BASE FOR A TOPOLOGY

Abstract. In this folowing article we describe two different ways of defining a basis for the topology . Then we describe how from giving a "base to a set" we can generate a topology on it . Further when we again give a base to it by the earlier methods we get back the "base on the set" that we started with .

1. Defining a Topolgy

Let X be a set and τ a subset of the power set of X . Then the a pair (X, τ) is said to define a topology on a the set X if τ satisfies the following properties :

(1) If ϕ and X is an element of τ.
(2) If union of any arbitrary number of elements of τ is also an element of τ.
(3) If intersection of a finite number of elements of τ is also an element of τ.

Now we define the following terms which arise as consequence of the above construction :

(1) We call the elements of τ as 'open sets'.
(2) We define all other subsets of X which are not contained in τ as "closed sets".
(3) After the set has been endowed with a topology we call X as the " underlying set" for the topology .
(4) Earlier what were called the elements of the set X will be called points of the topological space (X, τ) once a topology has been defined on X .

The motivation behind restricting the intersections to finitely many is that if arbitrary number is allowed then the inetersection of all intervals in $\Re$ of the form $[-\frac{1}{n}, \frac{1}{n}]$ being a single point that is 0 will be declared open! . This goes against our intuition about real numbers and hence this has been prevented by inserting the finiteness condition.

2. Base for the topology

We give here two definitions for the base for a topology (X, τ) .

(1) We call a subset $B1$ of τ as the "Base for the topology" if every set in τ can be obtained by union of some elements of B .

Chennai Mathematical Institute (CMI).

(2) We call a subset $B2$ of τ as the "Basis for the topology" if for every point $x \in U \subset \tau$ there exists an element of $B2$ which contains x and is a subset of U.

We now need to show that $B1 = B2$.

2.1. $B1 \subset B2$. First we show that $B1 \subset B2$

Let U be an open set under the topological structure considered . By definition 1 there exists a subset of $B1$ such that their union is equal to U. Hence every element of $B1$ which participates in this union is a subset of U.
Further every point in U is contained in atleast one of the sets of those who have participated in the union and by the last sentence each of them is a subset of U.
Finally since every point of X is contained in atleast one element of τ (lets call that U) (trivially X is always there!)and since that element of τ can be written as a union of elements on $B1$ by the first two sentences there exists an element of $B1$ containing that point and is a subset of U
Hence every element of $B1$ which contributes in the union to create U is an element of $B2$.Hence after scanning through all elements of τ to access all the elemnts of $B1$ we can see that $B1 \subset B2$.

2.2. $B2 \subset B1$. This is fairly simple. We look at at each point $x \in U$ and by definition there exists atleast one element of $B2$ which contains x and is a subset of U.So around every point in U we can find an element of $B2$ which is completely contained in U . So we take the union of all such elements of $B2$ and that will exactly be equal to U.Hence the elements of $B2$ which contribute fopr this U satisfy the condition of being in $B1$ with respect to the open set U
Since this can be done for every U and by doing this for every $U \subset \tau$ we can access all the elements of $B2$ .Thus defining criteria of $B1$ is collectively satisfied and hence $B2 \subset B1$.

Since $B1 \subset B2$ and $B2 \subset B1$, we can conclude that $B1 = B2$.

Hence the basis generated by the two definitions are equal.

3. Creating a topology from a given base on a set

3.1. **Base of a set.** We now have just a set X and we define that $B3$ (a subset of the power set of X) will be said to be a base for X if :

(1) If for every element x of X there exists a element of $B3$ containing it .
(2) If P and Q are 2 elements of $B3$ which contain the point x then there exists a another element R of $B3$ which contains x and is a subset of $P \cap Q$.

3.2. **Giving a topology on the set X through its base (as defined above).** Now given this way of defining a "base for a set" , there is no obvious way a topology can be endowed on X but there exists a natural way as follows :

Declare a subset U of the power set of X to be "open" if for every point $x \in U$ there exists a $B \subset B3$ such that $x \in B$ and $B \subset U$.

Now the set of all such U as defined above will be the topology τ on X and (X, τ) forms the topological space.

Lets check it!

3.3. **Consistency check.** That ϕ is an element of τ is vacuously true.

That $X \subset \tau$ is also obvious as the first criteria of a "base" guarantees existence of an element of $B3$ containing every point of X and all such elements of $B3$ are obviously subsets of X. Hence X is open

Let P and Q be elements of τ and hence there exists an element of $B3$ for every element $x \in P \cup Q$ which contains x and is is a subset of $P \cup Q$ since it is a subset of either P or Q (by definition). Hence $P \cup Q$ is an element of τ.This can be iterated arbitary number of times and hence arbitary unions are also elements of τ

By definition of a "base of a set" for every element x in $P \cap Q$ there exists atleast one element U in $B3$ such that $x \in U$ and $U \subset P \cap Q$. Hence $P \cap Q$ qualifies to be in τ .

Now we do induction on collection of n open sets indexed by i like U_i .
$i = 1$ is trivially true and let us aaume that it is true till $i = n - 1$. let J be the intersection till $n - 1$. Then the case of the intersection of J and U_n is the same as the case for 2 open sets as shown earlier. Hence induction folows.

Hence the collection of open sets defined above indeed satisfy the conditions to form a topology over X.

4. Lets reverse the above process!

Now given a topological space (X, τ) (as derived from the above construction) we create a base for it by using the definition of the last section that is :

$B3$ (a subset of the power set of X) will be said to define a base for X if :

(1) If for every element x of X there exists a element of $B3$ containing it .
(2) If P and Q are 2 elements of $B3$ which contain the point x then there exists a another element R of $B3$ which contains x and is a subset of $P \cap Q$.

4.1. **$B3 \subset B2$.** Let us take an openset U in τ and an arbitary point x in it . So by definition of $B3$ since $x \in X$ there exists some $B \in B3$ such that $x \in B$. Since these open sets are defined according to the construction that follows from $B3$ in the last section we can find a $B \in B3$ that not only contains x but also is a subset of U. So by scanning through all opensets in τ we have all the elements of $B3$ and each of them qualifies to be an element of $B2$.
Hence $B3 \subset B2$.

4.2. **$B2 \subset B3$.** This is slightly tricky! We again take an openset $U \subset \tau$ and there exists an element B of $B2$ such that $x \in B \subset U$.lrt C be another element in $B2$ which contains x. By definition $C \subset U$. Therefore $B \cap C$ is also a subset of U and it contains x . Now think of $B \cap C$ as another set which is a subset of τ . So by re-invoking the definition of $B2$ we are again guaranteed that there exists an element $D \subset B2$ which contains x and is a subset of $B \cap C$.
Hence the triplet B , C and D together satisfy the conditions to be in $B2$.
Hence by scanning all the opensets we can track down all the elements of $B2$ and in triplets they all are qualifying to be elements of $B3$. Hence $B2 \subset B3$.

Since $B2 \subset B3$ and $B3 \subset B2$ we have that $B2 = B3$. Hence we have that the two ways of giving a base to the same base to the topology.

5. A slight digression

6. Neighbourhood

Let (X, τ) be a topological space. Let $x \in X$ be a point in this space.

6.1. **Open Neighbourhood.** An openset $U \subset \tau$ containing x is called an **"open neighbourhood"** of x .

6.1.1. *Neighbourhood.* A set (element of the power set of X) is called a **"neighbourhood"** of x if it contains an "openset" containing x alternatively contains an *open neighbourhood* containing x.

6.2. **Open Cover.** An **Open Cover** of X is a family of opensets $\{U_\alpha\}$ such that each U_α is an element of τ and $X \subset \bigcup U_\alpha$.

6.2.1. *Compactness.* A topological space (X, τ) is called **"Compact"** if from *every* open cover of it a finite subcover can be extracted.

6.3. **Local compactness.** A topological space (X, τ) is called **"locally compact"** if every point in this space has a compact neibhbourhood.

6.4. **Fundamental system of Neighbourhoods.** A set of *neighbourhoods* of a point x ,$\{U_s\}\{s \in S \subset I\}$is called a **fundamental system of neighbourhoods** of x if *every neighbourhood* of x contains *atleast one* of the U_s.

6.5. **The important claim.** The claim is that *Compact Neighbourhoods of a point form a fundamental system of neighbourhoods for the point*

7. Conclusion

Hence we have shown that the 2 *ways of giving a base to a topology are the same as they give the same base and that the opensets obtained naturally from the process of giving a "base to the underlying set" indeed gives a topology to the set . Further we have shown that if one of the earlier methods of giving a base to it is repeated we get back the "base on the underlying set" that we had started with.*