_t(E) = 1 + s_1(E)t + s_2(E)t^2 + \ldots$$

We define the Chern polynomial to be the inverse power series

$$c_t(E) = 1 + c_1(E)t + c_2(E)t^2 + \ldots$$

It can be shown that $c_i(E) = 0$ for $i > r = \mathrm{rk}(E)$. The total Chern class is defined as

$$c(E) = 1 + c_1(E) + \cdots + c_r(E)$$

and $c_i(E)$ is called the $i$-th Chern class of $E$.

An important trick in proving the vanishing of Chern classes is the splitting principle which we recall below. For details of the proof we refer to Fulton.

Splitting principle. Given a vector bundle $E$ of rank $r$ on $X$, there exists a flat morphism $f : Y \longrightarrow X$ such that

(1) $F^* : A_*X \longrightarrow A_*Y$ is injective,
(2) $f^*E$ has a filtration by subbundles

$$f^*E = E_r \subset E_{r-1} \subset \cdots \subset E_1 \subset E_0 = 0$$

with line bundle quotients $L_i = E_i/E_{i-1}$.

The injectivity of $\pi^* : A_d(X) \longrightarrow A_{d+r}(E)$ follows from the following:

**Proposition 4.3** Let $f : P(E) \longrightarrow X$ be as before. Define $\theta : \oplus_{i=0}^{r-1} A_*(X) \longrightarrow A_*(P(E))$ as $\theta(a_0, a_1, \ldots, a_{r-1}) = \sum_{i=0}^{r-1} c_1(\mathcal{O}(1))^i f^*(a_i)$. Then $\theta$ is an isomorphism.

Proof. Surjectivity follows from surjectivity of $\pi^*$, once we reduce to the trivial bundle. Injectivity can be proved by fixing teh degree and using that negative Segree classes are zero. $\square$

Let $f : X \longrightarrow Y$ be an l.c.i. morphism of relative dimension $r$. Let $X \xrightarrow{i} M \xrightarrow{\pi} Y$ where $i$ is a regular embedding and $\pi$ is smooth of relative dimension $s$. Let $\widetilde{Y} \longrightarrow Y$ be a morphism and $\widetilde{X} := X \times_Y \widetilde{Y}$. Then $\widetilde{X} \hookrightarrow \widetilde{M} = M \times_Y \widetilde{Y}$ and $\widetilde{M} \longrightarrow \widetilde{Y}$ is smooth of relative dimension $s$. Let $\widetilde{g}$ be the map $\widetilde{X} \longrightarrow X$.

**Definition 4.4** We define $f^! : Z_d(\widetilde{Y}) \longrightarrow A_{d+r}(\widetilde{X})$ as a composition of three morphisms

$$Z_d(\widetilde{Y}) \longrightarrow Z_{d+s}(\widetilde{M}) \longrightarrow Z_{d+s}(C_{\widetilde{X}/\widetilde{M}}) \longrightarrow A_{d+r}(\widetilde{X})$$

where the morphisms are as described below.

(1) The first map is just the flat pull-back $\pi^* : Z_d(\widetilde{Y}) \longrightarrow Z_{d+s}(\widetilde{M})$.

(2) The second map is the specialization morphism. Let $V \subset \widetilde{M}$ be a $(d+s)$ -dimensional subvariety, $W = V \cap \widetilde{X}$ the scheme-theoretic intersection and $C_{W/V}$ the purely $(d+r)$-dimensional cone over $W$. We have the following commutative diagram with cartesian squares

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "\overline{g}"] & V \arrow[d] \\
\widetilde{X} \arrow[r] \arrow[d, "\widetilde{g}"] & \widetilde{M} \arrow[d] \\
X \arrow[r, "f"] & M
\end{tikzcd}$$

Now $\overline{g}$ is a closed embedding and so we have closed embeddings

$$C_{W/V} \hookrightarrow \overline{g}^* C_{\widetilde{X}/\widetilde{M}} \hookrightarrow C_{\widetilde{X}/\widetilde{M}}$$

As $C_{W/V}$ is pure dimensional, it has a fundamental class $[C_{W/V}] \in Z_{d+s}(C_{W/V})$. We push-forward this class via the closed embeddings above to get a class in $Z_{d+s}(C_{\widetilde{X}/\widetilde{M}})$.

(3) We have a closed embedding $C_{\widetilde{X}/\widetilde{M}} \hookrightarrow \widetilde{g}^* C_{X/M}$. Since $i : X \hookrightarrow M$ is a regular embedding of codimension $s-r$, we have $C_{X/M} = N_{X/M}$ a vector bundle of rank $s-r$. Let $E$ be the vector bundle $g^* C_{X/M}$ over $\widetilde{X}$ and $\phi : A_{d+s}(E) \longrightarrow A_{d+r}(\widetilde{X})$ be the inverse of the isomorphism $A_{d+r}(\widetilde{X}) \xrightarrow{\sim} A_{d+s}(E)$. The third map is defined to be the following composition

$$Z_{d+s}(C_{\widetilde{X}/\widetilde{M}}) \longrightarrow Z_{d+s}(E) \longrightarrow A_{d+s}(E) \xrightarrow{\phi} A_{d+r}(\widetilde{X})$$

where the first one is the push-forward map and the second one is quotienting by rational equivalence.

The following lemma completes the construction of $f^!$ and lists the standard properties.

**Lemma 4.2** (1 )$f^1$ respects rational equivalence and induces a homomorphism $f^! : A_d(\widetilde{Y}) \longrightarrow A_{d+r}(\widetilde{X})$.

(2) $f^!$ does not depend on the factorization.

(3) $f^!$ is compatible with proper push-forward and flat pullback.

(4) If we have morphisms of schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$ where $f$ is a regular embedding and $g$, $g \circ f$ are flat. Then $f^! \circ g^* = (g \circ f)^*$.

(5) If instead $f$, $g \circ f$ are regular embeddings and $g$ is smooth, then $f^! \circ g^* = (g \circ f)^!$.

CHAPTER 5

# Lecture 5

Recall the following key construction in Intersection theory. Let $f : X \longrightarrow Y$ be a l.c.i. morphism of relative dimension $r$, i.e. $f$ admits a factorization $X \stackrel{i}{\longrightarrow} M \stackrel{\pi}{\longrightarrow} Y$, where $\pi$ is smooth of relative dimension $s$ and $i$ is a regular embedding of codimension $s - r$. Then for any morphism $g : \widetilde{Y} \longrightarrow Y$, let $\widetilde{X} = X \times_Y \widetilde{Y}$ and $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$. We constructed a homomorphism $f^! : A_d(\widetilde{Y}) \longrightarrow A_{d+r}(\widetilde{X})$ for all $d \geq 0$, which commutes with proper push-forwards, flat pull-backs and Chern classes. We say that $[f^!] \in A^{-r}(X \stackrel{f}{\longrightarrow} Y)$ is a bivariant class.

We are can ask the following questions:

**Question 1.** What if $f$ is only locally l.c.i., i.e. such a factorization exists only locally?

**Question 2.** What if $i$ is just a closed embedding (not necessarily a regular embedding)?

We will answer Question 1 towards the end. Let us discuss Question 2. Recall that we defined $f^!$ as the following composition

$$A_d(\widetilde{Y}) \longrightarrow A_{d+s}(\widetilde{M}) \longrightarrow A_{d+s}(C_{\widetilde{X}/\widetilde{M}}) \longrightarrow A_{d+r}(\widetilde{X})$$

where a key ingredient in defining the last map was the embedding of the cone $C_{\widetilde{X}/\widetilde{M}} \hookrightarrow \widetilde{g}^* C_{X/M}$ into a vector bundle of rank $s - r$. We can generalize this to the following result from Fulton which addresses the case when $i$ is just a closed embedding.

Assume that $X \stackrel{i}{\hookrightarrow} M$ is a closed embedding and there is a closed embedding of cones $C_{X/M} \hookrightarrow E$, where $E$ is a rank $r$ vector bundle on $X$. Let $\widetilde{M} \longrightarrow M$ be a morphism and $\widetilde{X} = X \times_M \widetilde{M}$. We can define $i^!_E : A_d(\widetilde{M}) \longrightarrow A_{d-r}(\widetilde{X})$ as a composition of maps as done for $f^!$. Let the first map be given by the specialization to the normal cone and the second one is as follows. We have closed embedding $C_{\widetilde{X}/\widetilde{M}} \hookrightarrow \widetilde{g}^* C_{X/M} \hookrightarrow \widetilde{g}^* E$. We then take the composition

$A_d(C_{\widetilde{X}/\widetilde{M}}) \longrightarrow A_(\widetilde{g}^*E) \longrightarrow A_{d-r}(\widetilde{X})$, where the first map is the push-forward and the second one is the inverse of the isomorphism $A_{d-r}(\widetilde{X}) \xrightarrow{\sim} A_d(\widetilde{g}^*E)$.

**Remark 5.1** (i) This depends on the choice of the vector bundle $E$.
(ii) Note that an embedding $C_{X/M} \hookrightarrow E$ is equivalent to giving a surjective morphism $\mathcal{E} \longrightarrow I/I^2$ where $E = \mathrm{SpecSym}\mathcal{E}$ and $I = \mathcal{I}_{X/M}$.

Let us recall the cases where we have a construction of the Gysin homomorphism.

(1) If $i : X \hookrightarrow M$ is a regular closed embedding of codimension $r$. We have a map $i^! : A_d(\widetilde{M}) \longrightarrow A_{d-r}(\widetilde{X})$, for any morphism $\widetilde{M} \longrightarrow M$.

(2) If $i : X \hookrightarrow M$ is a closed embedding and we have a closed embedding of cones $C_{X/M} \hookrightarrow E$. Then for any morphism $\widetilde{M} \longrightarrow M$ and $\widetilde{X} = X \times_M \widetilde{M}$, we have $i^!_E : A_d(\widetilde{M}) \longrightarrow A_{d-r}(\widetilde{X})$, which depends on the choice of $E$.

(3) If $f : X \longrightarrow Y$ is an l.c.i. morphism of codimednsion $r$, with a factorization $X \overset{i}{\hookrightarrow} M \xrightarrow{\pi} Y$ where $i$ is a regular closed embedding and $\pi$ is smooth. Then for any $\widetilde{Y} \longrightarrow Y$ and $\widetilde{X} = X \times_Y \widetilde{Y}$, we defined $f^! : A_d(\widetilde{Y}) \longrightarrow A_{d+r}(\widetilde{X})$. This depends only on $f$ and is independent of the factorization.

We would like to generalize these constructions to the class of morphisms $f : X \longrightarrow Y$, which admits a factorization $X \overset{i}{\hookrightarrow} M \xrightarrow{\pi} Y$, where $i$ is just a closed embedding and $\pi$ is smooth. One possibility is to use (2), but in that case $f^!$ would depend on the factorization and the choice of $E$. We would like to know, what additional structure one needs to put on $f : X \longrightarrow Y$ such that there is an induced embedding $C_{X/M} \hookrightarrow E$ and $f^! := i^!_E \circ \pi^*$ is independent of the factorization. Further, we would like the vector bundle $E$ to have the following property. Let $X \overset{j}{\hookrightarrow} P \xrightarrow{\phi} Y$ be another factorization of $f$ and a morphism of factorizations $\rho : P \longrightarrow M$, i.e. TFDC

$$\begin{tikzcd}
 & P \arrow[d, "\rho"] \arrow[dr, "\phi"] & \\
X \arrow[ur, "j"] \arrow[r, "i"'] & M \arrow[r, "\pi"'] & Y
\end{tikzcd}$$

Suppose we have vector bundles $C_{X/M} \hookrightarrow E_1$ and $C_{X/P} \hookrightarrow E_2$ as above. Let $E_i = \mathrm{SpecSym}\mathcal{E}_i$. The embeddings above correspond to surjective morphisms $\mathcal{E}_1 \longrightarrow I/I^2$ and $\mathcal{E}_2 \longrightarrow J/J^2$. By, Lemma 3.3 we have an exact sequence of coherent sheaves

$$0 \longrightarrow I/I^2 \longrightarrow J/J^2 \longrightarrow j^*\Omega_{P/M} \longrightarrow 0.$$

We would like to have maps $\mathcal{E}_1 \longrightarrow \mathcal{E}_2$ and $\mathcal{E}_2 \longrightarrow j^*\Omega_{P/M}$ such that TFDC

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[d] & \mathcal{E}_2 \arrow[r] \arrow[d] & j^*\Omega_{P/M} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & I/I^2 \arrow[r] & J/J^2 \arrow[r] & j^*\Omega_{P/M} \arrow[r] & 0
\end{tikzcd}$$

Recall that, by Corollary 3.2., each factorization defines an object in $D(\mathrm{Coh}(X))$ as the class of the complex

$$0 \longrightarrow I/I^2 \longrightarrow i^*\Omega_{M/Y} \longrightarrow 0 \longrightarrow 0.$$

with $I/I^2$ in degree $(-1)$ and $i^*\Omega_{M/Y}$ in degree 0.

Let $D^{(-1,0)}(\mathrm{Coh}(X))$ be the full subcategory of $D(\mathrm{Coh}(X))$ of complexes $\mathcal{F}^\bullet$ such that $H^i(\mathcal{F}^\bullet) = 0$ for all $i \neq -1, 0$. For simplicity we will assume that $X$ has enough locally free sheaves.

We define the category $\mathcal{B}$ as follows:

- $\mathrm{Obj}\mathcal{B} = \{d : \mathcal{F}_{-1} \longrightarrow \mathcal{F}_0 | \mathcal{F}_i \in \mathrm{Coh}(X), d \in \mathrm{Hom}_{\mathrm{Sh_X}}(\mathcal{F}_{-1}, \mathcal{F}_0)\}$
- A morphism between $\mathcal{F}_{-1} \longrightarrow \mathcal{F}_0$ and $\mathcal{G}_{-1} \longrightarrow \mathcal{G}_0$ is defined as a commutative diagram

$$\begin{tikzcd}
\mathcal{F}_{-1} \arrow[r] \arrow[d, "\phi_{-1}"] & \mathcal{F}_0 \arrow[d, "\phi_0"] \\
\mathcal{G}_{-1} \arrow[r] & \mathcal{G}_0
\end{tikzcd}$$

Hence we have functor $\mathcal{B} \longrightarrow D^{(-1,0)}(\mathrm{Coh}(X))$ which maps $d : \mathcal{F}_{-1} \longrightarrow \mathcal{F}_0$ to the complex

$$\ldots \longrightarrow 0 \longrightarrow \mathcal{F}_{-1} \longrightarrow \mathcal{F}_0 \longrightarrow 0 \longrightarrow \ldots$$

It is easy to see that this is essentially surjective, but not an equivalence. We define the full subcategory $\mathcal{C}$ of $\mathcal{B}$ consisting of objects $\mathcal{F}_{-1} \longrightarrow \mathcal{F}_0$ with $\mathcal{F}_0$ is locally free. Again the functor $\mathcal{C} \longrightarrow D^{(-1,0)}(\mathrm{Coh}(X))$ is essentially surjective. Indeed, let $\mathcal{E}_0 \longrightarrow \mathcal{F}_0$ be s surjective morphism from a locally free sheaf $\mathcal{E}_0$ (this is possible since $X$ has enough locally frees). Then we define $\mathcal{E}_{-1} = \ker(\mathcal{F}_{-1} \oplus \mathcal{E}_0 \xrightarrow{\psi} \mathcal{F}_0)$ where $\psi(a,b) = b - a$. We have a morphism of compelxes

$$\begin{tikzcd} \mathcal{E}_{-1} \arrow[r] \arrow[d, "\phi_{-1}"] & \mathcal{E}_0 \arrow[d, "\phi_0"] \\ \mathcal{F}_{-1} \arrow[r] & \mathcal{F}_0 \end{tikzcd}$$

which is a quasi-isomorphism.

Therefore, if we have a morphism $f : X \longrightarrow Y$, which admits a factorization $X \hookrightarrow M \longrightarrow Y$, then we have an object associated to $f$ in $\mathcal{B}$, namely $I/I^2 \longrightarrow i^*\Omega_{M/Y}$ (since $\Omega_{M/Y}$ is locally free). Note that the $\Omega_{X/Y} = \mathrm{Coker}(I/I^2 \longrightarrow i^*\Omega_{M/Y})$. Therefore if $\Omega_{X/Y} = 0$, for example if $f : X \hookrightarrow Y$ is a closed embedding, then we have an object in $D^{(-1,-1)}(\mathrm{Coh}(X))$. Here $D^{(-1,-1)}(\mathrm{Coh}(X))$ is the full subcategory of objects $\mathcal{F}^\bullet$ such that $H^i(\mathcal{F}^\bullet) = 0$ for $i \neq -1$. There is an equivalence of categories $H^0 : D^{(-1,-1)}(\mathrm{Coh}(X)) \longrightarrow \mathrm{Coh}(X)$. Recall that we have an equivalence of categories $\mathrm{Coh}(X) \longrightarrow (\mathfrak{ab}_X)^o$. Philosophically, the complexes concentrated at one degree have Abelian cones as their geometric counterparts. We would like to have such an equivalence of categories for $D^{(-1,-1)}(\mathrm{Coh}(X))$ too. For this, we will have to use "Picard stacks"defined by Deligne in SGA $4\frac{1}{2}$. We will see that there is a natural equivalence of categories $D^{(-1,0)}(\mathfrak{ab}_X) \xrightarrow{\sim} \mathrm{Ho}(\text{Picard stacks})$, where the second object above is the homotopy category of Picard stacks.

Recall that for a smooth projective variety $V$, the Picard scheme $\mathrm{Pic}(V)$ represents the following functor.

$$(\mathfrak{Sch})^o \longrightarrow (\mathrm{Sets})$$

defined as

$$S \mapsto \{\alpha \in \mathrm{Pic}(V \times S)\}/\sim$$

where $L_1 \sim L_2$ iff there exists $M \in \mathrm{Pic}(S)$ such that $L_1 \cong L_2 \otimes p_S^* M$.

Now what is a "Picard stack"? Recall that a groupoid is a category where all morphisms are isomorphisms. A stack is a "sheaf of groupoids ". We will briefly discuss stacks in the following lectures. For an exposition on stacks one can refer to [Fan]. If we think of stacks to be an analogue of sheaf of sets, then Picard stacks can be thought of analogous to sheaves of Abelian groups. Then we abelian groups would generalize to Picard stacks over a point. Note that in order to give a set $S$ an abelian group structure, we need to have a map $m : S \times S \longrightarrow S$ which is commutative, associative and for all $s \in S$, the multiplication map $S \xrightarrow{m_X} S$, given by $x \mapsto s.x$, is bijective. Motivated by this, we have the following definition.

**Definition 5.1** A Picard groupoid (i.e. Picard stack over a point) is a groupoid $G$ with a functor $m : G \times G \longrightarrow G$ and two 2-arrows corresponding to commutativity and associativity, such that for all $g \in \mathrm{Obj}G$, the functor $m(g, \_)$ is an equivalence of groupoids.

CHAPTER 6

# Lecture 6

Recall that for a morphism $f : X \longrightarrow Y$ with a factorization $X \stackrel{i}{\hookrightarrow} M \longrightarrow Y$, we have an associated complex

$$\widetilde{L}_f := [I/I^2 \longrightarrow i^*\Omega_{M/Y}] \in D^{(-1,0)}(\mathrm{Coh}(X))$$

In fact $\widetilde{L}_f$ exists in $D^{(-1,0)}(X)$ of $\mathcal{O}_X$-modules for any morphism of schemes and if $f$ is locally of finite type, it exists in $D^{(-1,0)}(\mathrm{Coh}(X))$. We can define it as a truncation $\widetilde{L}_f := \tau_{\geq -1} L_f$, where $L_f$ is the cotangent complex of $f$. But we will mostly be interested in the situation when $f$ admits a factorization. Our approach to Question 2. in the last lecture leads us to the following:

**Goal:** To find conditions on morphsims $\mathcal{E}^\bullet \longrightarrow \widetilde{L}_f$ in $D^{(-1,0)}(\mathrm{Coh}(X))$ such that $\mathcal{E}^\bullet$ is isomorphic to $\mathcal{E} \longrightarrow i^*\Omega_{M/Y}$ in $D^{(-1,0)}(\mathrm{Coh}(X))$ with $\mathcal{E}$ is locally free and the morphism is given by

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "\phi"] & i^*\Omega_{M/Y} \arrow[d, "id"] \\ I/I^2 \arrow[r] & i^*\Omega_{M/Y} \end{tikzcd}$$

with $\phi : \mathcal{E} \longrightarrow I/I^2$ surjective.

**Lemma 6.1** Let $\phi : \mathcal{E}^\bullet \longrightarrow \mathcal{F}^\bullet$ be a morphism in $\mathcal{C}$ which induces an isomorphism in $D(\mathrm{Coh}(X)$. Then $\mathcal{E}_{-1}$ is locally free iff $\mathcal{F}_{-1}$ is locally free.

Proof. Since the question is local, we may assume $X = \mathrm{Spec} A$ and that $\mathcal{E}^0$, $\mathcal{F}^0$ are free. We can replace $\mathcal{E}^{-1} \longrightarrow \mathcal{E}^0$ by $\mathcal{E}^{-1} \oplus \widetilde{A}^{\oplus r} \longrightarrow \mathcal{E}^0 \oplus \widetilde{A}^{\oplus r}$, for some $r$, to assume that $\mathcal{E}^0 \longrightarrow \mathcal{F}^0$ is surjective. Since $\phi$ is a quasi-isomorphism, it follows that we have an exact sequence

$$0 \longrightarrow \mathcal{E}^{-1} \longrightarrow \mathcal{F}^{-1} \oplus \mathcal{E}^0 \longrightarrow \mathcal{F}^0$$

Then the claim follows since $\mathcal{E}^0$, $\mathcal{F}^0$ are free. $\square$

**Exercise 6.1** Let $\pi : Z \longrightarrow X$ be a flat morphism of relative dimension $r$. Then there exists a well-defined $R\pi_*\mathcal{F} \in D(\mathrm{Coh}(X))$ which is perfect in $[0,r]$, i.e. quasi-isomorphic to a complex of locally free sheaves with non-zero objects only in degrees $[0,r]$. Moreover, if $H^r(Z_x, \mathcal{F}_{Z_x}) = 0$ for all $x \in X$ then $R\pi_*\mathcal{F}$ is perfect in $[0, r-1]$. This is a derived version of cohomology and base change. (for details we refer to [Mum])

**Exercise 6.2** Suppose we have a commutative diagram

$$\begin{tikzcd} \mathcal{E}_{-1} \arrow[r, "d_E"] \arrow[d, "\phi_{-1}"] & \mathcal{E}_0 \arrow[d, "\phi_0"] \\ \mathcal{F}_{-1} \arrow[r, "d_F"] & \mathcal{F}_0 \end{tikzcd}$$

of coherent sheaves with $\phi_0$ an isomorphism. Then $\phi^{-1}$ is surjective iff $\mathrm{coker} d_E \longrightarrow \mathrm{coker} d_F$ is an isomorphism and $\ker d_E \longrightarrow \ker d_F$ is surjective.

**Exercise 6.3** A morphism $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ of coherent sheaves is surjective iff the induced map $\mathrm{F} \otimes_{\mathcal{O}_{X,x}} k(x) \longrightarrow \mathrm{G} \otimes_{\mathcal{O}_{X,x}} k(x)$ is surjective for all $x \in X$.

From the rest of the lecture, all schemes are assumed to be of finite type over $\mathbb{C}$ ( or an algebraically closed field of char 0).

**Definition 6.1** A fat point is a scheme $S$ such that $S_{red} \longrightarrow \mathrm{Spec}\mathbb{C}$ is an isomorphism.

**Remark 6.1** The category of fat points is equivalent to the category $\mathcal{A}^o$ where A is the catego of local $\mathbb{C}$-algebras $A$ such that the following equivalent conditions hold

- $A$ is finitely generated and Artinian
- $\dim A$ is finite
- $A$ is finitely generated and the maximal ideal $m_A$ is nilpotent.

**Definition 6.2** Let $S_1 \subset S_2$ be a closed embedding of fat points and $\phi : A_2 \longrightarrow A_1$ be the corresponding surjection in $\mathcal{A}^o$. Let $I = \ker\phi$. We say that the embedding is sqaure zero if $I^2 = 0$. It is semi-small if $I.m_A = 0$ and it is said to be small if $\dim_{\mathbb{C}} I = 1$.

**Lemma 6.2** Let

$$0 \longrightarrow I \longrightarrow A_2 \longrightarrow A_1 \longrightarrow 0$$

be a square-zero extension. Let $R$ be a $\mathbb{C}$-algebra and $P$ an $R$-algebra. Suppose we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "g"] & A_2 \arrow[d, "\phi"] \\ P \arrow[r, "h"] & A_1 \end{tikzcd}$$

Then the set $D = \{\alpha \in \mathrm{Hom}(P, A_2) | f = \alpha \circ g, h = \phi \circ \alpha\}$, if non-empty, is a principal homogeneous space under $\mathrm{Hom}_P(\Omega_{P/R}, I)$.

Proof. Suppose $D$ is non-empty. Let $\alpha_0 \in D$. Then for any $\alpha : P \longrightarrow A_2$ let $\lambda = \alpha - \alpha_0$. Then $\lambda \in \mathrm{Hom}_P(P, I)$. It is easy to check that $\lambda$ is an $R$-derivation iff $\alpha$ makes the diagram commute. Therefore we have a bijection $\mathrm{Hom}_P(\Omega_{P/R}, I) \longrightarrow D$ given by $\lambda \mapsto \alpha_0 + \lambda$. $\square$

**Proposition 6.1** Let $f : X \longrightarrow Y$ be a morphism of schemes with a factorization $X \overset{i}{\hookrightarrow} M \longrightarrow Y$. Let $S_1 \hookrightarrow S_2$ be a semi-small closed embedding of fat points. Suppose we have a commutative diagram

$$\begin{tikzcd} S_1 \arrow[r] \arrow[d] & X \arrow[d] \\ S_2 \arrow[r] & Y \end{tikzcd}$$

Let $x$ be the point defined by $x : S_{1,\mathrm{red}} \longrightarrow S_1 \longrightarrow X$. Let $J$ be the ideal sheaf of $X$ in $M$. Then there is a natural obstruction to the existence of $\alpha : S_2 \longrightarrow X$, making the diagram commute, which lies in $\mathrm{coker}((\mathrm{Hom}_{\mathbb{C}}(x^* i^* \Omega_{M/Y}, I) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(x^* J/J^2, I))$ where the map is induced by $J/J^2 \longrightarrow i^*\Omega_{M/Y}$.

Proof. We may assume that $X$ ,$Y$ are affine. Let $Y = \mathrm{Spec}R$, $M = Y \times \mathbb{A}^n = \mathrm{Spec}P$ with $P = R[x_1, \ldots, x_n]$ and $X = \mathrm{Spec}P/J$. Let $D$ be the set of $\alpha : S_2 \longrightarrow X$ and $N$ the set of $\beta : S_2 \longrightarrow M$ such that the following diagram commutes

$$\begin{tikzcd}
S_1 \arrow[r] \arrow[dd] & X \arrow[d, hook] \\
 & M \arrow[d] \\
S_2 \arrow[r] \arrow[uur, dashed, "\alpha"] \arrow[ur, dashed, "\beta"'] & Y
\end{tikzcd}$$

Then the map $D \longrightarrow E$ given by $\alpha \mapsto i \circ \alpha$ is injective as $i : X \hookrightarrow M$ is a closed embedding. We have the following commutative diagram of algebras

$$\begin{tikzcd}
A_1 & P/J \arrow[l] \arrow[ddl, dashed, "\alpha"'] \\
 & P \arrow[u] \arrow[dl, dashed, "\beta"] \\
A_2 \arrow[uu] & R \arrow[l] \arrow[u]
\end{tikzcd}$$

By abuse of notation we define $D$ (and $N$) to be the set of maps $\alpha$ (and $\beta$ resp.) such that the diagram above commutes. We have the following commuattive diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & P/J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] & A_2 \arrow[r] & A_1 \arrow[r] & 0
\end{tikzcd}$$

where $\beta$ induces the map $J \longrightarrow I$ above. Then we have a map $\theta : N \longrightarrow \mathrm{Hom}_{P/J}(J, I) = \mathrm{Hom}_{P/J}(J/J^2, I)$. By the Lemma above, $\mathrm{Hom}(\Omega_{P/R}, I)$ acts on $E$ and it acts on $\mathrm{Hom}_{P/J}(J/J^2, I)$ by the map $\phi : \mathrm{Hom}_{\mathbb{C}}(x^* i^* \Omega_{M/Y}, I) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(x^* J/J^2, I)$ induced by $J/J^2 \longrightarrow i^*\Omega_{M/Y}$. Also the map $\theta$ is $\mathrm{Hom}(\Omega_{P/R}, I)$-equivariant. The result then follows from the following Lemma. $\square$

**Lemma 6.3** Suppose we have a homomorphism of Abelian groups $\phi : M_1 \longrightarrow M_2$. Let $N$ be a principal homogeneous space over $M_1$ and $\theta : N \longrightarrow M_2$ a $M_1$-equivariant map. Let $\pi : M_2 \longrightarrow \mathrm{coker}\phi$ be the quotient map. Then

- $\pi(\theta(n_1)) = \pi(\theta(n_2)) := c_0$ for all $n_1, n_2 \in N$.

- Given $m \in M_2$ tehre exists $n \in N$ such that $\theta(n) = m$ iff $\pi(m) = c_0$.
- Any $r \in M_2$ is in the image of $\theta$ iff $c_0 = 0$.

**Corollary 6.1** Let $f : X \longrightarrow Y$ be a smooth morphism. The for any commutative diagram

$$\begin{tikzcd} S_1 \arrow[r] \arrow[d, hook] & X \arrow[d] \\ S_2 \arrow[r] \arrow[ur, dashed, "\alpha"] & Y \end{tikzcd}$$

where $S_1 \hookrightarrow S_2$ is a closed embedding of fat points. There exists $\alpha : S_2 \longrightarrow X$ such that the above diagram commutes.

Proof. Note that we can factor $S_1 \hookrightarrow S_2$ into a sequence of semi-small embeddings. Indeed, we have $A_2 = A_2/m^n I \longrightarrow A_2/M^{n-1}I \longrightarrow \ldots \longrightarrow A_2/I = A_1$. Since $f$ is smooth we can take $M = X$ in the factorization. The result follows from the above Propsition since smoothness implies that coker$\phi = 0$. $\square$

Let $f : X \longrightarrow Y$ be a morphism of schemes of finite type over $\mathbb{C}$. We define the vector space

$$T^2_{X,f} := \text{coker}(\text{Hom}_{\mathbb{C}}(x^* i^* \Omega_{M/Y}, \mathbb{C}) \xrightarrow{\phi} \text{Hom}_{\mathbb{C}}(x^* J/J^2, \mathbb{C}))$$

where $\phi$ is induced by $J/J^2 \longrightarrow i^*\Omega_{M/Y}$. It has the property that for any commutative diagram as above, with $S_1 \hookrightarrow S_2$ semi-small with kernel $I$, the obstruction to the existence of $\alpha$ lies in $T^2_{X,f} \otimes_{\mathbb{C}} I$. Further this obstruction is functorial. We say that $T^2_{X,f}$ is an obstruction space for $X$ and $Y$ at the point $x : \text{Spec}\mathbb{C} = \text{Spec}(S_{1,red}) \longrightarrow X$. If $V$ is any other $\mathbb{C}$-vector space which is an obstruction space for $X$ and $Y$ at $x$, then there exists an inclusion $T_{X,f} \hookrightarrow V$ inducing the obstruction.

Therefore we have the following conclusion:

Given $\mathcal{E}^\bullet \longrightarrow \widetilde{L}_f$ in $D^{(-1,0)}(\text{Coh}(X))$ the condition that $\mathcal{E}^\bullet$ is isomorphic to $\mathcal{E} \longrightarrow i^*\Omega_{M/Y}$ in $D^{(-1,0)}(\text{Coh}(X))$ with $\mathcal{E}$ is locally free, is equivalent to requiring that coker$(\text{Hom}(x^*\mathcal{E}^0, \mathbb{C}) \longrightarrow \text{Hom}(x^*\mathcal{E}^{-1}, \mathbb{C}))$ is an obstruction space for $X$ and $Y$ for all $x \in X$.

CHAPTER 7

# Lecture 7

Let $d : B \longrightarrow A$ be a homomorphism of Abelian groups. Define the groupoid $X := [A/B]$ as follows

- $\mathrm{Obj} X = A$
- $\mathrm{Mor}(a, a') = \{b \in B | a' = a + d(b)\}$ for all $a, a' \in X$.

The composition of morphisms is defined by addition in $B$.

We have a multilplication functor $m : X \times X \longrightarrow X$ such that $m(a_1, a_2) = a_1 + a_2$ and on morphisms $m(b_1, b_2) = b_1 + b_2$. The functor $m$ is strictly commutative, i.e. we have a commutative diagram

$$\begin{tikzcd} X \times X \arrow[rr, "j"] \arrow[dr, "m"'] & & X \times X \arrow[dl, "m"] \\ & X & \end{tikzcd}$$

In other words, $[A/B]$ is an Abelian group object in the category of groupoids.

Suppose we have a morphism $f$ in $D(Ab)$, the derived category of Abelian groups, which gives a commutative diagram

$$\begin{tikzcd} B_1 \arrow[r, "d_1"] \arrow[d, "f_B"] & A_1 \arrow[d, "f_A"] \\ B_2 \arrow[r, "d_2"] & A_2 \end{tikzcd}$$

Then we have an associated functor

$$F : [A_1/B_1] \longrightarrow [A_2/B_2]$$

which respects multiplication.Indeed, we define the functor as $a_1 \mapsto f_A(a_1)$ on for all objects $a \in A_1$ and on morphisms as $b_1 \mapsto f_B(b_1)$. It is easy to check that this is well-defined and commutes with multiplication.

**Exercise 7.1** The morphism $f$ is a quasi-isomorphism iff the functor $F$ is an equivalence of groupoids.

**Exercise 7.2** Let

$$f, g : (B_1 \longrightarrow A_1) \longrightarrow (B_2 \longrightarrow A_2)$$

be two morphisms in $D(Ab)$ and $h : A_1 \longrightarrow B_2$ a homotopy between $f$ and $g$. Then $h$ induces a natural equivalence between the associated functors $F$ and $G$.

**Remark 7.1** The converses are also true, i.e. every Abelian group object in the category of groupoids can be realized as $[A/B]$ for some morphism $f : B \longrightarrow A$ and similarly equivalences of groupoids arise from homotopies. In other words, studying complexes of length 2 of Abelian groups is equivalent to studying Abelian group objects in the category of groupoids. In the same spirit, to study complexes of length 2 of sheaves, one needs to study sheaves of groupoids, i.e. stacks!

Next we discuss some examples of stacks. For the definition of a stack we refer to [Fan].

Recall, that a scheme $X$ is determined (up to a canonical isomorphism) by its functor of points

$$h_X : \mathfrak{Sch}^o \longrightarrow (\text{Sets})$$

defined as

$$h_X(S) = \text{Mor}_{\mathfrak{Sch}}(S, X).$$

For affine schemes we have $h_X : (\text{Rings}) \longrightarrow (\text{Sets})$. The functor of points $h_X$ is a sheaf in the étale topology. An algebraic stack is a pseudo-functor (Rings) $\longrightarrow$ (Groupoids) which is a sheaf of groupoids in the étale/smooth topology.

**Example 7.1** Fix $g \geq 2$. We define the functor $M_g : \mathfrak{Sch}^o \longrightarrow$ (Groupoids) as follows:

- $\text{Obj}(M_g(S))$ consists of $\pi : C \longrightarrow S$ such that $\pi$ is smooth, projective and the geometric fibres of $\pi$ are connected genus $g$ curves.
- $\text{Mor}((\pi : C \longrightarrow S), (\pi_1 : C_1 \longrightarrow S)) = \{f : C \xrightarrow{\sim} C_1 | \pi = \pi_1 \circ f\}$.

For a morphism $\epsilon : \widetilde{S} \longrightarrow S$, we have the pull-back

$$\epsilon^*(C \xrightarrow{\pi} S) = (C \times_S \widetilde{S} \longrightarrow \widetilde{S}).$$

Now the category $M_g(\mathbb{C})$ consists of non-singular compelx projective curves of genus $g$ and the morphisms are isomorphisms. Note that this is not a rigid groupoid, as any curve of genus $g$ has non-trivial automorphsims (Recall that a rigid groupoid is a groupoid where all the automorphisms are trivial). We define $M_g^0(S) \subset M_g(S)$ to be the full-subcategory of objects such that the geometric fibres are rigid. Note that if $g = 2$, then $M_g^0$ is empty.

Suppose we have a map $S \longrightarrow M_g$ where $S$ is a scheme. This corrresponds to a family $C \longrightarrow S$. Let $S^0$ be the set of all geometric points $s \in S$ such that $\mathrm{Aut}(C_s) = 0$. Then $S^0 \subset S$ is an open subscheme. Let $\widetilde{S}$ is a scheme with morphisms $\epsilon : \widetilde{S} \longrightarrow S$ and $\widetilde{S} \longrightarrow M_g^0$ such that TFDC

$$\begin{tikzcd} \widetilde{S} \arrow[r] \arrow[d] & M_g^0 \arrow[d] \\ S \arrow[r] & M_g \end{tikzcd}$$

The map $\widetilde{S} \longrightarrow M_g^0$ corresponds to $\widetilde{C} \longrightarrow \widetilde{S}$ and the diagram above gives an isomorphism $\alpha : \widetilde{C} \xrightarrow{\sim} \epsilon^* C$ such that TFDC

$$\begin{tikzcd} \widetilde{C} \arrow[rr, "\alpha"] \arrow[dr] & & \epsilon^* C \arrow[dl] \\ & S & \end{tikzcd}$$

Therefore the fibres of $\epsilon^* C$ are also rigid and $\epsilon(\widetilde{S}) \subset S^0$. Hence $S^0 = S \times_{M_g} M_g^0$. We say that $M_g^0 \longrightarrow M_g$ is an open embedding as every base change by a scheme is an open embedding. It can be shown that $M_g^0(S)$ is a rigid groupoid for all schemes $S$ and the corresponding functor to (Sets) is representable by a smooth quasi-projective scheme of dimension $3g - 3$.

The stack $M_g$ can be viewed as a quotient stack $[V/G]$ for a scheme $V$ and a group scheme $G$ over $\mathrm{Spec}\mathbb{Z}$. Let $C \in M_g(\mathbb{C}$ and $L_C := K_c^{\otimes 3}$. Then $H^1(L_C) = 0$ and let $h^0(L_C) = N + 1$ , $d = \mathrm{def} L_c$. We define the category $V(\mathbb{C})$ as follows

- $\mathrm{Obj} V(\mathbb{C}) = \{[C; v_0, \ldots, v_N] | C \in M_g(\mathbb{C}), H^0(C, L_C) = \mathbb{C}\langle v_0, \ldots, v_N \rangle\}$.
- $\mathrm{Mor}([C, \{v_i\}], [\widetilde{C}, \{\widetilde{v}_i]\}) = \{\phi : C \longrightarrow \widetilde{C} | \phi^* \widetilde{v}_i = v_i\}$.

Although we defined $V(\mathbb{C})$, one can define $V(S)$ for all schemes $S$. For simplicity we stick with $\mathbb{C}$. The groupoid $V(\mathbb{C})$ is a rigid groupoid. We have a map $V(\mathbb{C}) \longrightarrow \mathrm{Hilb}^P(\mathbb{P}^N)$ where $P = dt + 1 - g$. The image $U \subset \mathrm{Hilb}^P(\mathbb{P}^N)$ is a locally closed subsheme and $V(\mathbb{C}) \longrightarrow U$ is a $\mathbb{C}^*$-bundle. Thus we have a scheme sturcture on $V(\mathbb{C})$. We have

$$M_g(\mathbb{C}) = [V(\mathbb{C})/\mathrm{GL}(N + 1)].$$

One can do this in families over any base scheme $S$, and it can be shown that indeed $M_g = [V/\mathrm{GL}(N+1)]$.

An algebraic stack $X$ which is locally of finite type over a field $k$ is a Deligne-Mumford stack ifff for all geometric points $x$, we have a cartesian diagram

$$\begin{array}{ccc} \mathrm{Aut}(x) & \longrightarrow & \mathrm{Spec}k \\ \downarrow & & \downarrow x \\ \mathrm{Spec}k & \xrightarrow{x} & X \end{array}$$

such that $\mathrm{Aut}(x) \longrightarrow \mathrm{Spec}k$ is étale. In our case $X = M_g$ and if $x$ corresponds to a curve $C$ , then we have $T\mathrm{Aut}(x) = H^0(C, T_C) = 0$. Therefore $\mathrm{Aut}(x) \longrightarrow \mathrm{Spec}k$ is étale and $M_g$ is a DM-stack.

**Example 7.2** Similarly, we define $\mathfrak{M}_g$ for all $g \geq 0$. Let $\mathfrak{M}_g(S)$ consist of morphisms $\pi : C \longrightarrow S$ such that $\pi$ is flat, projective and geometric fibres are connected nodal curves of genus $g$. The morphisms and pull-backs are defined as done earlier for $M_g$. Then $\mathfrak{M}_g$ is an algebraic stack locally of finite type over $\mathbb{C}$. To show that it is locally of finite type, it is enough to show that for all $C \in \mathfrak{M}_g(\mathbb{C})$, there exists a smooth open morphism $U \longrightarrow \mathfrak{M}_g$ from a scheme $U$ such that the image contains the point $C$. Let $L$ be a very ample line bundle on $C$ such that $H^1(C, L) = 0$. We choose a basis of $H^0(C, L)$ and have a corresponding embedding $C \hookrightarrow \mathbb{P}^N$. Let $\mathcal{U} \longrightarrow \mathrm{Hilb}^P(\mathbb{P}^N)$ be the open subscheme parametrizing nodal genus $g$ curves $C_u$ such that $H^1(C_u, \mathcal{O}(1)) = 0$, where $P = \mathrm{deg}Lt+1-g$.

The universal family defines a morphism $\mathcal{U} \longrightarrow \mathfrak{M}_g$. Since $C_u$ is l.c.i., we have an exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{\mathbb{P}^N}|_{C_u} \longrightarrow \Omega_{C_u} \longrightarrow 0.$$

By dualizing this we obtain

$$\mathrm{Hom}(I/I^2, \mathcal{O}_{C_u}) \longrightarrow \mathrm{Ext}^1(\Omega_{C_u}, \mathcal{O}_{C_u}) \longrightarrow \mathrm{Ext}^1(\Omega_{\mathbb{P}^N}|_{C_u}, \mathcal{O}_{C_u}).$$

Note that $\mathrm{Ext}^1(\Omega_{\mathbb{P}^N}|_{C_u}, \mathcal{O}_{C_u}) = 0$. We have $\mathrm{Hom}(I/I^2, \mathcal{O}_{C_u}) = T_{[u]}\mathrm{Hilb} = T_u\mathcal{U}$ and $\mathrm{Ext}^1(\Omega_{C_u}, \mathcal{O}_{C_u} = T_u\mathfrak{M}_g$. Then surjectivity of the first map implies that $\mathcal{U} \longrightarrow \mathfrak{M}_g$ is formally smooth. Since base change by any scheme is again a scheme we have that the map is smooth as desired.

**Example 7.3** Let $B$ be a base scheme, $V$ a $B$-scheme and $G$ a smooth group scheme over $B$ acting on $V$. We define the stack $[V/G] : (\mathfrak{Sch}/B)^o \longrightarrow$ (Groupoids) as follows:

- $\mathrm{Obj}[V/G](S)$ consists of principal $G$-bundles $P \longrightarrow S$ with a $G$-equivariant morphism $f : P \longrightarrow V$.
- $\mathrm{Mor}((P,f),(P_1,f_1))$ is the set of morphisms $\epsilon : P \longrightarrow P_1$ such that the following diagram commutes as $G$-bundles

$$\begin{tikzcd} P \arrow[rr, "\epsilon"] \arrow[dr, "f"'] & & P_1 \arrow[dl, "f_1"] \\ & V & \end{tikzcd}$$

Note that there is a map $V \longrightarrow [V/G]$ corresponding to the trivial principal $G$-bundle $G \times V \longrightarrow V$ and the action map $a : G \times V \longrightarrow V$.

**Exercise 7.3** If $V$ is a principal $G$-bundle over $W$. The the groupoid $[V/G]$ is canonically equivalent to the set $\mathrm{Mor}(S,W)$ compatible with pull-backs.

**Exercise 7.4** For any object $(P \longrightarrow S, f)$ in $[V/G](S)$ we have a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & V \arrow[d] \\ S \arrow[r] & {[V/G]} \end{tikzcd}$$

**Remark 7.2** If we set $B = \mathrm{Spec}\mathbb{C}$ and $G = \mathrm{GL}_N(\mathbb{C})$. Then $[B/G]$ is the stack of all principal bundles.

CHAPTER 8

# Lecture 8

Recall, if $X$ is a scheme with enough locally free sheaves then every object in $D^{(-1,0)}(\mathrm{Coh}X)$ can be written as $\mathcal{E}^{-1} \longrightarrow \mathcal{E}^0$ with $\mathcal{E}^{-1}$ locally free and $\mathcal{E}^0$ coherent. We would like to associate an algebraic stack with an $\mathbb{A}^1$-action to each such object $\mathcal{E}^{-1} \longrightarrow \mathcal{E}^0$. Let $E_i = \mathrm{SpecSym}\mathcal{E}^{-i}$. Then we have a morphism of Abelian group schemes $E_0 \longrightarrow E_1$, which defiens an action of $E_0$ on $E_1$. Since $E_0$ is an affine group scheme over $X$, we can construct the quotient stack $[E_1/E_0]$, which we denote by

$$\mathfrak{E} := (h^{-1}/h^0)^\vee(\mathcal{E}^\bullet) := h(\mathcal{E}^\bullet) := [E_1/E_0].$$

Here $\mathbb{A}^1$ is a group scheme under multiplication. One can think of the action of $\mathbb{A}^1$ on $E_0$, $E_1$ as the extension of the action of the multiplicative group $\mathbb{G}_m$. Suppose we have an object

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & E_1 \\ S & \end{tikzcd}$$

in $\mathfrak{E}(S)$, where $S \longrightarrow X$ is an $X$-scheme and $P \longrightarrow S$ is a principal $s^*E_0$-bundle. Let $a : s^*E_0 \times P \longrightarrow P$ be the action map. Let $\lambda \in \mathbb{A}^1(S)$. We define the action of $\lambda$ as $\lambda(a) : s^*E_0 \times P \longrightarrow P$ as the composition

$$\begin{tikzcd} s^*E_0 \times P \arrow[r, "a"] & P \\ s^*E_0 \times P \arrow[u, "(\lambda \mathrm{id}, \mathrm{id})"] \arrow[ur, "\lambda(a)"] & \end{tikzcd}$$

Similarly one can define an equivariant map to $E_1$. Thus we have an action of $\mathbb{A}^1$ on $\mathfrak{E}$.

**Proposition 8.1** Suppose we have a commutative diagram of coherent sheaves on $X$ Suppose we have a commutative diagram

$$\begin{tikzcd} \mathcal{E}_{-1} \arrow[r, "d_E"] \arrow[d, "\phi_{-1}"] & \mathcal{E}_0 \arrow[d, "\phi_0"] \\ \mathcal{F}_{-1} \arrow[r, "d_F"] & \mathcal{F}_0 \end{tikzcd}$$

with $\mathcal{E}^0$, $\mathcal{F}^0$ locally free. We have a corresponding commuative diagram of Abelian cones

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d] & F_1 \arrow[d] \\ E_0 \arrow[r] & E_1 \end{tikzcd}$$

Then

(1) The diagram induces an $\mathbb{A}^1$-equivariant morphism of algebraic stacks

$$h(\phi) : \mathfrak{E} \longrightarrow \mathfrak{E}$$

(2) $h(\phi)$ is strongly representable (i.e. base change bya scheme is a scheme) iff $h^0(\phi)$ is surjective.
(3) $h(\phi)$ is a closed embedding iff $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.
(4) $h(\phi)$ induces an isomorphism of algebraic stacks iff $\phi$ is a quasi-isomorphism.
(5) If $\alpha : \mathcal{E}^0 \longrightarrow \mathcal{F}^{-1}$ is a morphism in $\mathrm{Coh}(X)$ defining $\psi_{-1},\psi_{)}$ homotopic to $\phi_{-1},\phi_0$. Then $\alpha$ induces a 2-morphism

$$h(\alpha) : h(\phi) \Rightarrow h(\psi).$$

**Definition 8.1** Let $X$ be a scheme. An Abelian cone stack over $X$ is an algebraic stack $\mathfrak{E} \longrightarrow X$ wihci is Zariski-locally isomorphic ($\mathbb{A}^1$-equivariantly) to $h(\mathcal{E}^\bullet)$ for some $\mathcal{E}^{-1} \longrightarrow \mathcal{E}^0$ as above.

The following result can be obtained by arguing along the lines of Deligne's proof of $D^{(-1,0)}(\mathfrak{ab}_X) \xrightarrow{\sim} \mathrm{Ho}(\mathrm{Picard\ stacks})$.

**Theorem 8.1** ([BF]) Let $X$ be a scheme (not necessarily with enough locally frees).There is an equivalence of categories

$$D^{(-1,0)}_{\mathrm{Coh}}(X)^o \xrightarrow{\sim} \mathrm{Ho}(\mathrm{Abelian-cone\ stacks})$$

which is compatible with pull-back and extending $(h^{-1}/h^0)^\vee$.

**Corollary 8.1** Let $f : X \longrightarrow Y$ be a morphism of schemes. We can associate to $f$, an Abelian-cone stack $\mathfrak{N}_{X/Y}$ corresponding to $\widetilde{L}_f = \tau_{\geq 1} L_f$. This Abelian-cone stack $\mathfrak{N}_{X/Y}$ is flat local on $Y$ and étale local on $X$.

**Definition 8.2** An Abelian-cone stack $\mathfrak{E}$ is a vector bundle stack of rank $r \in \mathbb{Z}$ if $\mathfrak{E} \longrightarrow X$ is flat of relative dimension $r$.

**Lemma 8.1** $\mathfrak{E}$ is a vector bundle stack of rank $r$ iff $\mathfrak{E} = h(\mathcal{E}^{-1} \longrightarrow \mathcal{E}^0)$ with both $\mathcal{E}^{-1}$ and $\mathcal{E}^{)}$ locaaly free of rank $r_{-1}$ and $r_0$ respectively and $r = r_{-1} - r_0$.

**Corollary 8.2** Under the correspondence of the Theorem above, vector bundle stacks correspond to perfect compelxes of perfect amplitude in $[-1, 0]$.

The following result enables us to do intersection theory on stacks. One can see [Vis], [EG] for details and the proof.

**Theorem 8.2** One can define Chow groups $A_d(\mathfrak{X})$ for an algebraic stack $\mathfrak{X}$, which extends the definitions for schemes and proper push forwards , flat pullbcaks. If $\mathfrak{X}$ is reduced and irreducible of dimension $d_0 \in \mathbb{Z}$ then we have a fundamental class $[\mathfrak{X}] \in A_{d_0}(\mathfrak{X})$.

**Example 8.1** Let $\mathfrak{X} = B\mathbb{G}_m = [\mathrm{Spec}\mathbb{C}/\mathbb{G}_m]$ which is non-singular, irreducible of dimension $-1$. We write $\mathbb{C}^0 = \mathrm{Spec}\mathbb{C}$. We have a map

$$[\mathbb{C}^n/\mathbb{G}_m] \longrightarrow [\mathbb{C}^0/\mathbb{G}_m]$$

and an open substack

$$\mathbb{P}^{n-1} = [\mathbb{C}^n \setminus 0/\mathbb{G}_m] \subset [\mathbb{C}^n/\mathbb{G}_m].$$

We have maps

$$A_d(B\mathbb{G}_m) \longrightarrow A_{d+n}([\mathbb{C}^n/\mathbb{G}_m]) \twoheadrightarrow A_{d+n}(\mathbb{P}^{n-1})$$

Therefore $A_d \neq 0$ for $d \in \{-n, \ldots, -1\}$ for all $n$. Hence $A_d \neq 0$ for all $d \leq -1$. If we set $n = 1$, then $[\mathbb{C}/\mathbb{G}_m] \longrightarrow B\mathbb{G}_m$ is a line bundle. We denote the line bundle by $L$. Then $\mathbb{Z}[c_1(L)]$ acts on $A_*(B\mathbb{G}_m)$ and it is a free rank 1 module generated by $[B\mathbb{G}_m]$.

Let $f : X \longrightarrow Y$ be a morphism. We have the associated normal cone stack $\mathfrak{N}_{X/Y} = h(\widetilde{L}_f)$. Suppose locally on $X$ we have a factorization $X \overset{i}{\hookrightarrow} M \xrightarrow{\pi} Y$. Then by definition $\mathfrak{N}_{X/Y} = [N_{X/M}/i^*T_\pi]$.

**Lemma 8.2** (1) Given such a factorization $C_{X/M}$ is invariant under the $i^*T_\pi$-action and we have locally $\mathfrak{C}_{X/Y} := [C_{X/M}/i^*T_\pi] \subset \mathfrak{N}_{X/Y}$.

(2) These local constructions glue to give globally $\mathfrak{C}_{X/Y} \subset \mathfrak{N}_{X/Y}$.

(3) If $Y$ is reduced and irreducible then $\mathfrak{C}_{X/Y}$ is pure dimensional.

(4) If $f$ is l.c.i., then $\mathfrak{C}_{X/Y} = \mathfrak{N}_{X/Y}$ is a vector bundle stack.

Recall, if $f : X \longrightarrow Y$ is a closed embedding of schemes with $N_{X/Y} \hookrightarrow E$ , where $E$ is a rank $r$ vector bundle, then we can define $f^!_E \in A^r(X \longrightarrow Y)$. We can generalize this to stacks.

**Proposition 8.2** If $f : X \longrightarrow Y$ is a morphism of algebraic stacks which is of DM-type (base change by a scheme is a DM-stack) and $\mathfrak{N}_{X/Y} \hookrightarrow \mathfrak{E}$ a closed embedding of Abelian cone stacks where $\mathfrak{E}$ is a vector bundle of rank $r$. Then we can define $f^!_E \in A^r(X \longrightarrow Y)$.

**Remark 8.1** Degeneration to the normal cone can also be generalized to the case of stacks.

**Definition 8.3** Let $f : X \longrightarrow Y$ be a DM-type morphism of Artin stacks. A morphism $\phi : \mathcal{E}^\bullet \longrightarrow \widetilde{L}_f$ in $D^{(-1,0)}_{\mathrm{Coh}}(X)$ is called an Obstruction theory if the induced homomorphism $\mathfrak{N}_{X/Y} \longrightarrow \mathfrak{E} = h(\mathcal{E}^\bullet)$ is a closed embedding.

**Proposition 8.3** Given a morphism v as above. It is an obstruction theory if $\mathcal{E}^\bullet$ is 'as good as '$\widetilde{L}_f$ for computing tangent spaces and obstructions at geometric ppoints of $X$.